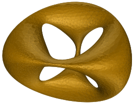
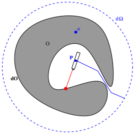
Mesh Generation

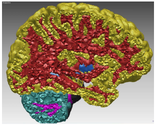
Jean-Daniel Boissonnat
DataShape, INRIA
http://www-sop.inria.fr/datashape

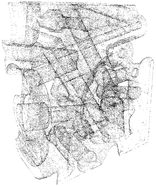
Meshing surfaces and 3D domains

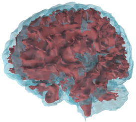
- visualization and graphics applications
- CAD and reverse engineering
- geometric modelling in medecine, geology, biology etc.
- autonomous exploration and mapping (SLAM)
- scientific computing : meshes for FEM











Mesh generation : from art to science

Grid methods

Lorensen & Cline [87] : marching cube
Lopez & Brodlie [03] : topological consistency
Plantiga & Vegter [04] : certified topology using interval arithmetic

Morse theory

Stander & Hart [97]
B., Cohen-Steiner & Vegter [04] : certified topology

Delaunay refinement

Hermeline [84]
Ruppert [95]
Shewchuk [02]

Chew [93]
B. & Oudot [03,04]
Cheng et al. [04]

Main issues

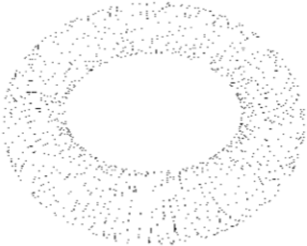
Sampling

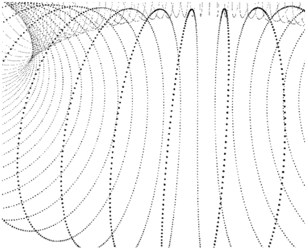
- How do we choose points in the domain ?
- What information do we need to know/measure about the domain ?

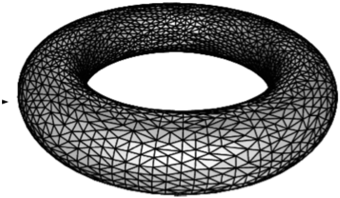
Meshing

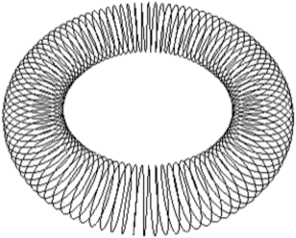
- How do we connect the points ?
- Under what sampling conditions can we compute a good approximation of the domain ?

Scale and dimension



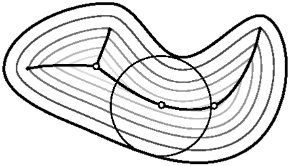






Medial axis of $\mathbb{M}$: $\text{axis}(\mathbb{M})$

set of points with at least two closest points on $\mathbb{M}$



Local feature size and reach

$\forall x \in \mathbb{M}, \quad \text{lfs}(x) = d(x, \text{axis}(\mathbb{M}))$

$\text{rch}(\mathbb{M}) = \inf_{x \in \mathbb{M}} \text{lfs}(x)$

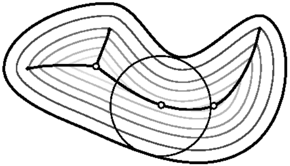
$(\epsilon, \bar{\eta})$-net of $\mathbb{M}$

1. $\mathcal{P} \subset \mathbb{M}, \forall x \in \mathbb{M} : \quad d(x, \mathcal{P}) \leq \epsilon \, \text{lfs}(x)$
2. $\forall p, q \in \mathcal{P}, \quad \|p - q\| \geq \bar{\eta}\varepsilon \, \min(\text{lfs}(p), \text{lfs}(q))$

Sampling conditions [Federer 1958], [Amenta & Bern 1998]

