Mesh Generation

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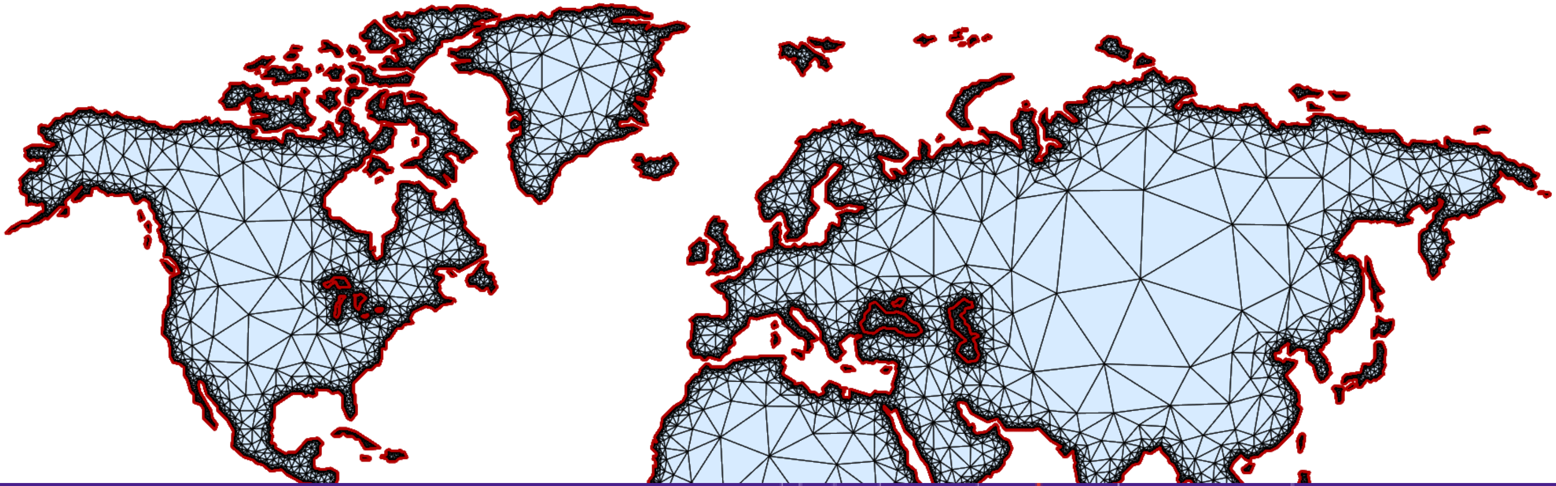
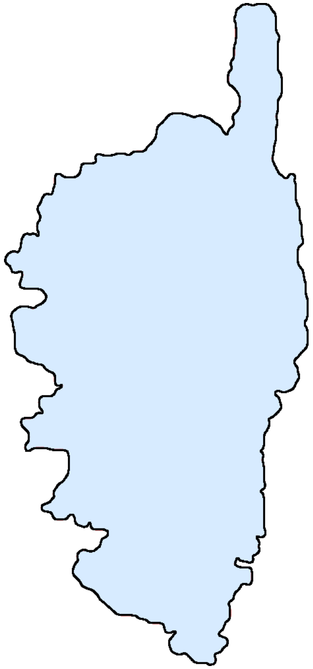
Definitions

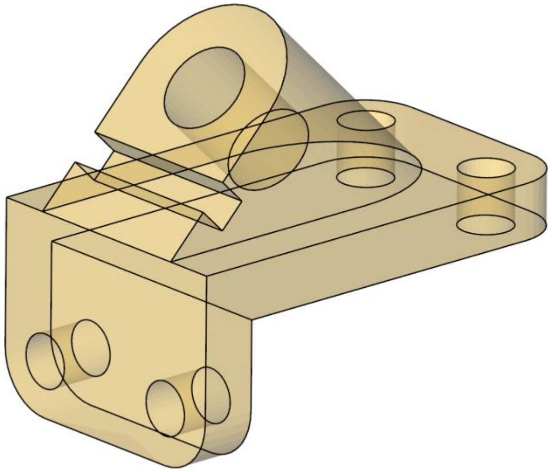
- **Mesh:** Cellular complex partitioning an input domain into elementary cells.
- **Elementary cell**: Admits a bounded description.
- **Cellular complex**: Two cells are disjoint or share a lower dimensional face.



Variety

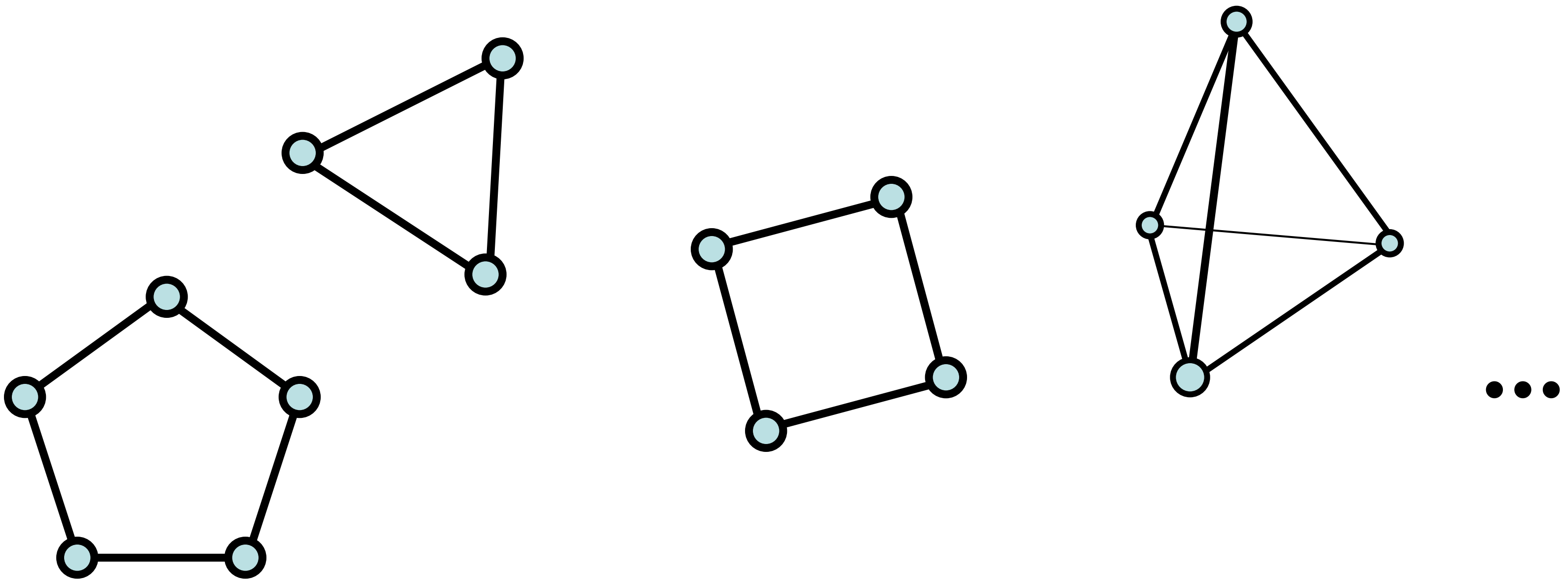
- **Domains**: 2D, 3D, nD, surfaces, manifolds, periodic vs non-periodic, etc.





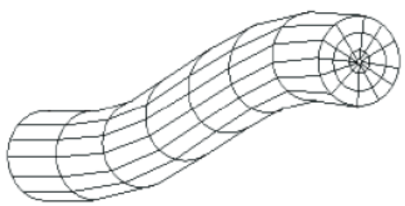
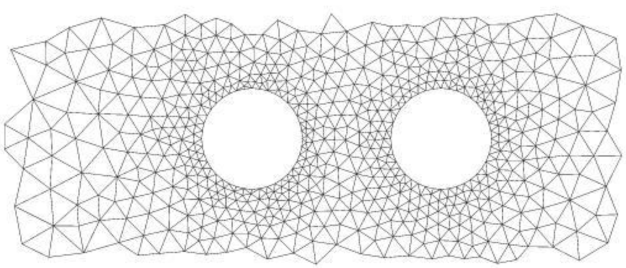
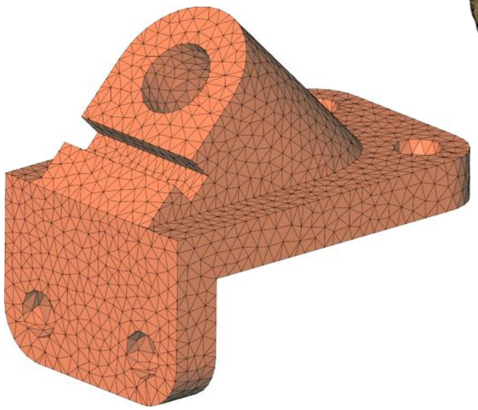
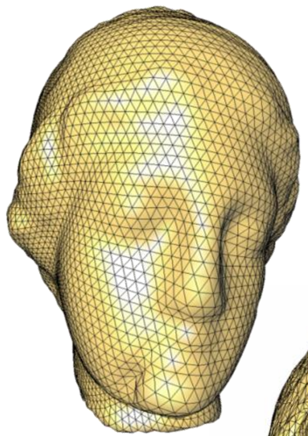
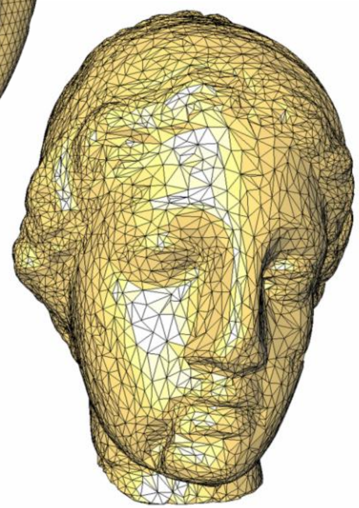
Variety

- **Elements**: simplices (triangles, tetrahedra), quadrangles, polygons, hexahedra, arbitrary cells.



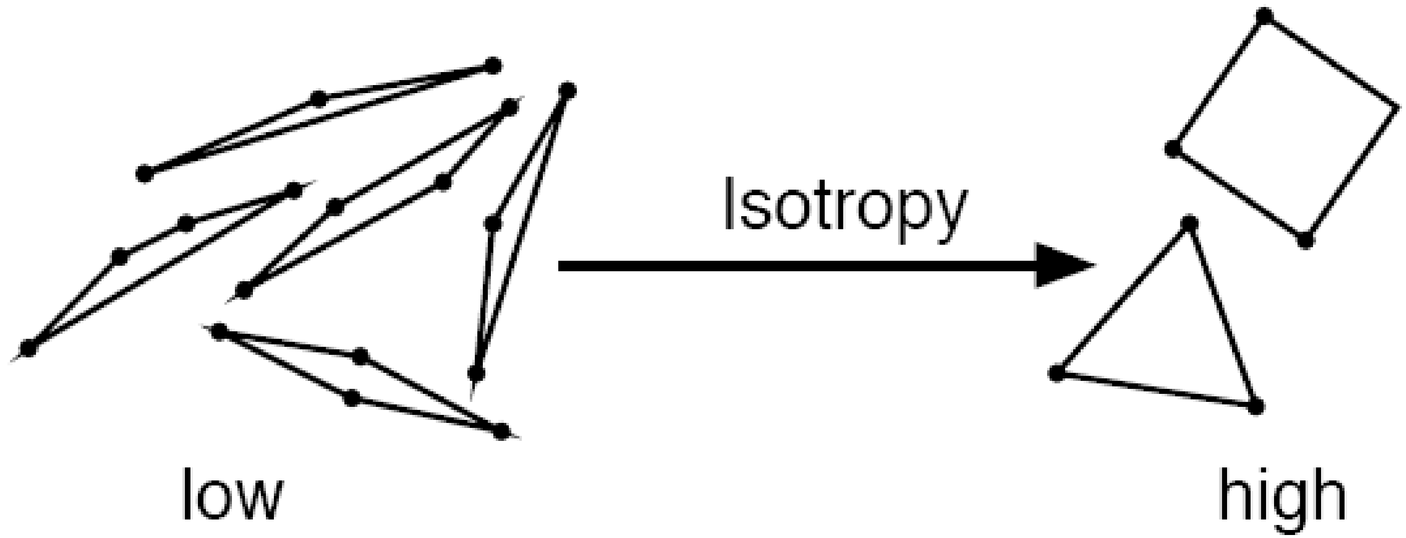
Variety

- **Structured mesh**: All interior nodes have an equal number of adjacent elements.
- **Unstructured mesh**: Any number of elements can meet at a single vertex.

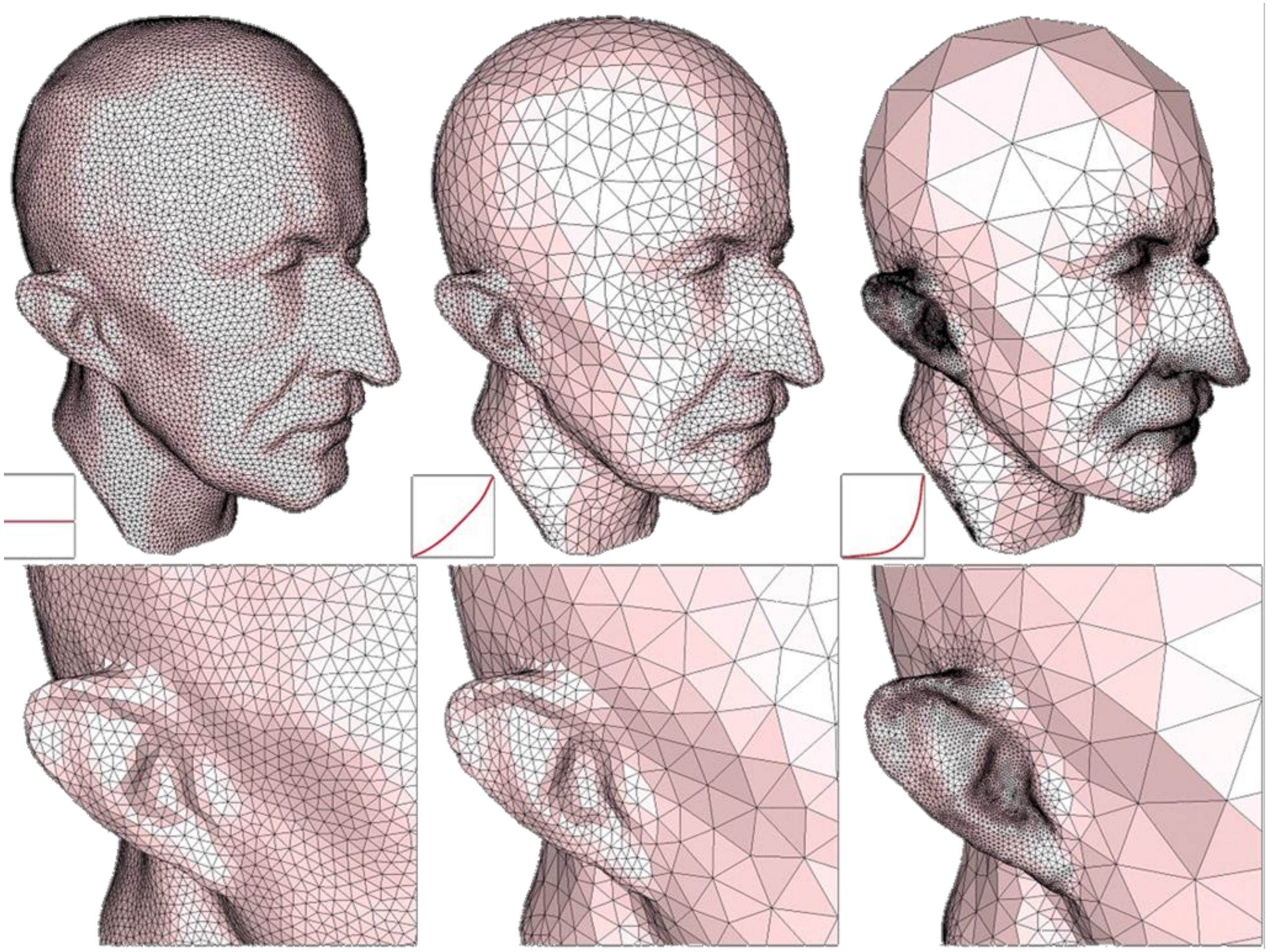
Variety

- **Anisotropic vs Isotropic**



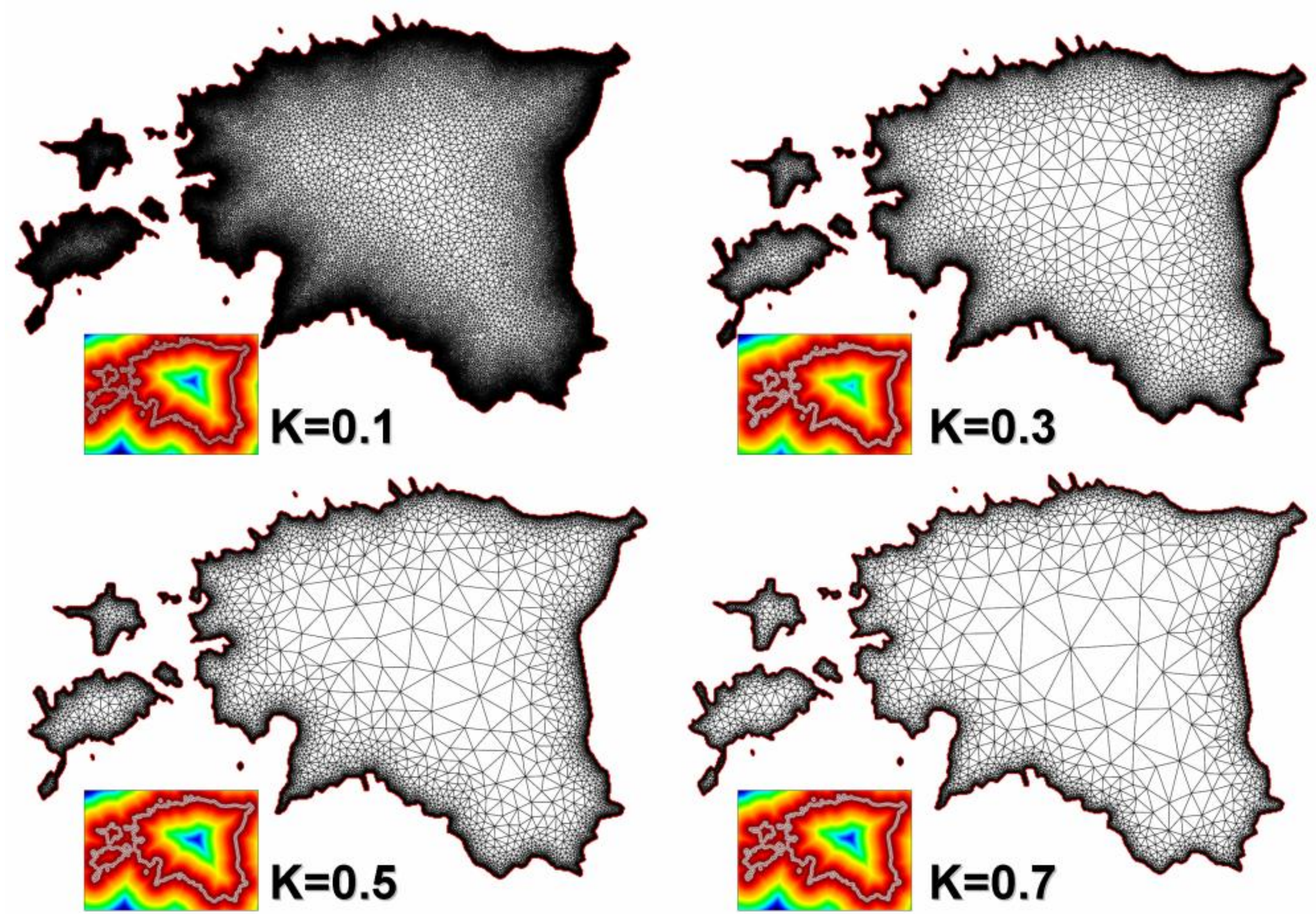
Variety

- **Element distribution:** Uniform vs adapted



Variety

- **Grading:** Smooth vs fast

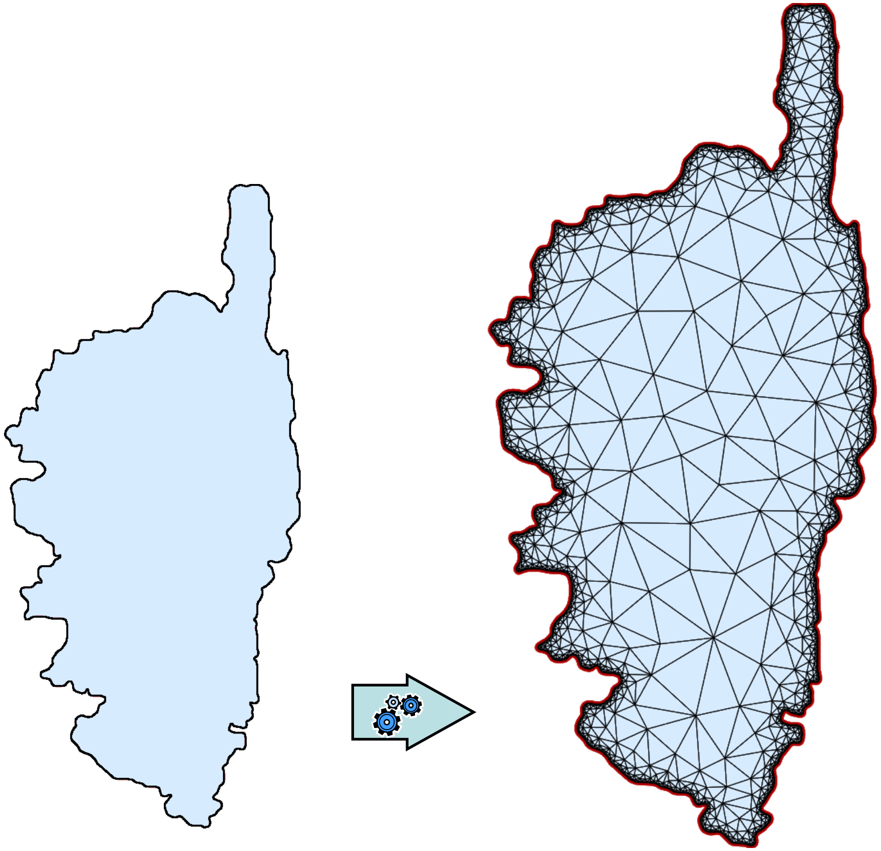


Required Properties

- **control/optimization over:**
  - #elements
  - shape
  - orientation
  - size
  - grading
  - Boundary approximation error
  - … (application dependent)

Mesh Generation

- **Input:**
  - Domain boundary + internal constraints
  - Constraints
    - Sizing
    - Grading
    - Shape
    - Topology
    - …



2D Delaunay Refinement

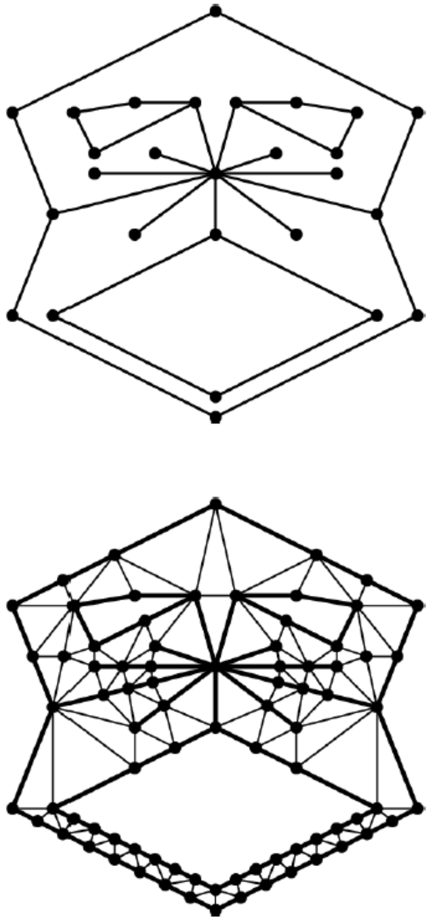
2D Triangle Mesh Generation

Input:

- PSLG C (planar straight line graph)
- Domain Ω bounded by edges of C

Output:

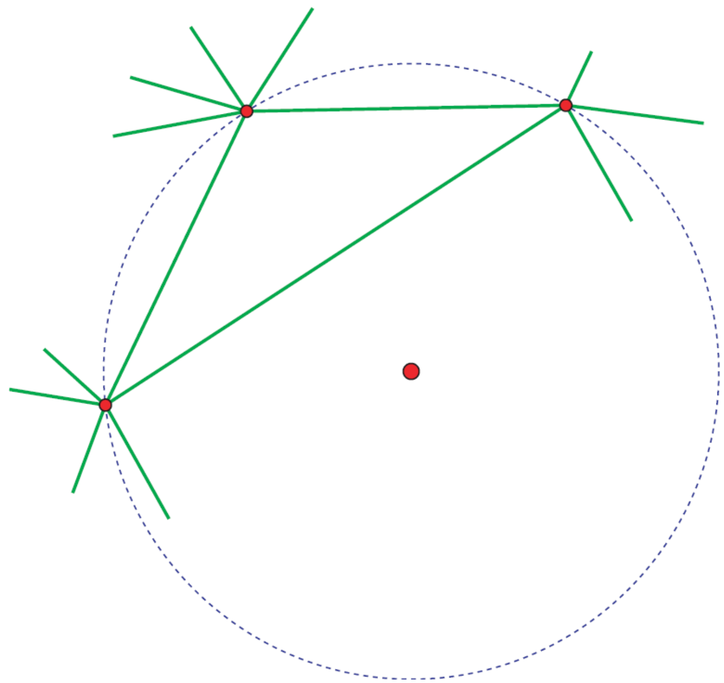
- triangle mesh T of Ω such that
  - vertices of C are vertices of T
  - edges of C are union of edges in T
  - triangles of T inside Ω have controlled size and quality



Key Idea

- Break bad elements by inserting circumcenters (Voronoi vertices) [Chew, Ruppert, Shewchuk,...]

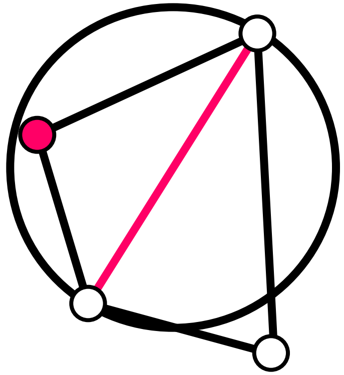
  "bad" in terms of size or shape



Basic Notions

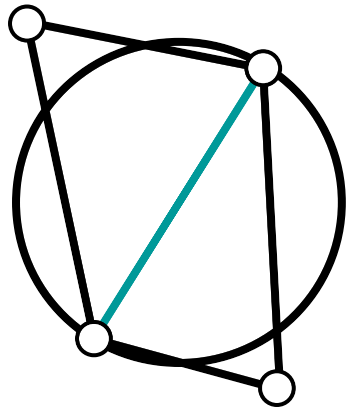
C: PSLG describing the constraints

T: Triangulation to be refined



Respect of the PSLG

- Edges a C are split until constrained subedges are edges of T
- Constrained subedges are required to be Gabriel edges



- An edge of a triangulation is a **Gabriel** edge if its smallest circumcirle encloses no vertex of T
- An edge e is **encroached** by point p if the smallest circumcirle of e encloses p.

Refinement Algorithm

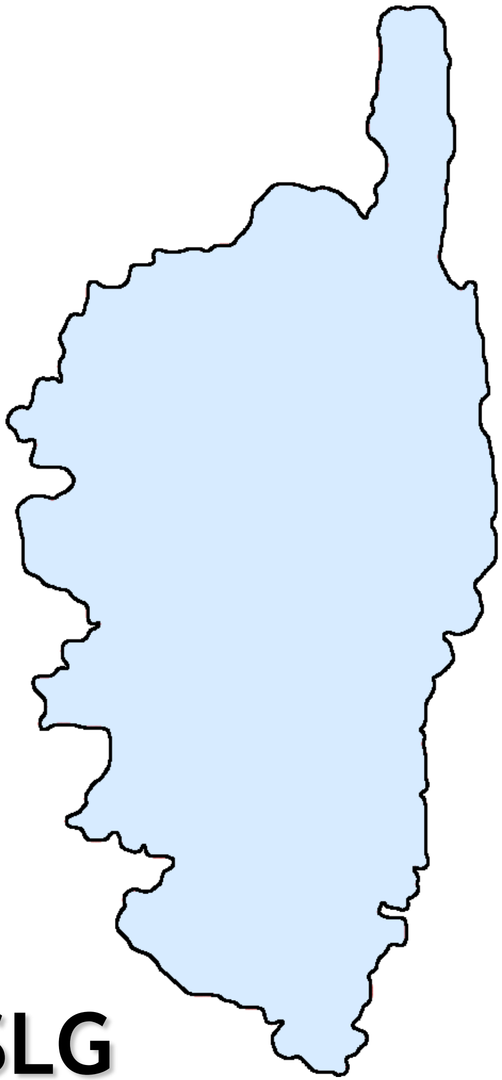
C: PSLG bounding the domain to be meshed.
T: Delaunay triangulation of the current set of vertices
$T_{|\Omega}$: $T \cap \Omega$
Constrained subedges: subedges of edges of C