Medial axis of $\mathbb{M}$: $\mathrm{axis}(\mathbb{M})$

set of points with at least two closest points on $\mathbb{M}$



Local feature size and reach

$\forall x \in \mathbb{M}, \quad \mathrm{lfs}(x) = d(x, \mathrm{axis}(\mathbb{M}))$

$\mathrm{rch}(\mathbb{M}) = \inf_{x \in \mathbb{M}} \mathrm{lfs}(x)$

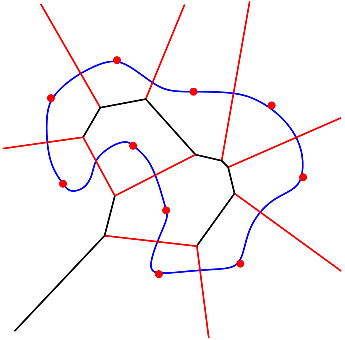
$(\epsilon, \bar{\eta})$-net of $\mathbb{M}$

1. $\mathcal{P} \subset \mathbb{M}, \forall x \in \mathbb{M} : \quad d(x, \mathcal{P}) \leq \epsilon \, \mathrm{lfs}(x)$
2. $\forall p, q \in \mathcal{P}, \quad \|p - q\| \geq \bar{\eta}\varepsilon \, \min(\mathrm{lfs}(p), \mathrm{lfs}(q))$

Restricted Delaunay triangulation [Chew 93]

Definition

The restricted Delaunay triangulation $\mathrm{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\mathrm{Vor}(\mathsf{P}) \cap X$

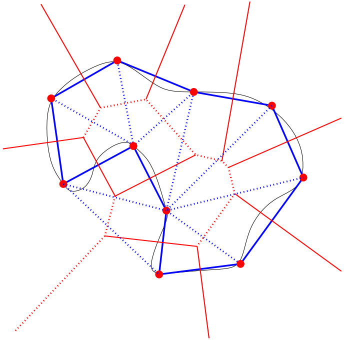


If P is an ε-sample, any ball centered on X that circumscribes a facet f of $\mathrm{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon \,\mathrm{lfs}(c_f)$

Restricted Delaunay triangulation [Chew 93]

Definition

The restricted Delaunay triangulation $\mathrm{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\mathrm{Vor}(\mathsf{P}) \cap X$

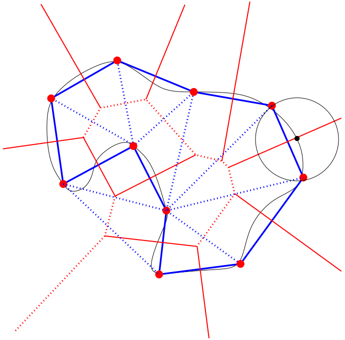


If P is an ε-sample, any ball centered on X that circumscribes a facet f of $\mathrm{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon\,\mathrm{lfs}(c_f)$

Restricted Delaunay triangulation [Chew 93]

Definition

The restricted Delaunay triangulation $\mathrm{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\mathrm{Vor}(\mathsf{P}) \cap X$

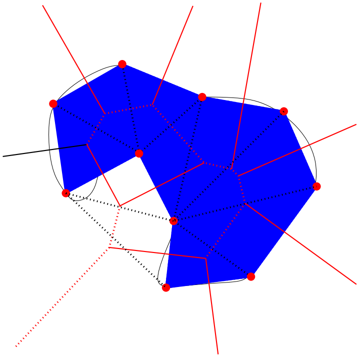


If P is an ε-sample, any ball centered on X that circumscribes a facet f of $\mathrm{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon\, \mathrm{lfs}(c_f)$

Restricted Delaunay triangulation [Chew 93]

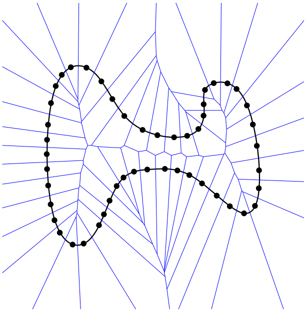
Definition

The restricted Delaunay triangulation $\mathrm{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\mathrm{Vor}(\mathsf{P}) \cap X$