Initialise with T = Delaunay triangulation of vertices of C

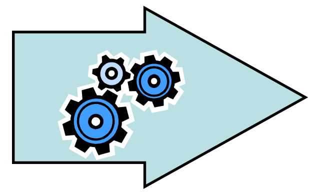
Refine until no rule apply

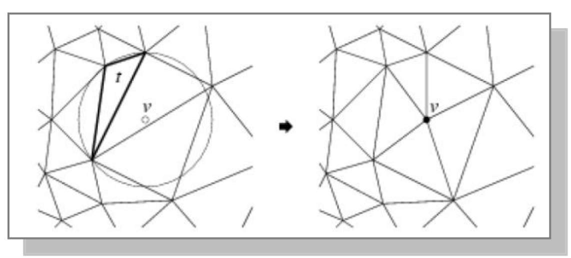
- **Rule 1**
  if there is an encroached constrained subedge e
  insert c = midpoint(e) in T (refine-edge)
- **Rule 2**
  if there is a bad facet f in $T_{|\Omega}$
  c = circumcenter(f)
  if c encroaches a constrained subedge e
  refine-edge(e).
  else
  insert(c) in T

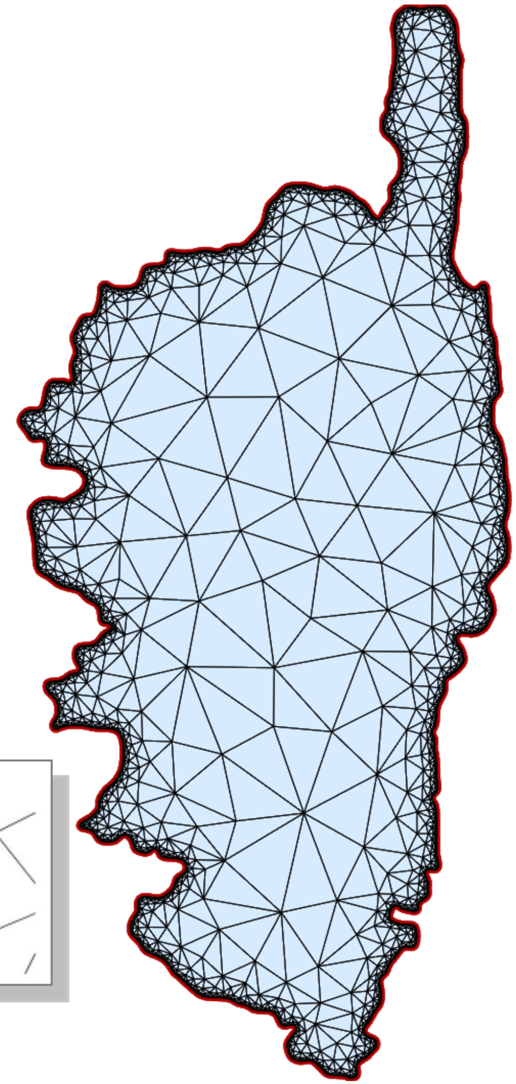
2D Delaunay Refinement



PSLG

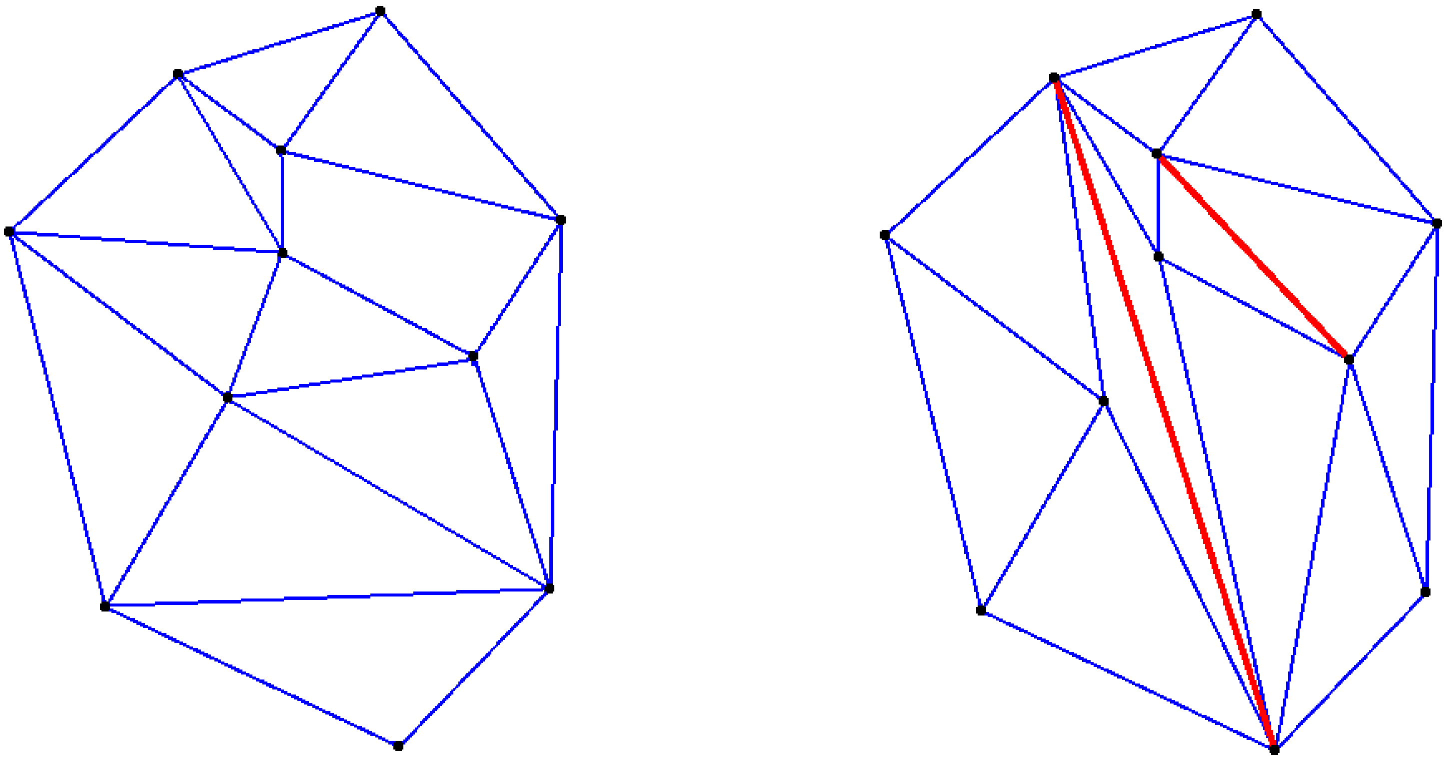




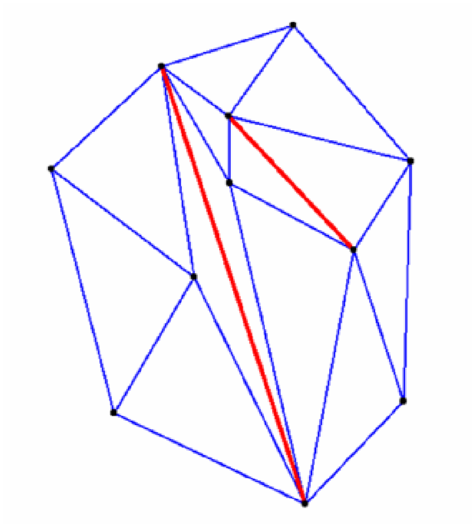


Background

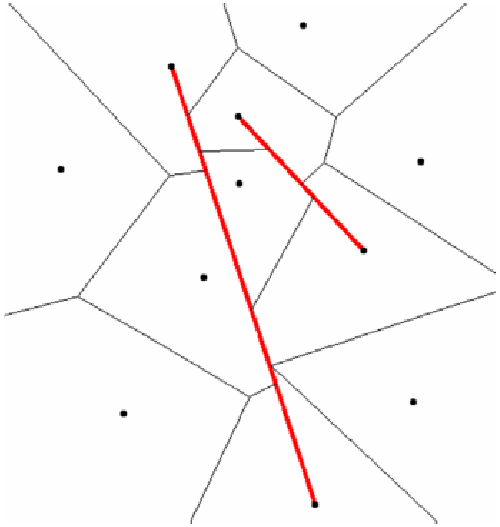
Constrained Delaunay Triangulation



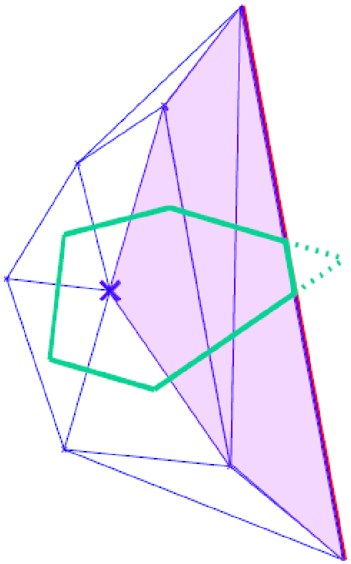
Pseudo-dual: Bounded Voronoi Diagram



constrained



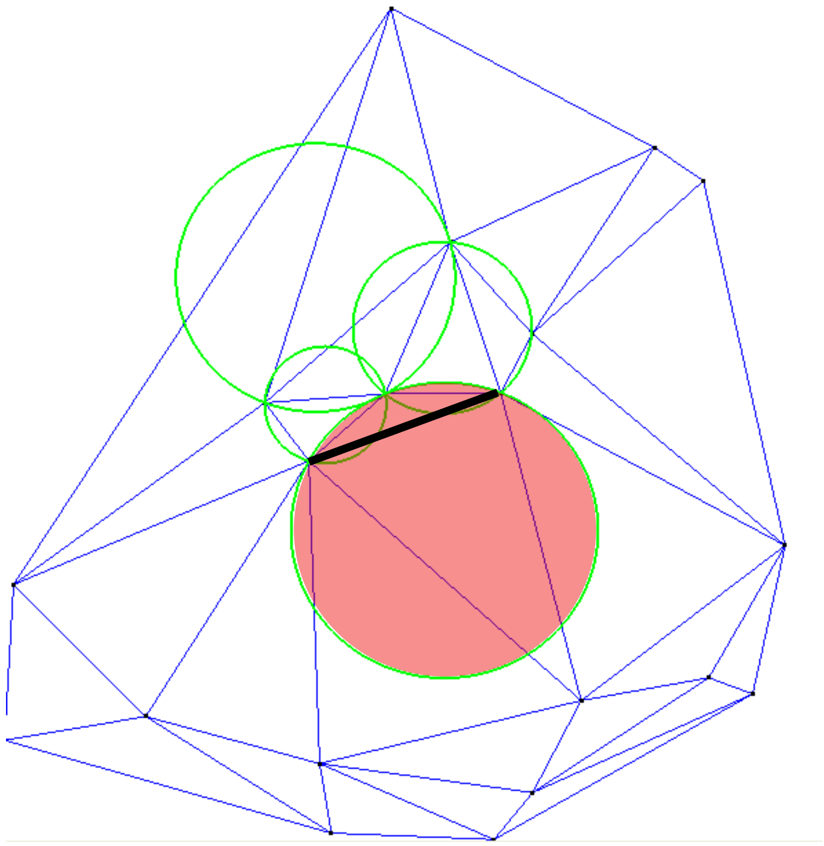
Bounded Voronoi diagram



“blind” triangles

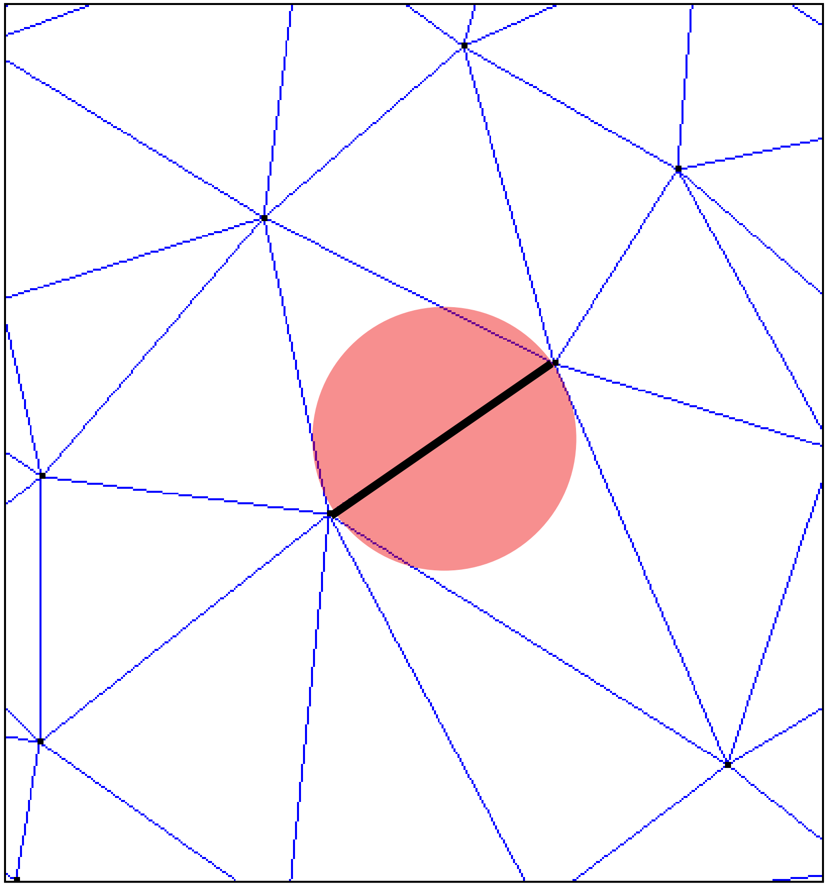
Delaunay Edge

An edge is said to be a Delaunay edge, if it is inscribed in an empty circle



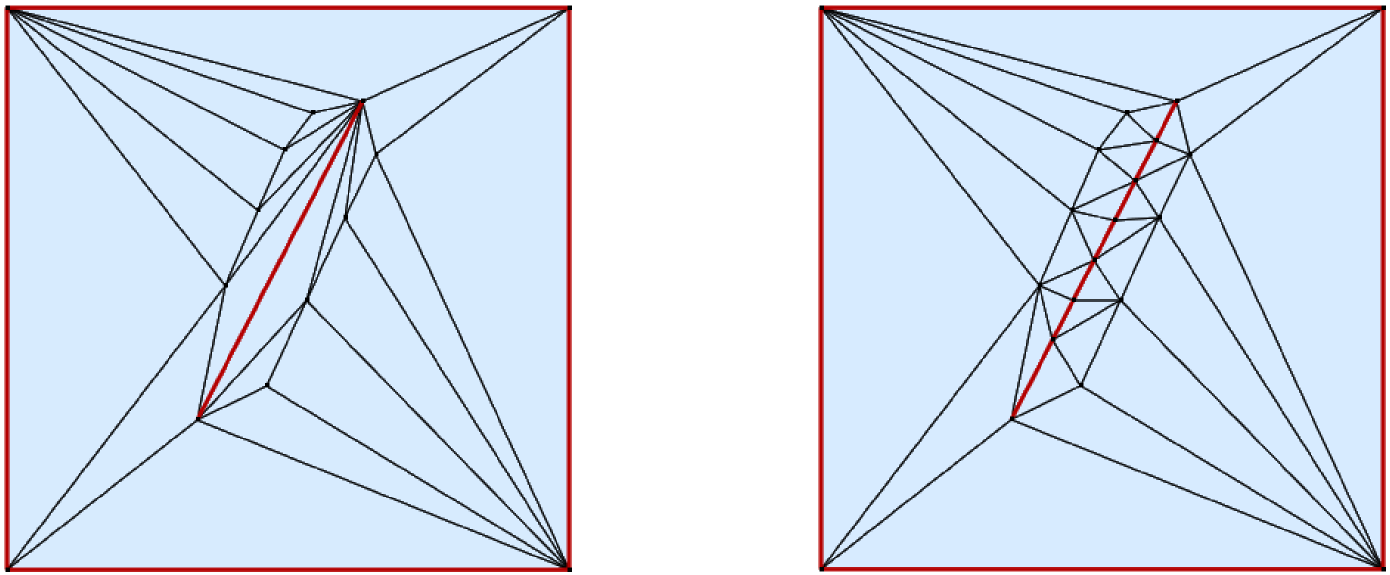
Gabriel Edge

An edge is said to be a Gabriel edge, if its diametral circle is empty



Conforming Delaunay Triangulation

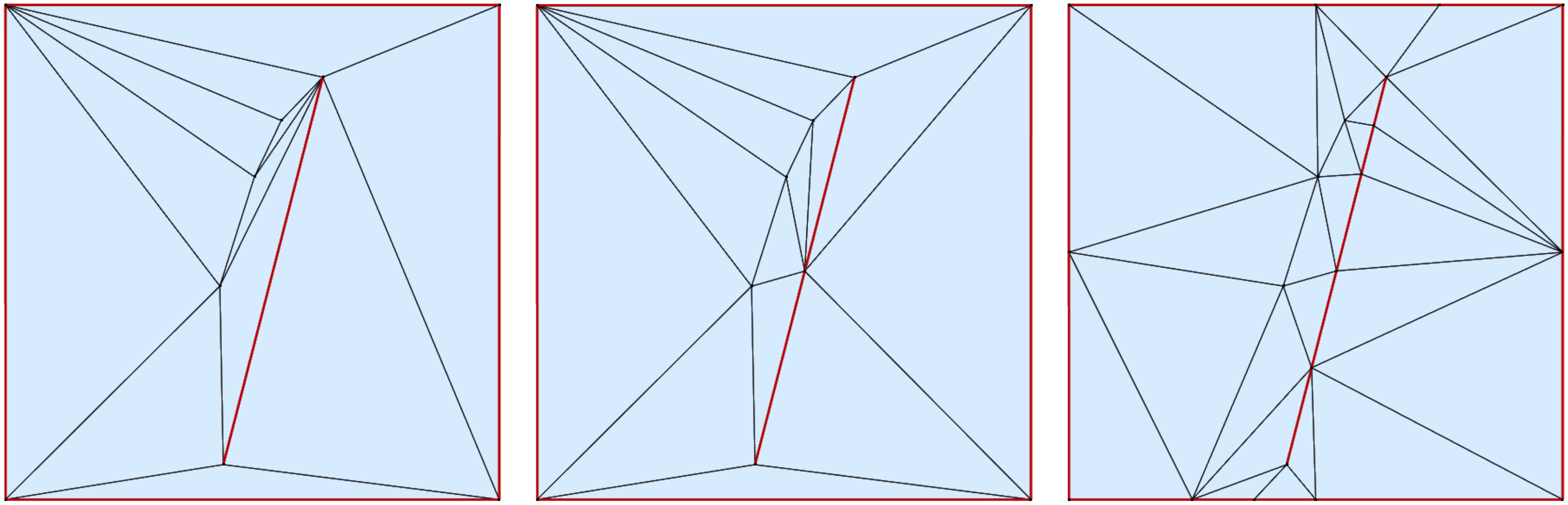
A constrained Delaunay triangulation is a conforming Delaunay triangulation, if every constrained edge is a Delaunay edge



non conforming   conforming

Conforming Gabriel Triangulation

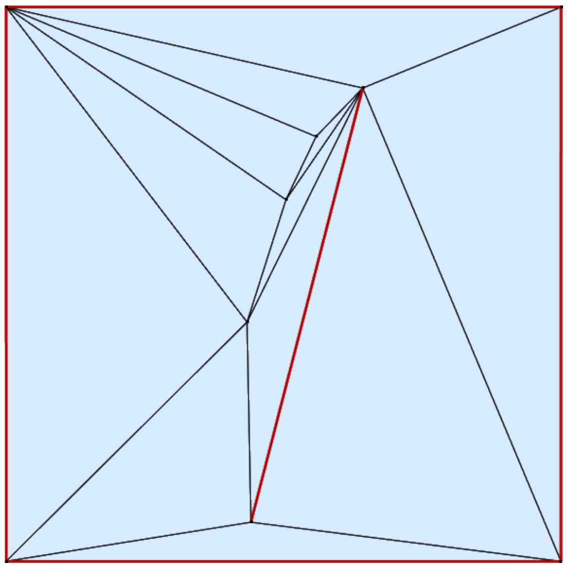
A constrained Delaunay triangulation is a conforming Gabriel triangulation, if every constrained edge is a Gabriel edge



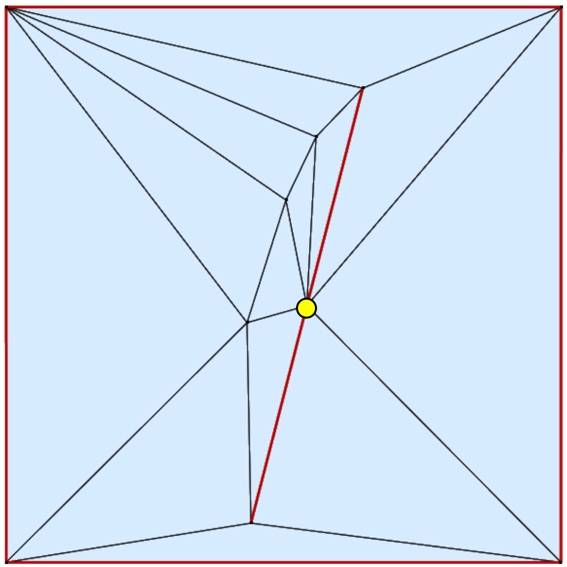
non conforming | conforming | Gabriel

Steiner Vertices

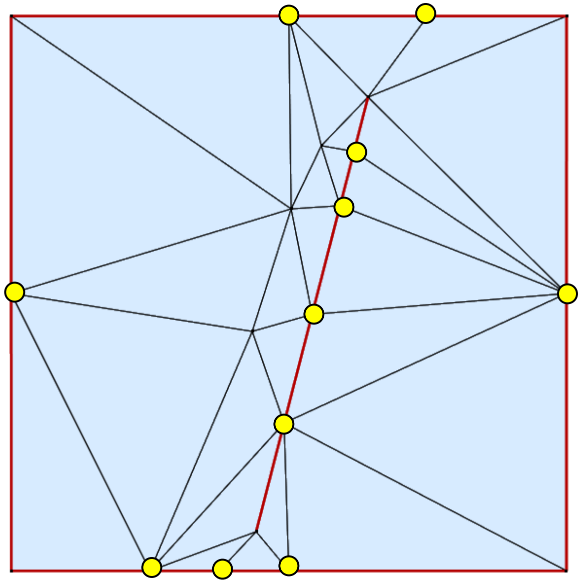
Any constrained Delaunay triangulation can be refined into a conforming Delaunay or Gabriel triangulation by adding Steiner vertices.



non conforming



conforming

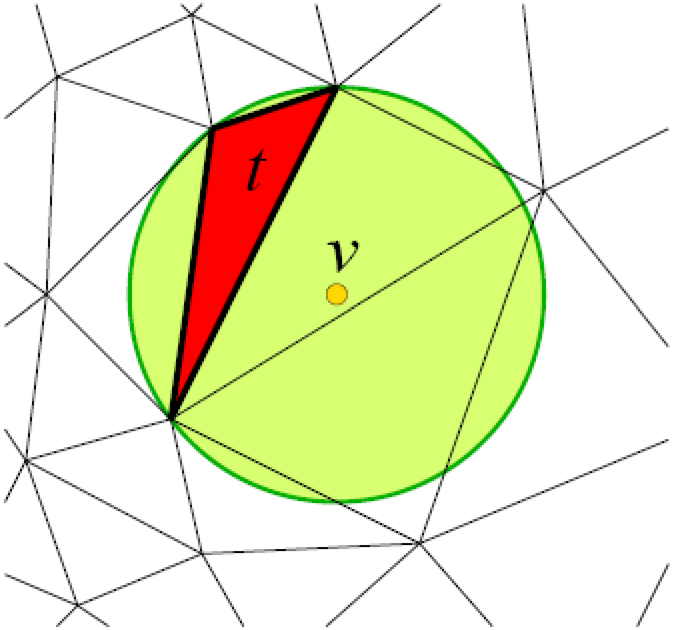


Gabriel

Delaunay Refinement

Rule #1: break bad elements by inserting circumcenters (Voronoi vertices)

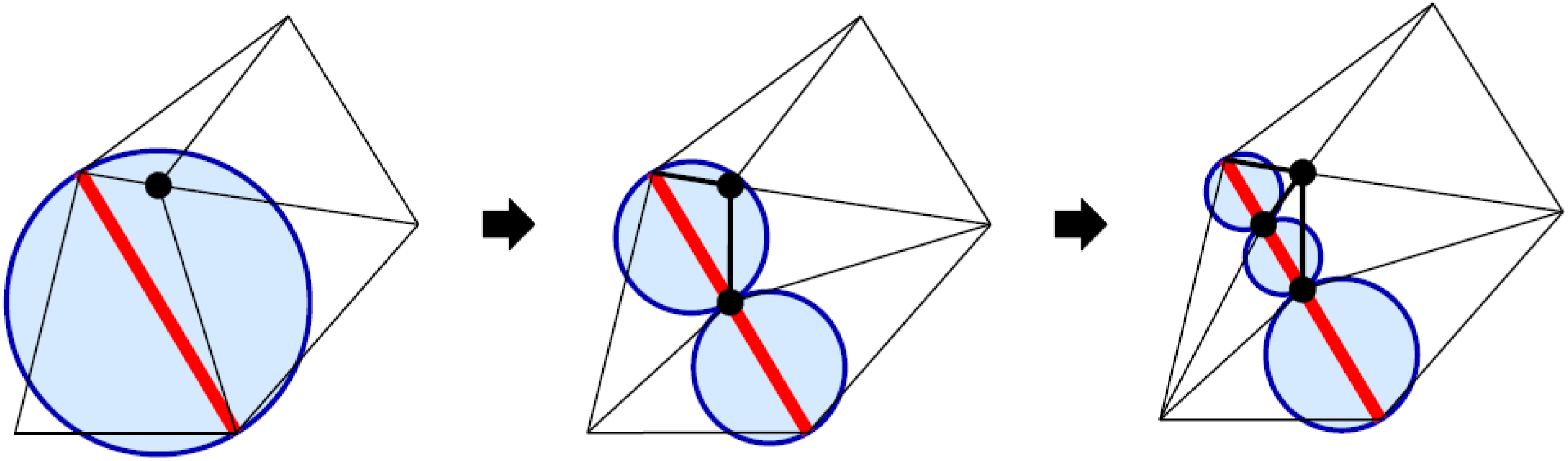
- "bad" in terms of size or shape (too big or skinny)



Delaunay Refinement

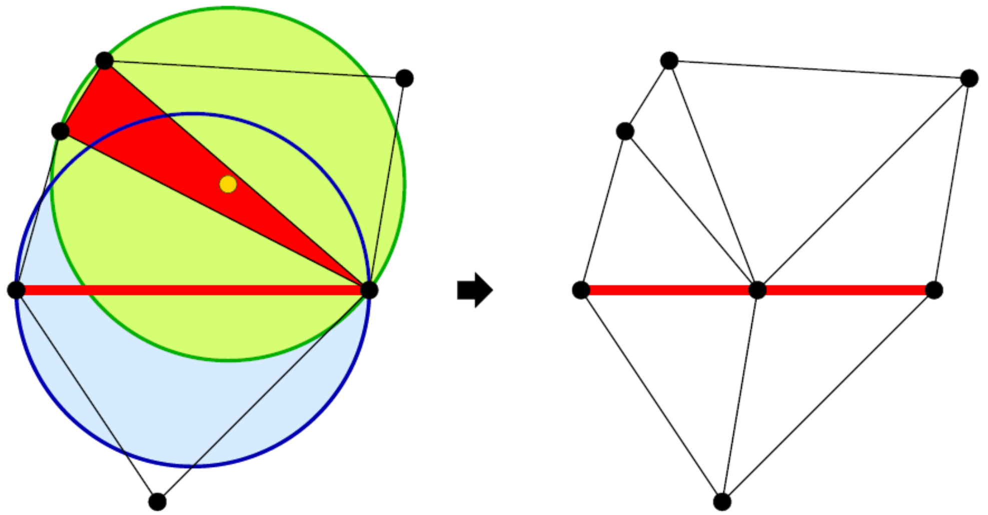
Rule #2: Midpoint vertex insertion

A constrained segment is said to be encroached, if there is a vertex inside its diametral circle



Delaunay Refinement

Encroached subsegments have priority over skinny triangles



Surface Mesh Generation

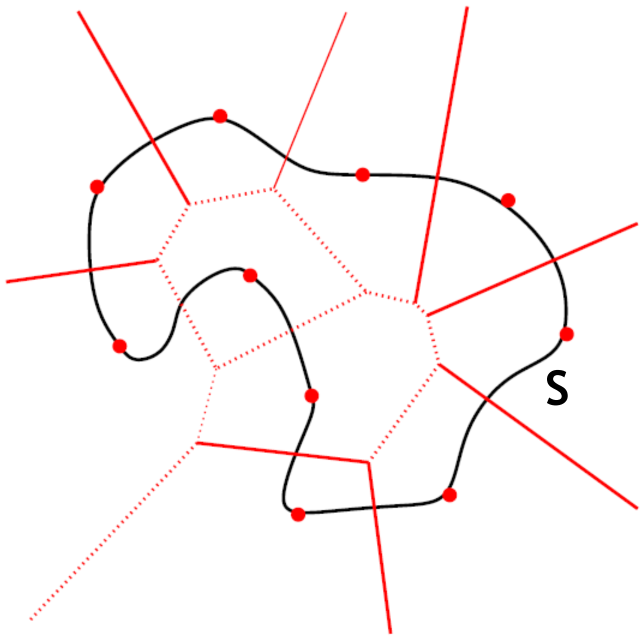
Mesh Generation

Key concepts:

- Voronoi/Delaunay filtering
- Delaunay refinement

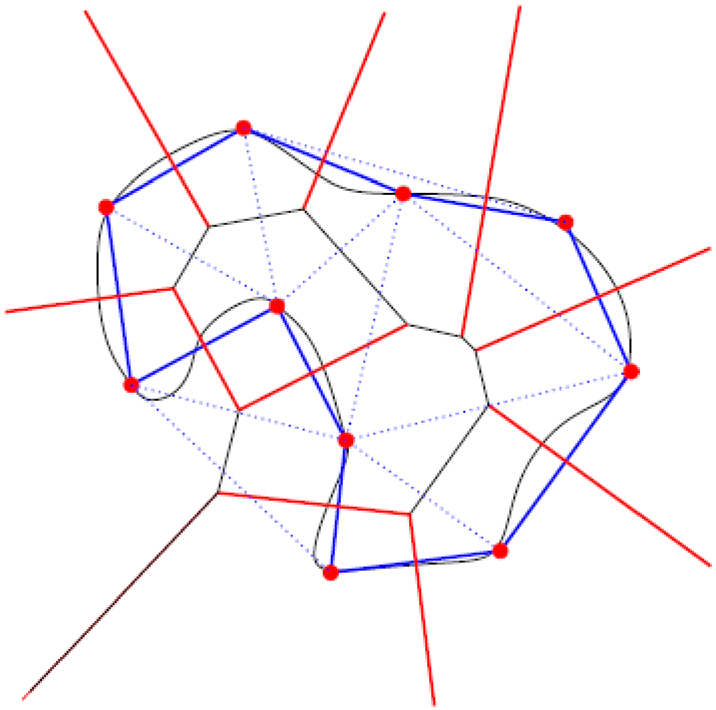
Voronoi Filtering

- The Voronoi diagram **restricted** to a curve S, $Vor_{|S}(E)$, is the set of edges of Vor(E) that intersect S.