Restricted Delaunay triang. of $(\varepsilon, \bar{\eta})$-nets

[Amenta et al. 1998-], [B. & Oudot 2005]



If

- $\mathcal{S} \subset \mathbb{R}^3$ is a compact surface of positive reach without boundary
- $\mathcal{P}$ is an $(\varepsilon, \bar{\eta})$-net with $\bar{\eta} \geq cst$ and ε small enough

then

- $\mathrm{Del}_{|S}(\mathcal{S})$ provides good estimates of normals
- There exists a homeomorphism $\phi : \mathrm{Del}_{|S}(\mathcal{P}) \to \mathcal{S}$
- $\sup_x(\|\phi(x) - x\|) = O(\varepsilon^2)$

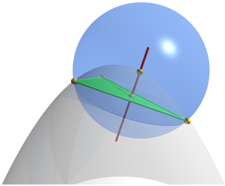
Surface mesh generation by Delaunay refinement

[Chew 1993, B. & Oudot 2003]

$\phi : S \to \mathbb{R}$ = Lipschitz function
$\forall x \in S,\ 0 < \phi_0 = \bar{\eta}_0\, \varepsilon\, \mathrm{rch}(\mathcal{S}) \le \phi(x) < \varepsilon\, \mathrm{lfs}(x)$

ORACLE : For a facet f of $\mathrm{Del}_{|S}(\mathcal{P})$,
return c_f, r_f and $\phi(c_f)$

A facet f is bad if $r_f > \phi(c_f)$



Algorithm

INIT compute an initial (small) sample $\mathcal{P}_0 \subset S$

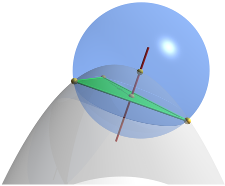
REPEAT IF f is a bad facet
insert_in_Del3D(c_f) ,
update $\mathcal{P}$ and $\mathrm{Del}_{|S}(\mathcal{P})$

UNTIL no bad facet remains

Surface mesh generation by Delaunay refinement

[Chew 1993, B. & Oudot 2003]

$\phi : S \to \mathbb{R}$ = Lipschitz function
$\forall x \in S,\; 0 < \phi_0 = \bar{\eta}_0\, \varepsilon\, \mathrm{rch}(\mathcal{S}) \leq \phi(x) < \varepsilon\, \mathrm{lfs}(x)$



ORACLE : For a facet f of $\mathrm{Del}_{|S}(\mathcal{P})$,
return c_f, r_f and $\phi(c_f)$

A facet f is bad if $r_f > \phi(c_f)$

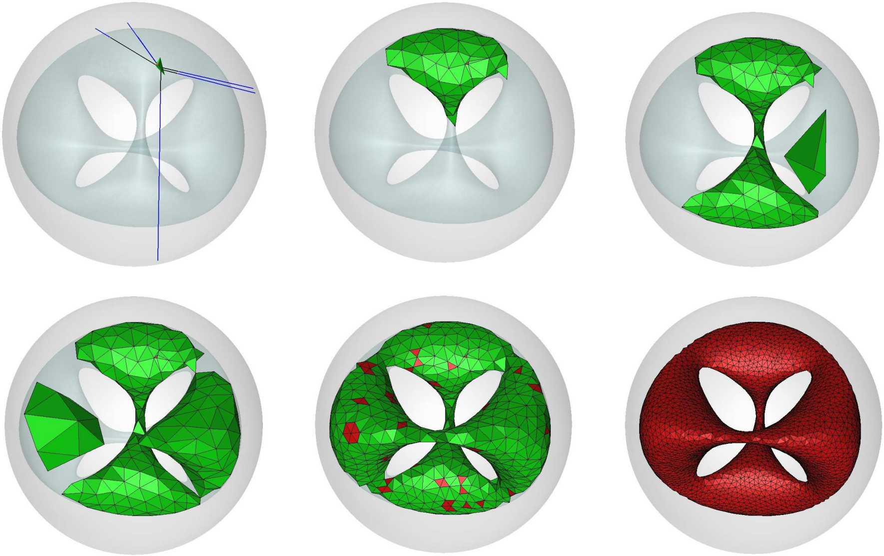
Algorithm

INIT compute an initial (small) sample $\mathcal{P}_0 \subset S$

REPEAT IF f is a bad facet
insert_in_Del3D(c_f) ,
update $\mathcal{P}$ and $\mathrm{Del}_{|S}(\mathcal{P})$

UNTIL no bad facet remains

The meshing algorithm in action



The algorithm outputs a loose net of $\mathcal{S}$

Separation

$$\begin{aligned} \forall p \in \mathcal{P}, d(p, \mathcal{P} \setminus \{p\}) &= \|p-q\| \\ &\geq \min(\phi(p), \phi(q)) \\ &\geq \phi(p) - \|p-q\| \qquad \text{(ϕ is Lipschitz)} \end{aligned}$$

$$\Rightarrow \quad \|p-q\| \geq \frac{1}{2}\,\phi(p) \geq \frac{1}{2}\,\phi_0 > 0 \qquad \Rightarrow \text{the algorithm terminates}$$

Density

- Each facet has a $\mathcal{S}$-radius $r_f \leq \phi(c_f) < \varepsilon\,\mathrm{lfs}(c_f)$
- If correctly initialized, $\mathrm{Del}_{|S}(\mathcal{P})$ has a vertex on each cc of $\mathcal{S}$

Size of the sample $= \Theta\left(\int_S \frac{dx}{\phi^2(x)}\right)$

Upper bound

$B_p = B(p, \frac{\phi(p)}{2}),\ p \in \mathcal{P}$

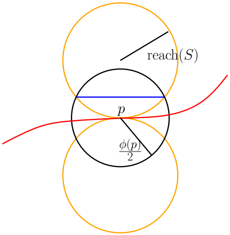
$$\int_S \frac{dx}{\phi^2(x)} \geq \sum_p \int_{(B_p \cap S)} \frac{dx}{\phi^2(x)}$$

(the B_p are disjoint)

$$\geq \frac{4}{9} \sum_p \frac{\text{area}(B_p \cap S)}{\phi^2(p)}$$

$$\phi(x) \leq \phi(p) + \|p - x\| \leq \frac{3}{2}\ \phi(p))$$

$$\geq \sum_p C \ = \ C\ |\mathcal{P}|$$



Lower bound

Use a covering instead of a packing

Size of the sample $= \Theta\left(\int_S \frac{dx}{\phi^2(x)}\right)$

Upper bound

$B_p = B(p, \frac{\phi(p)}{2}), \; p \in \mathcal{P}$

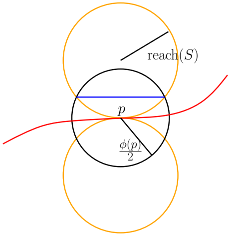
$$\int_S \frac{dx}{\phi^2(x)} \geq \sum_p \int_{(B_p \cap S)} \frac{dx}{\phi^2(x)}$$

(the B_p are disjoint)

$$\geq \frac{4}{9} \sum_p \frac{\text{area}(B_p \cap S)}{\phi^2(p)}$$

$$\phi(x) \leq \phi(p) + \|p - x\| \leq \frac{3}{2}\,\phi(p))$$

$$\geq \sum_p C \; = \; C\,|\mathcal{P}|$$



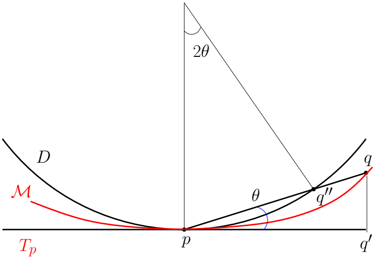
Lower bound

Use a covering instead of a packing

Chord lemma

Let x and y be two points of $\mathbb{M}$. We have

1. $\sin\angle(xy, T_x) \leq \frac{\|x-y\|}{2\,\mathrm{rch}(\mathbb{M})}$;
2. the distance from y to T_x is at most $\frac{\|x-y\|^2}{2\,\mathrm{rch}(\mathbb{M})}$.