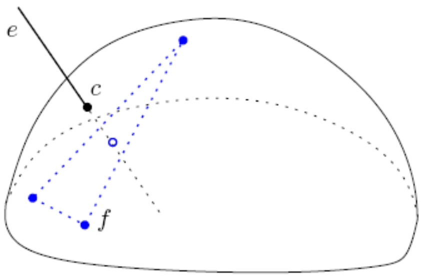
Delaunay Filtering

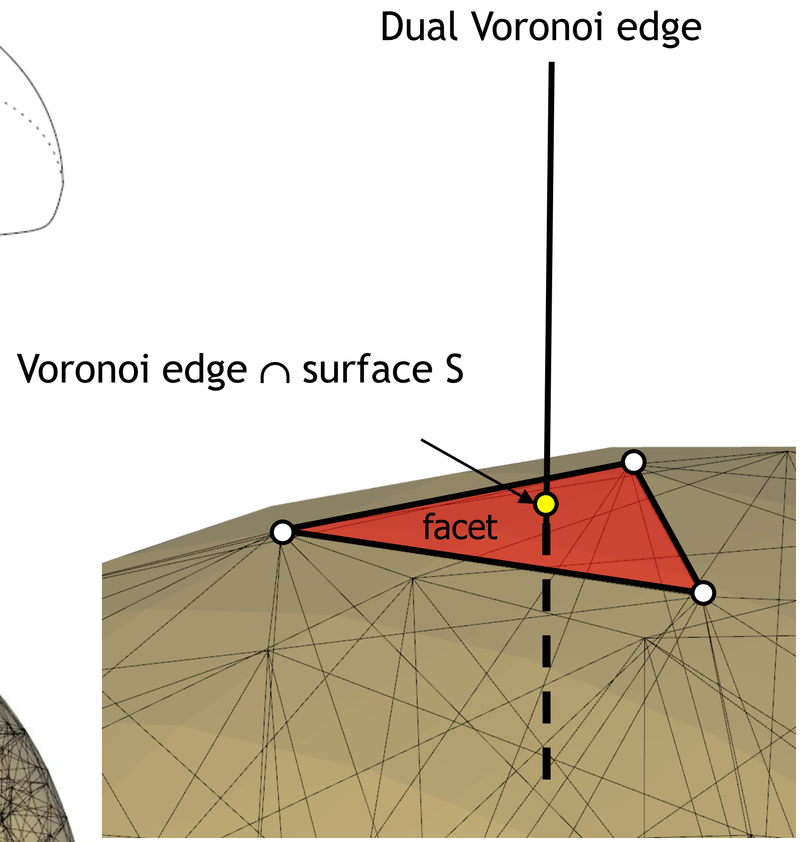
- The restricted Delaunay triangulation restricted to a curve S is the set of **edges** of the Delaunay triangulation whose dual edges **intersect** S.



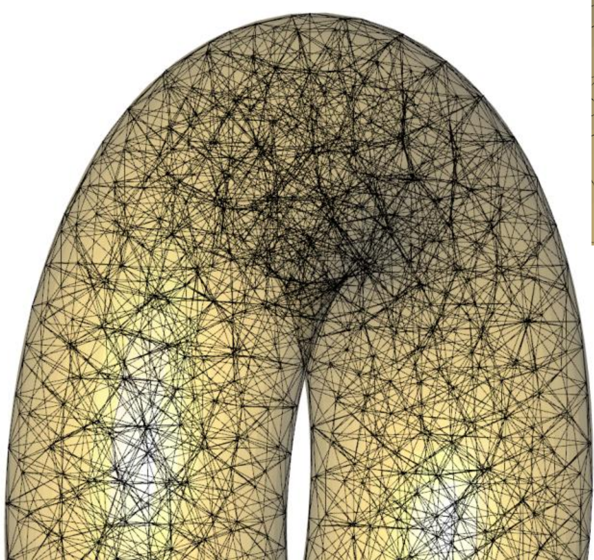
(2D)

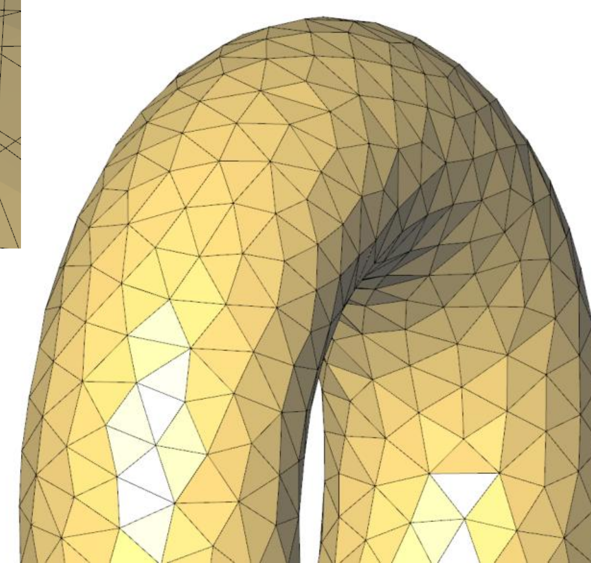
Delaunay Filtering





Delaunay triangulation restricted to surface S

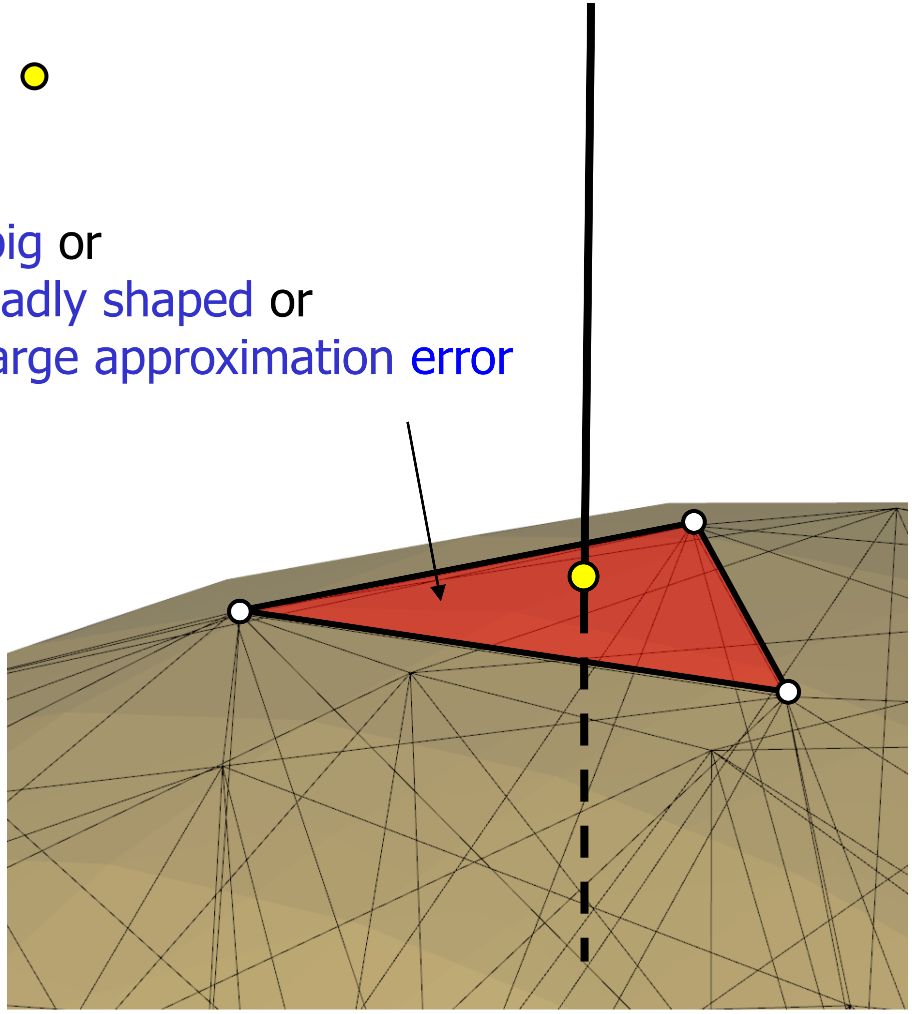




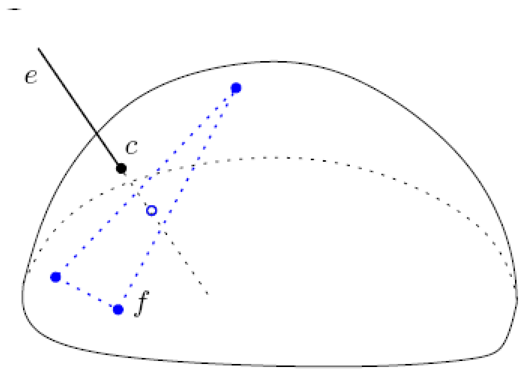
Delaunay Refinement

Steiner point

Bad facet = big or
badly shaped or
large approximation error



Surface Mesh Generation Algorithm



repeat

{

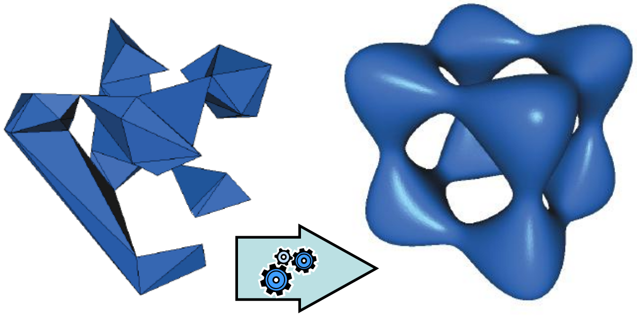
pick bad facet **f**

insert furthest (dual(**f**) $\cap$ **S**) in Delaunay triangulation

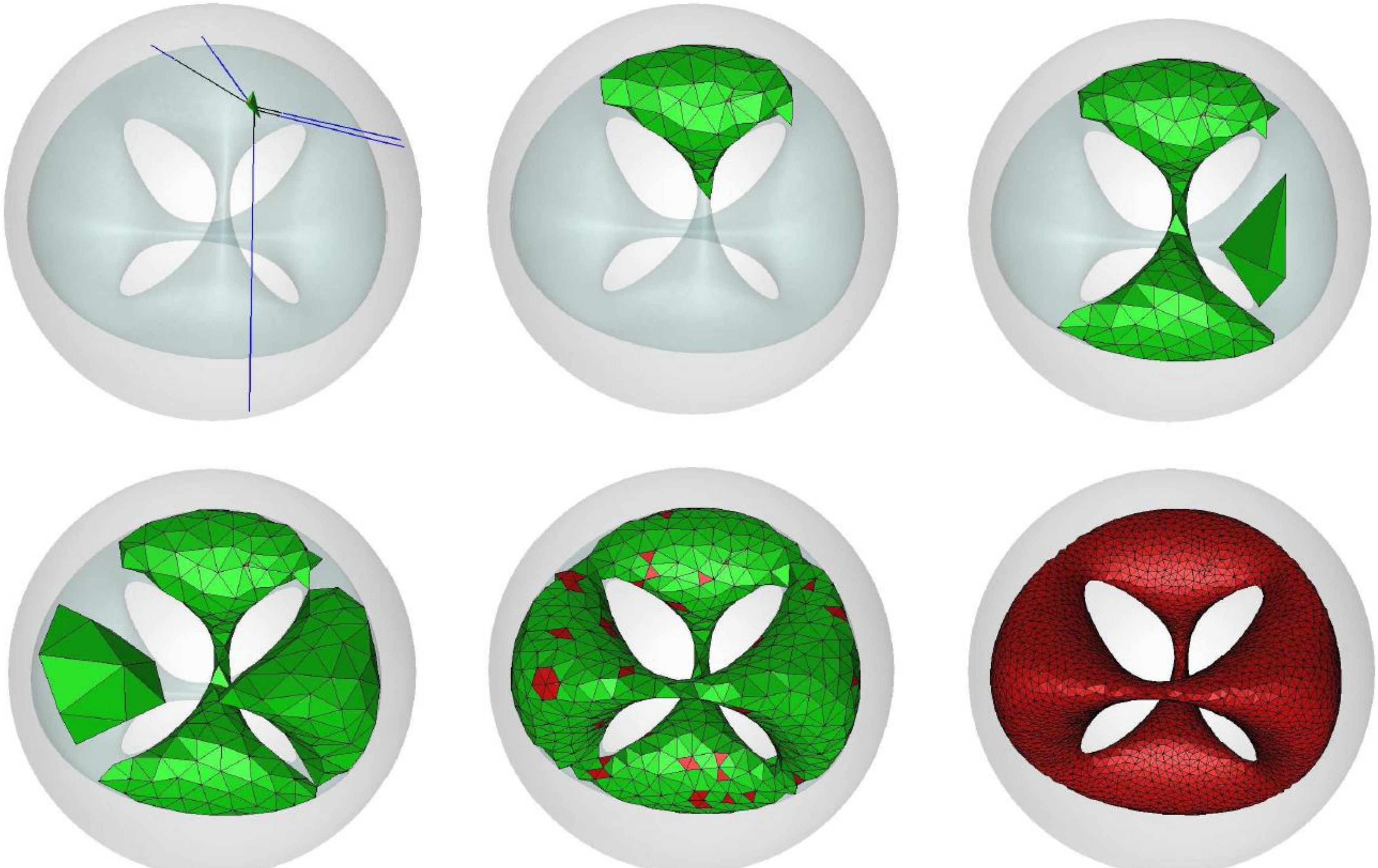
update Delaunay triangulation restricted to **S**