$$\|x-y\| \geq \|x-y''\| = 2\,\mathrm{rch}(\mathbb{M})\ \sin\angle(xy, T_x)$$

$$\|y-y'\| = \|x-y\|\ \sin\angle(xy, T_x) \leq \frac{\|x-y\|^2}{2\,\mathrm{rch}(\mathbb{M})}$$

Tangent space approximation

Lemma

[Whitney 1957]

If σ is a j-simplex whose vertices all lie within a distance h from a hyperplane $H \subset \mathbb{R}^d$, then

$$\sin \angle(\text{aff}\,(\sigma), H) \leq \frac{2j\,h}{D(\sigma)} = \frac{2h}{\Theta(\sigma)\,\Delta(\sigma)}$$

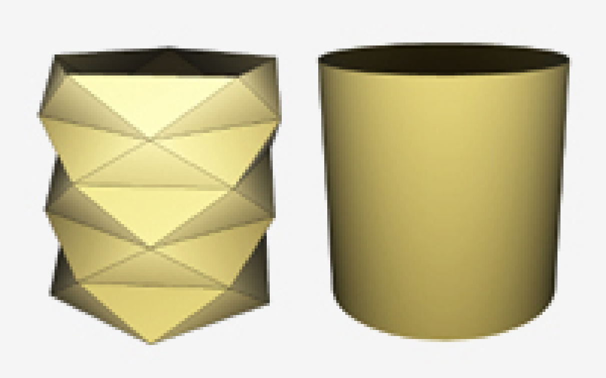
Corollary

If σ is a j-simplex, $j \leq k$, vert $(\sigma) \subset \mathbb{M}$, $\Delta(\sigma) \leq 2\varepsilon\,\text{rch}(\mathbb{M})$

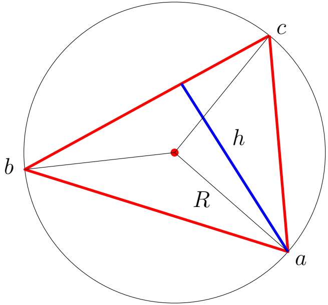
$$\forall p \in \sigma, \quad \sin \angle(\text{aff}(\sigma), T_p) \leq \frac{2\varepsilon}{\Theta(\sigma)}$$

($h \leq \frac{\Delta(\sigma)^2}{2\,\text{rch}(\mathbb{M})}$ by the Chord Lemma)

Schwarz lantern



Thickness and angle bounds for **triangles**



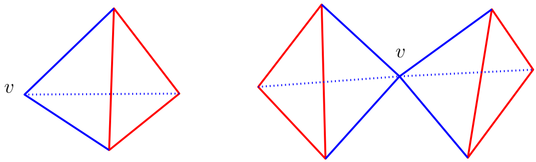
$$h = ab \ \times \ sinb = \frac{ab \times ac}{2R}$$

$$\Theta(abc) = \frac{h}{2\Delta} \geq \frac{h}{4R} \geq \frac{\phi_0^2}{32R} \geq \frac{{\phi_0}^2}{32\varepsilon\,\mathrm{rch}(\mathcal{S})}$$

$$\sin b \geq \frac{\phi_0}{2\varepsilon\,\mathrm{rch}(\mathcal{S})}$$

Triangulated pseudo-surface

For ε small enough, $\mathrm{Del}_{|S}(\mathcal{P})$ is a triangulated pseudo-surface

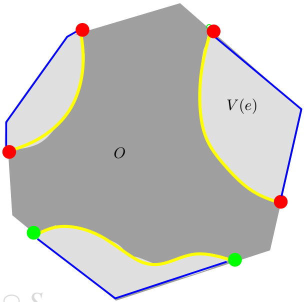


- Any edge e of $\mathrm{Del}_{|S}(\mathcal{P})$ has exactly two incident facets
- The adjacency graph of the facets is connected

Triangulated pseudo-surface

e an edge of $\hat{\mathbb{M}}$, f any face incident on e

1. $V(p)$ bounded $\Rightarrow$ $|V(e) \cap S|$ is even
2. $V(f)$ intersects S at most once



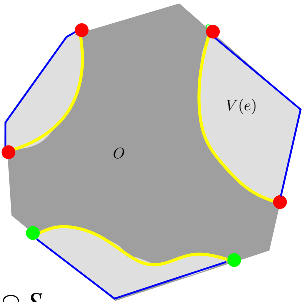
Proof of 2 by contradiction: $c, c' \in V(f) \cap S$

a. $\|c - c'\| \leq 2\varepsilon\, \mathrm{rch}(S) \Rightarrow \sin \angle(cc', T_c) \leq \varepsilon$ (Chord L.)
b. $cc' \perp f$ (dual face)
c. $T_c \approx \mathit{aff}(f)$ (Whitney's L.)

Triangulated pseudo-surface

e an edge of $\hat{\mathbb{M}}$, f any face incident on e

1. $V(p)$ bounded $\Rightarrow$ $|V(e) \cap S|$ is even
2. $V(f)$ intersects S at most once



Proof of 2 by contradiction: $c, c' \in V(f) \cap S$

a. $\|c - c'\| \leq 2\varepsilon \, \mathrm{rch}(S) \Rightarrow \sin \angle(cc', T_c) \leq \varepsilon$ (Chord L.)
b. $cc' \perp f$ (dual face)
c. $T_c \approx \mathit{aff}(f)$ (Whitney's L.)

The main result

The Delaunay refinement algorithm produces

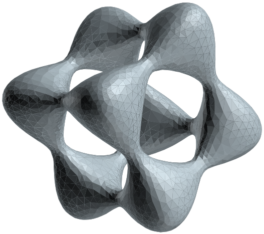
- an ε- net $\mathcal{P}$ of $\mathcal{S}$
- a triangulated surface $\hat{S}$
  - homeomorphic to S
  - close to $\mathcal{S}$ (Hausdorff/Fréchet distance $O(\varepsilon^2)$, approximation of normals $O(\varepsilon)$)

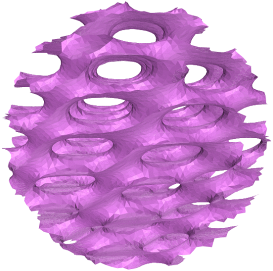
Applications

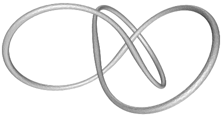
- Implicit surfaces $f(x, y, z) = 0$
- Isosurfaces in a 3d image (Medical images)
- Triangulated surfaces (Remeshing)
- Point sets (Surface reconstruction)

see cgal.org, CGALmesh project

Results on smooth implicit surfaces



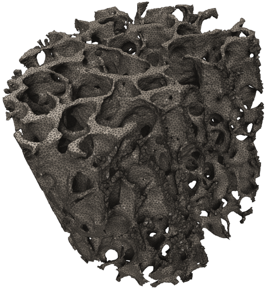




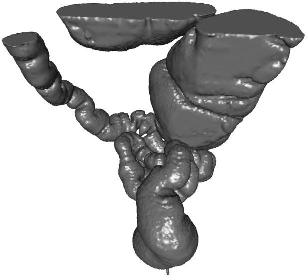
Meshing 3D domains

Input from segmented 3D medical images

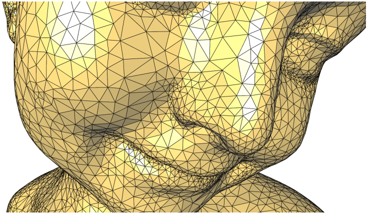
[INSERM]