}

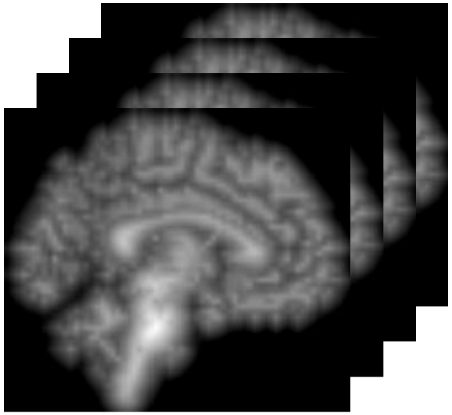
until all facets are good



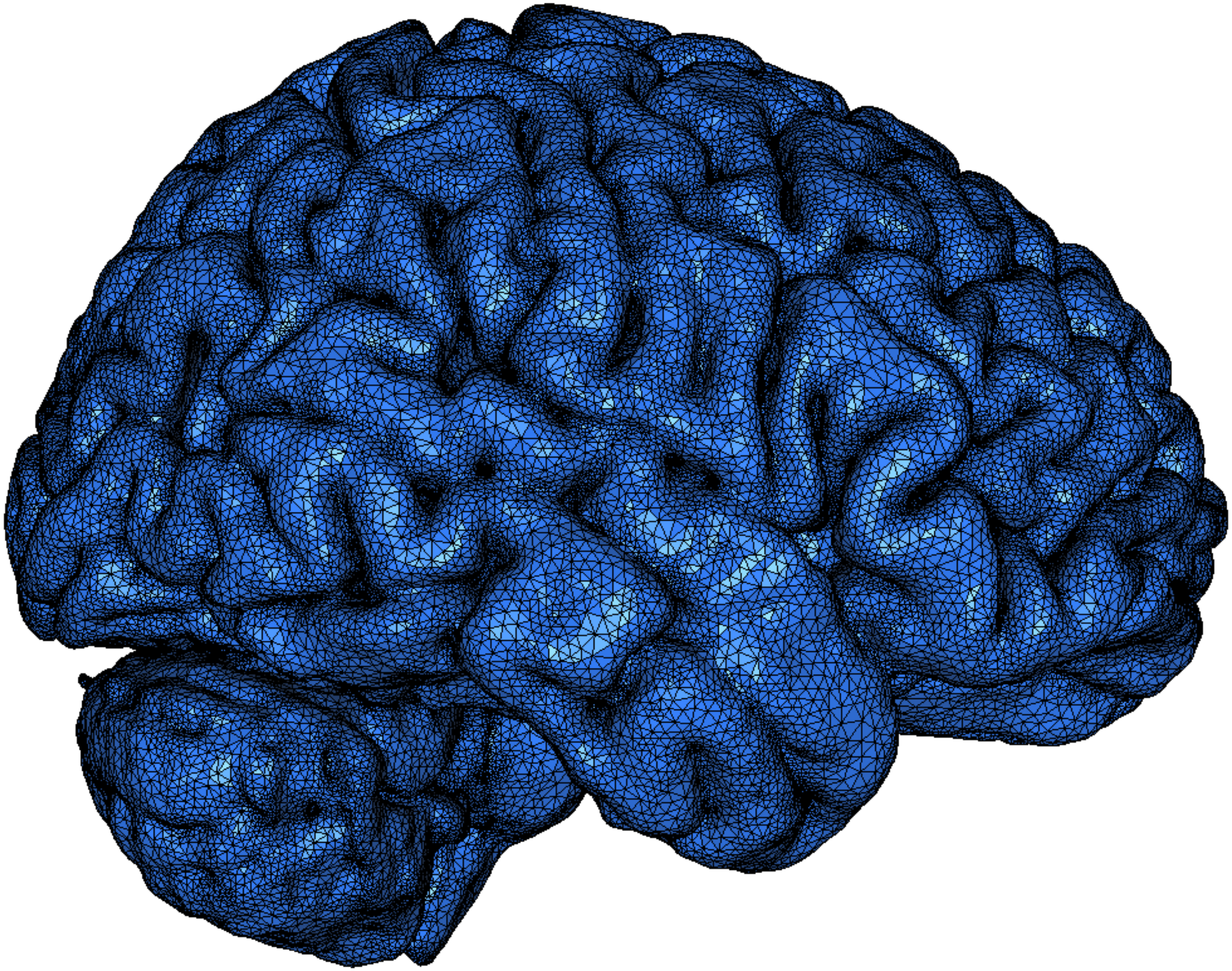
Surface Meshing at Work



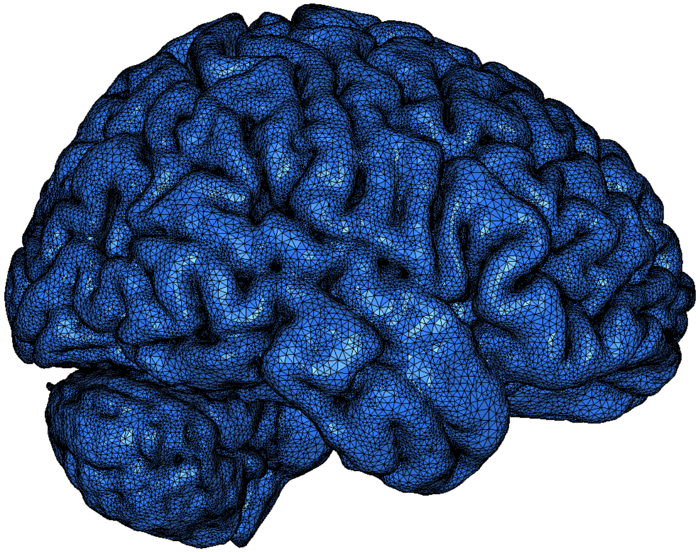
Isosurface from 3D Grey Level Image



input



Output Mesh Properties



Termination

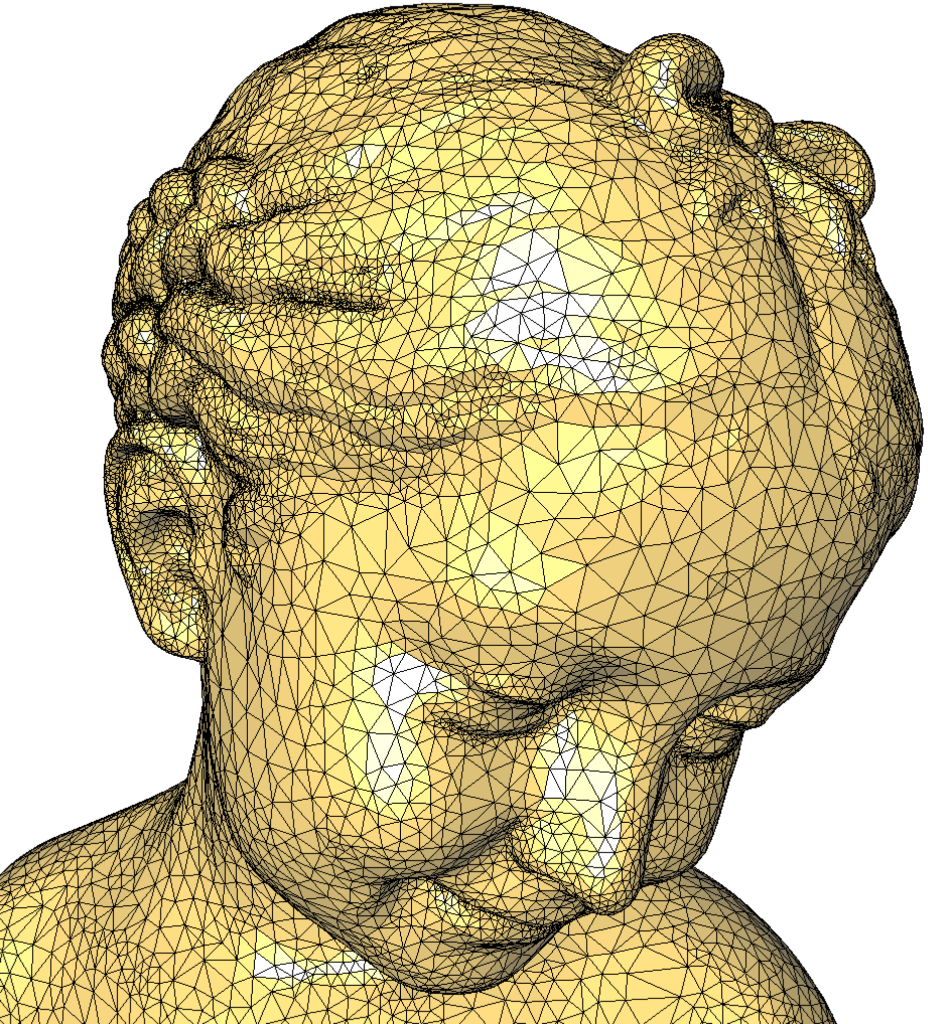
Parsimony

Output mesh properties:

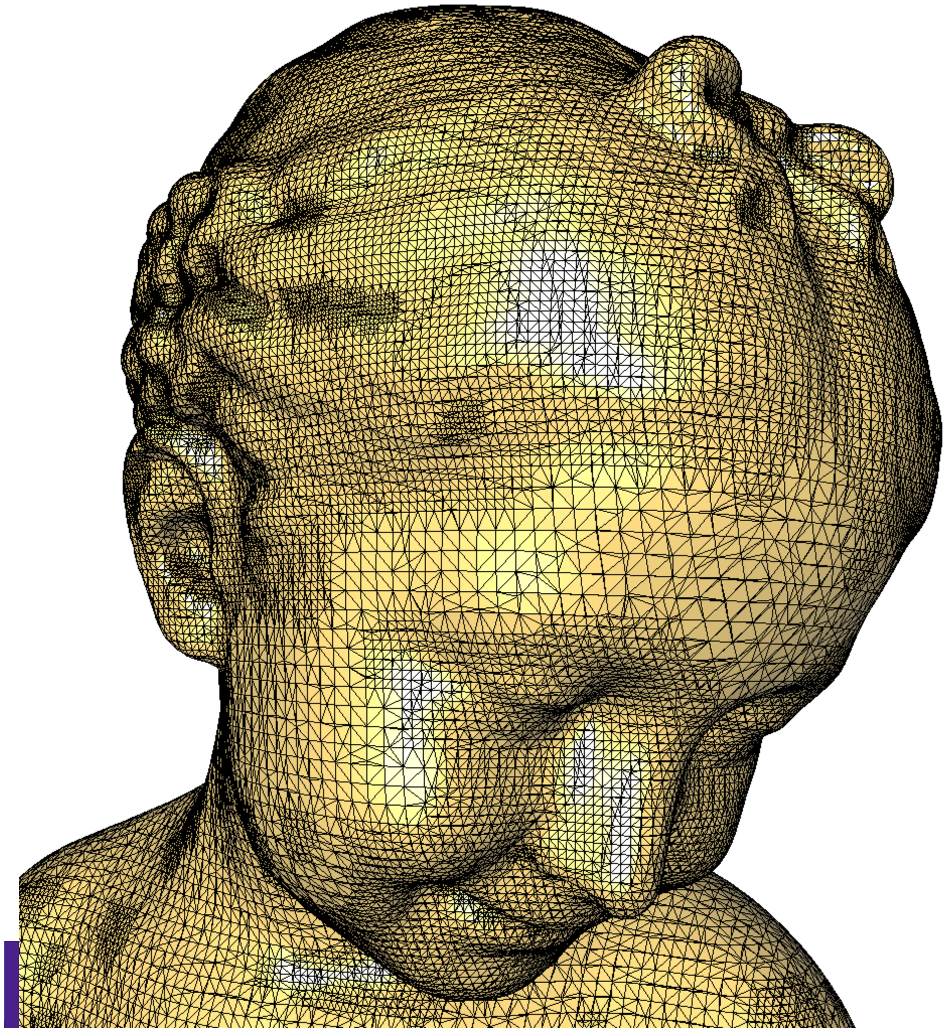
- Well shaped triangles
  - Lower bound on triangle angles
- Homeomorphic to input surface
- Manifold
  - not only combinatorially, i.e., no self-intersection
- Faithful Approximation of input surface
  - Hausdorff distance
  - Normals

Delaunay Refinement vs Marching Cubes

Delaunay refinement



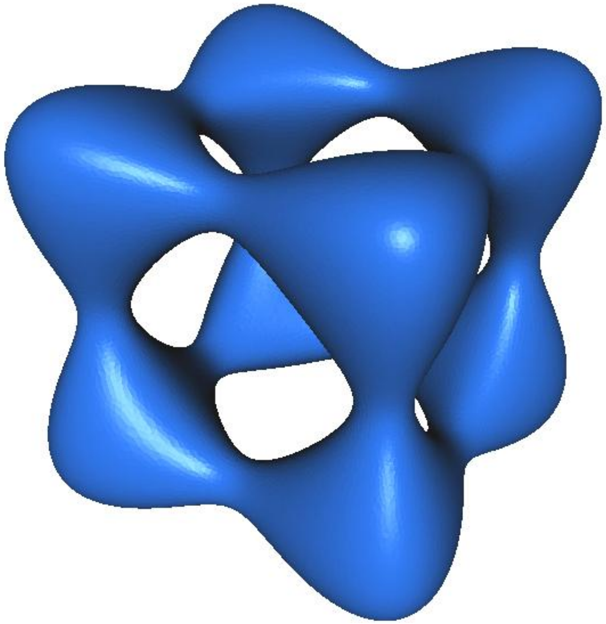
Marching cubes in octree

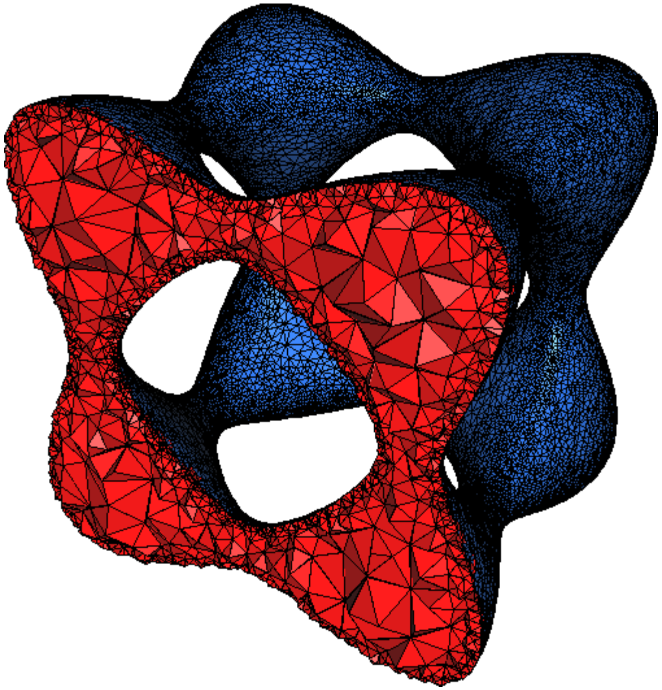


Guarantees

- Produces a good approximation of the surface
  - $\hat{S}$ is isotopic to S
  - $d_H(\hat{S}, S) = O_S(\varepsilon^2)$, error on normals, area $= O(\varepsilon)$
  - S is covered by the surface Delaunay balls of $\hat{S}$
- Produces sparse samples of optimal size
  - the set E of vertices of $\hat{S}$ is a sparse 2ε-sample of S
  - $|E| = O(\frac{\mathrm{area}(S)}{\varepsilon^2})$
- The aspect ratio of the facets can be controlled

Volume Mesh Generation





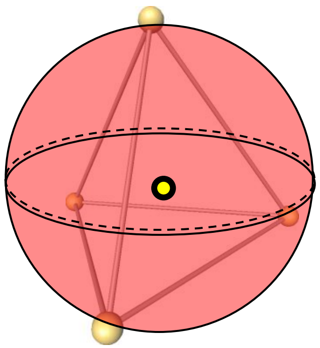
Volume Meshing

- Couple the latter algorithm with 3D Delaunay refinement

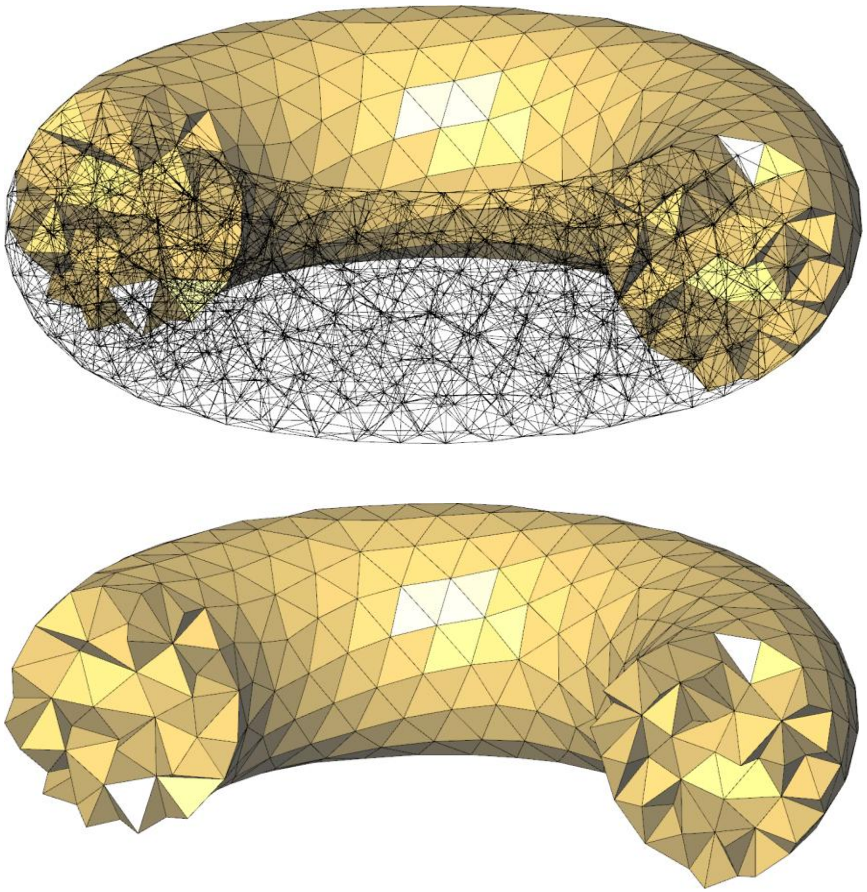
  (insert circumcenters of “bad” tetrahedra)
- Remove slivers at post-processing with sliver exudation

More Delaunay Filtering

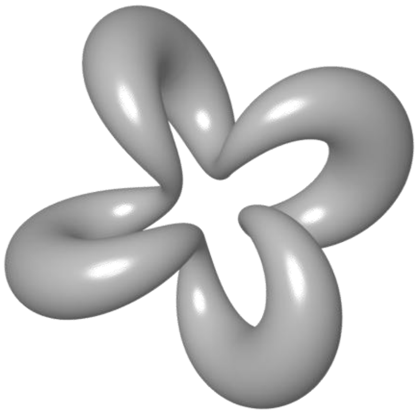
Delaunay triangulation restricted to **domain**



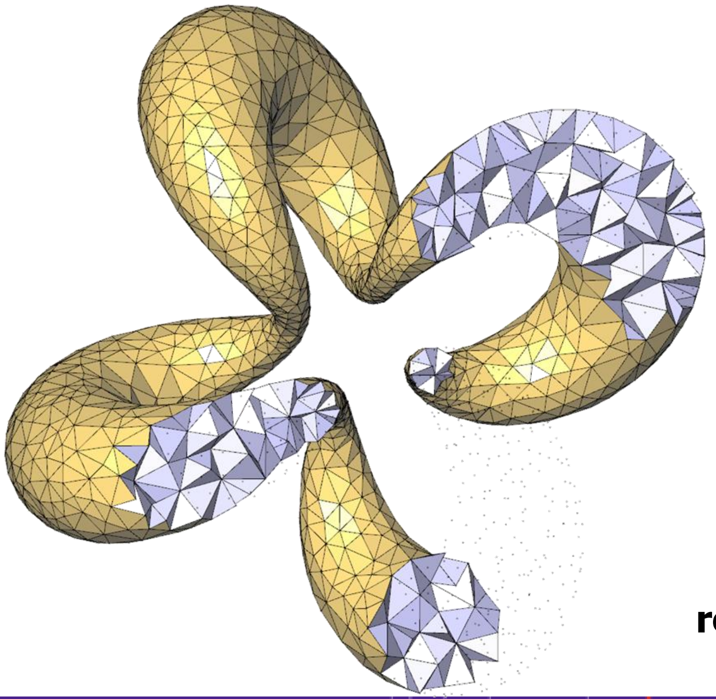
Dual Voronoi vertex inside domain (“oracle”)

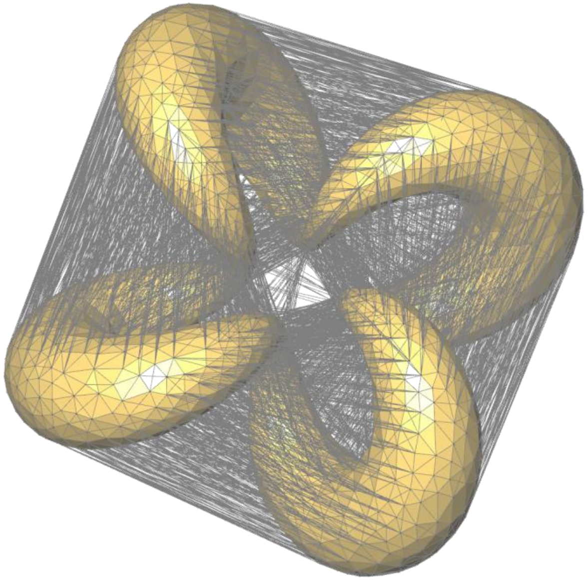


Delaunay Filtering



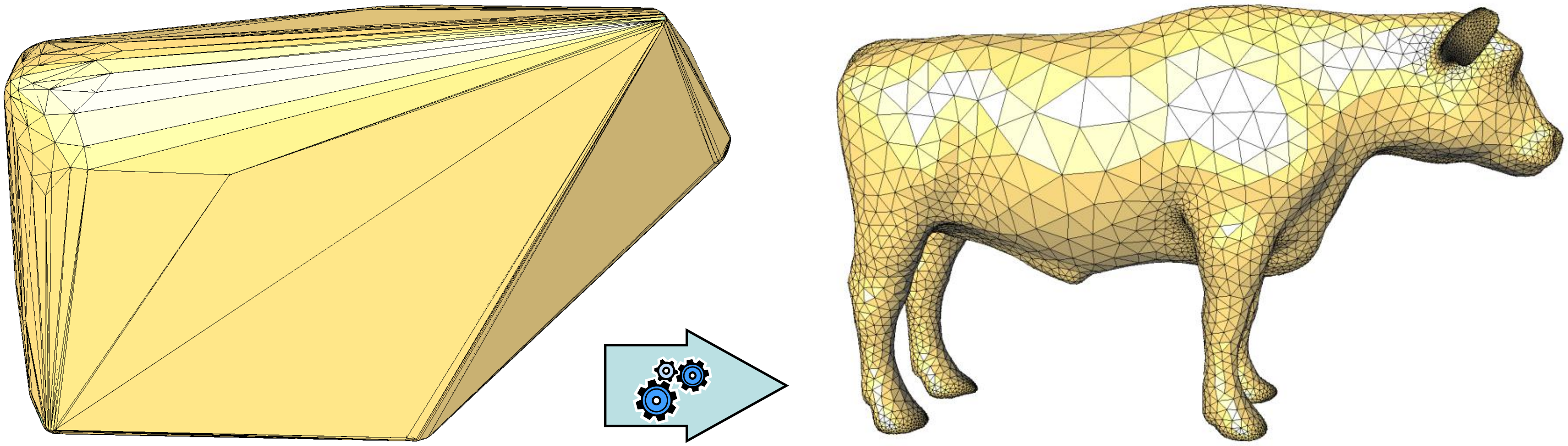
domain boundary





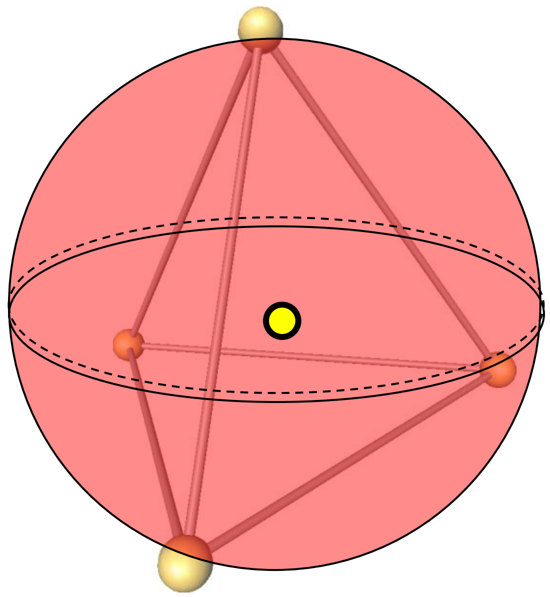
restricted Delaunay triangulation

3D Restricted Delaunay Triangulation



Delaunay Refinement

Steiner point



Volume Mesh Generation Algorithm

```
repeat
{
    pick bad simplex
    if(Steiner point encroaches a facet)
       refine facet
    else
       refine simplex
    update Delaunay triangulation restricted to domain
}
until all simplices are good
Exude slivers
```

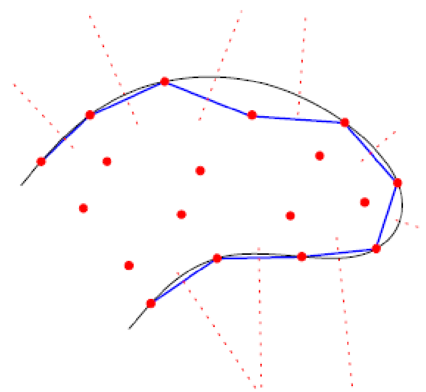
Delaunay Refinement

Apply the following rules with priority order

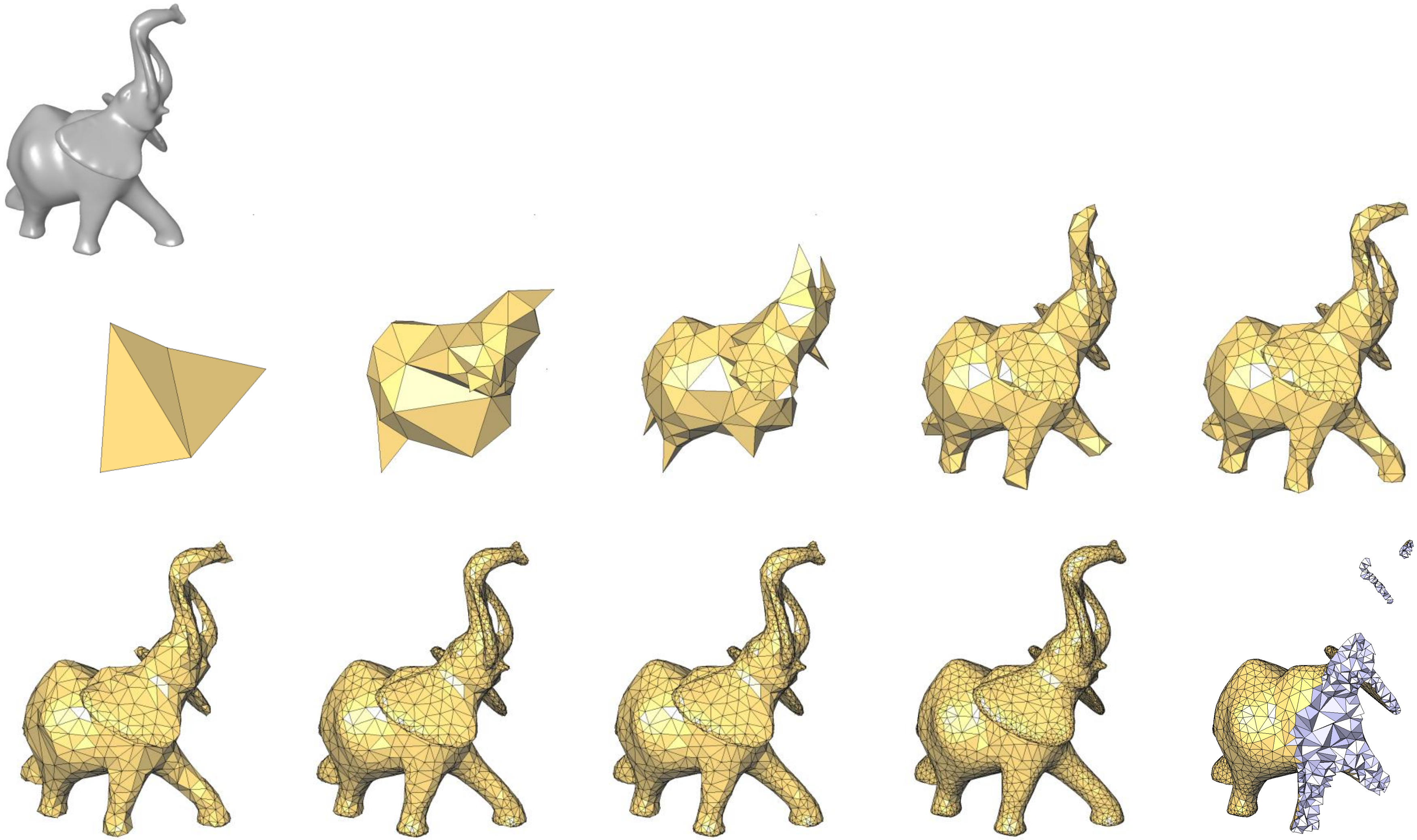
Rule 1: While there is a facet f in $\mathrm{Del}_{|\mathrm{bd}O}(\mathcal{P})$
with vertices $\notin \mathrm{bd}O$
refine_facet(f)

Rule 2: While there is a bad facet f in $\mathrm{Del}_{|\mathrm{bd}O}(\mathcal{P})$
refine_facet(f)