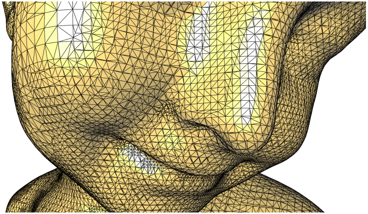
[SIEMENS]



Comparison with the Marching Cube algorithm



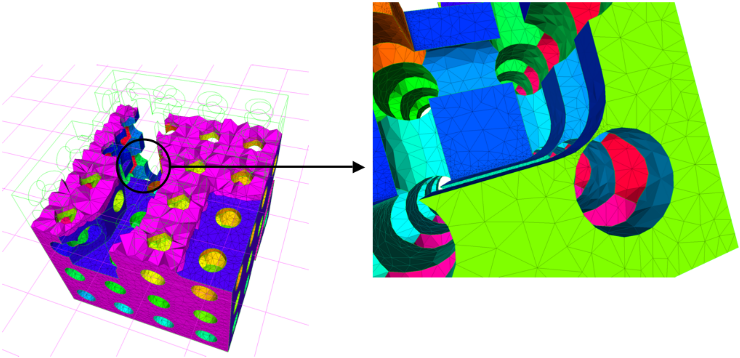
Delaunay refinement



Marching cube

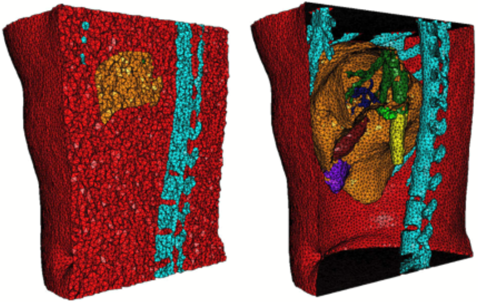
Meshing with sharp features

A polyhedral example



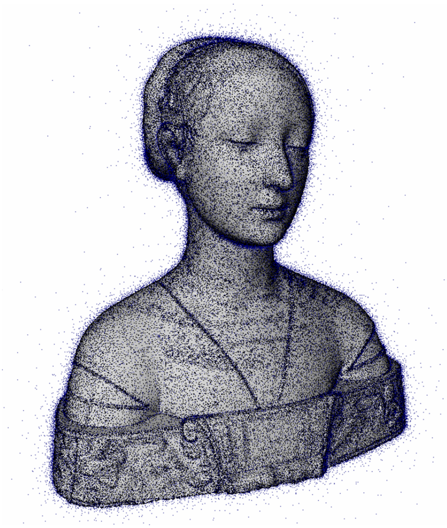
Meshing 3D multi-domains

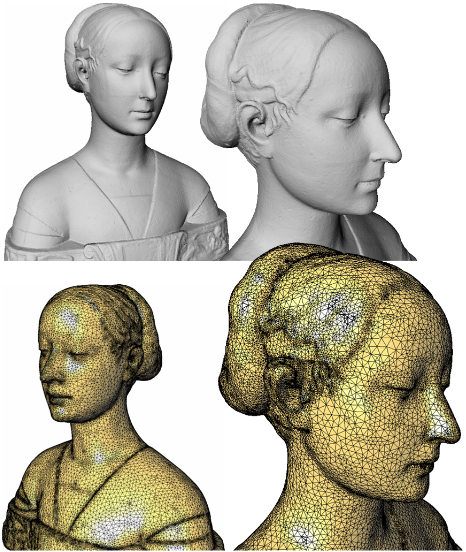
Input from segmented 3D medical images [IRCAD]



Size bound (mm)	vertices nb	facets nb	tetrahedra nb	CPU Time (s)
16	3,743	3,735	19,886	0.880
8	27,459	19,109	159,120	6.97
4	199,328	76,341	1,209,720	54.1
2	1,533,660	311,420	9,542,295	431

Surface reconstruction from unorganized point sets





Courtesy of P. Alliez