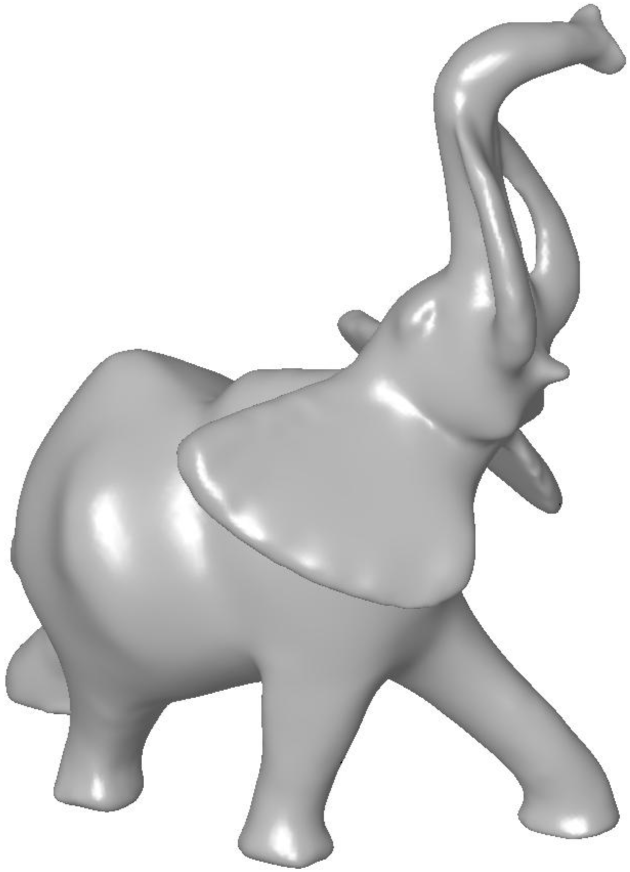
Rule 3: While there is a bad tetrahedron t in $\mathrm{Del}_{|O}(\mathcal{P})$
refine_tetrahedron_or_facet(t)

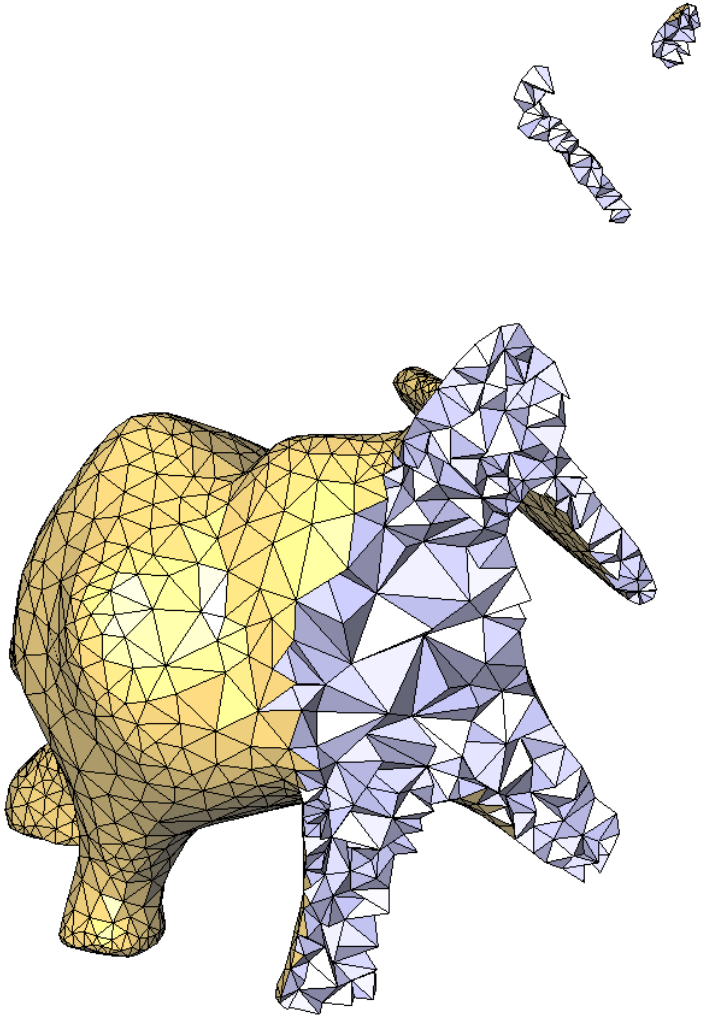


Example

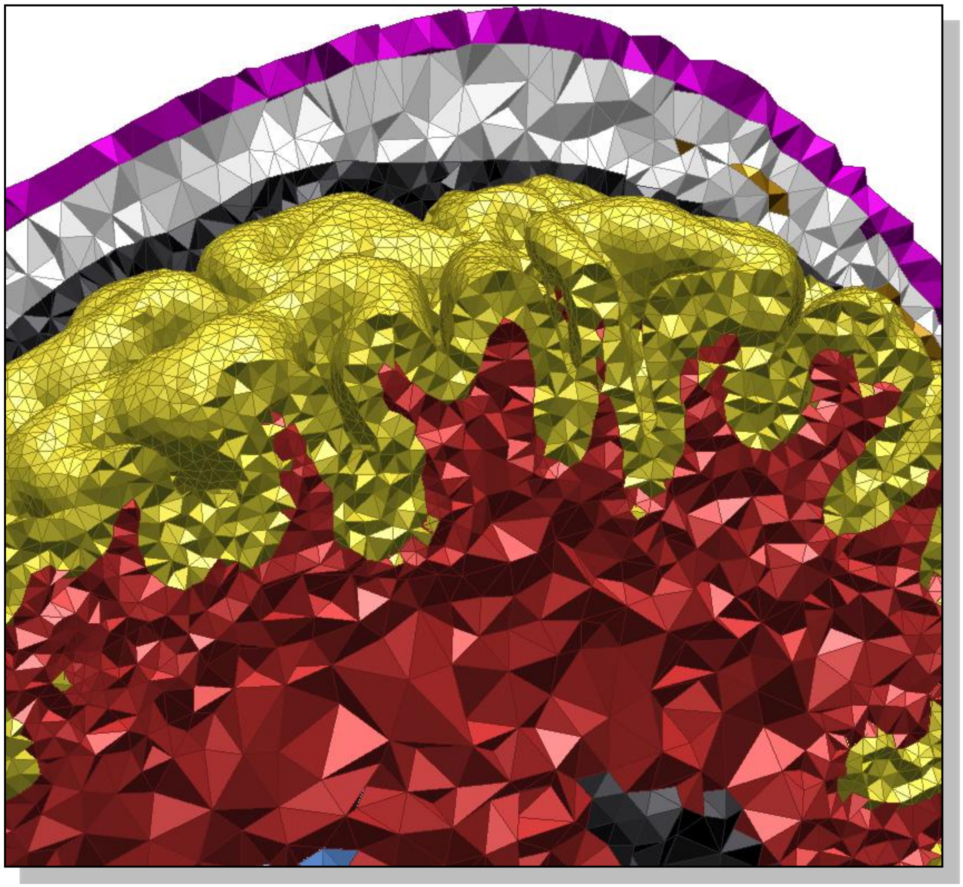


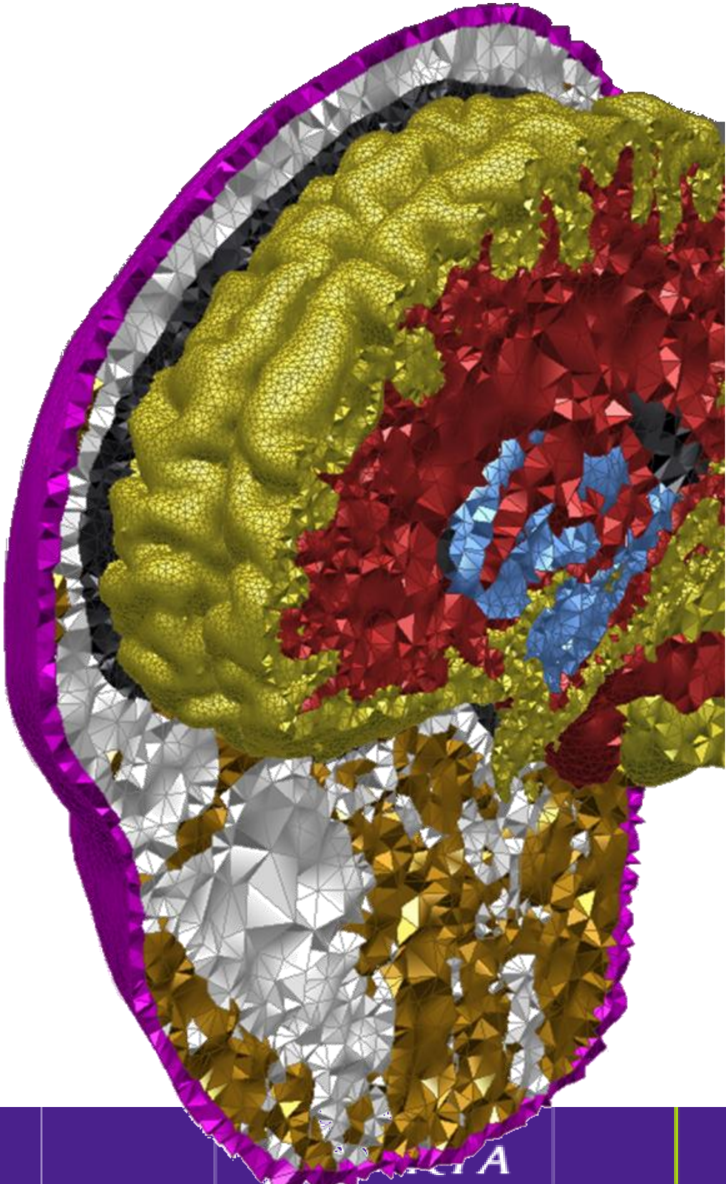
Example



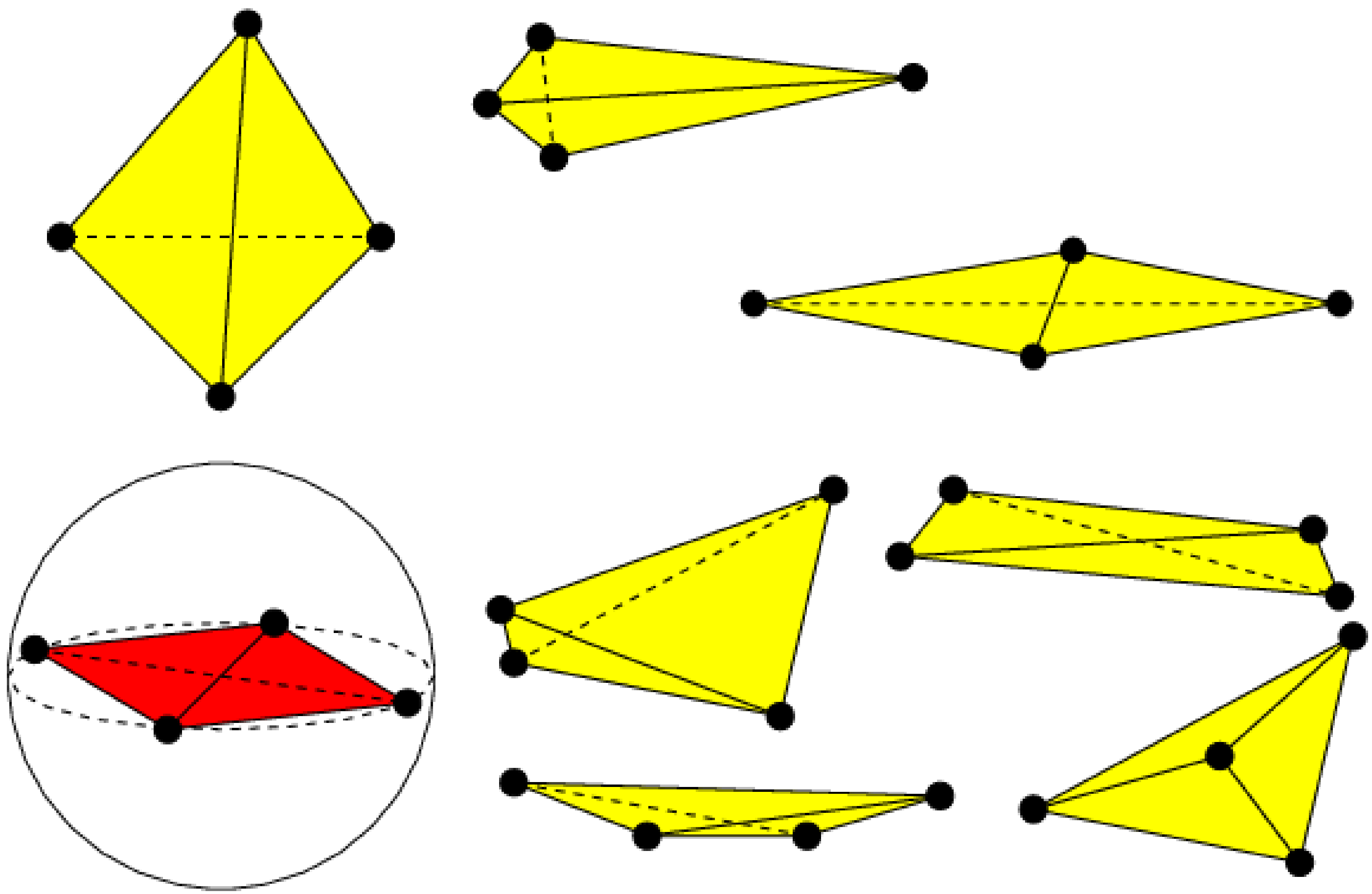


Multi-Domain Volume Mesh

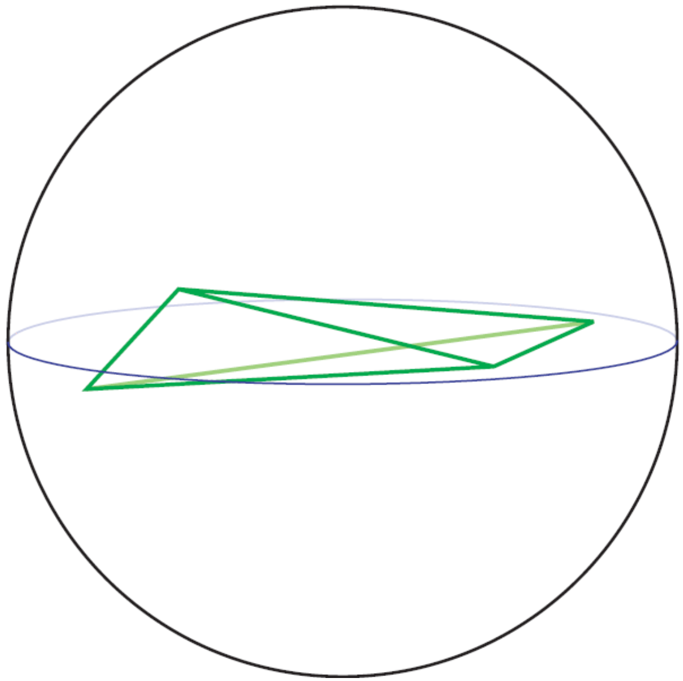




Tetrahedron Zoo

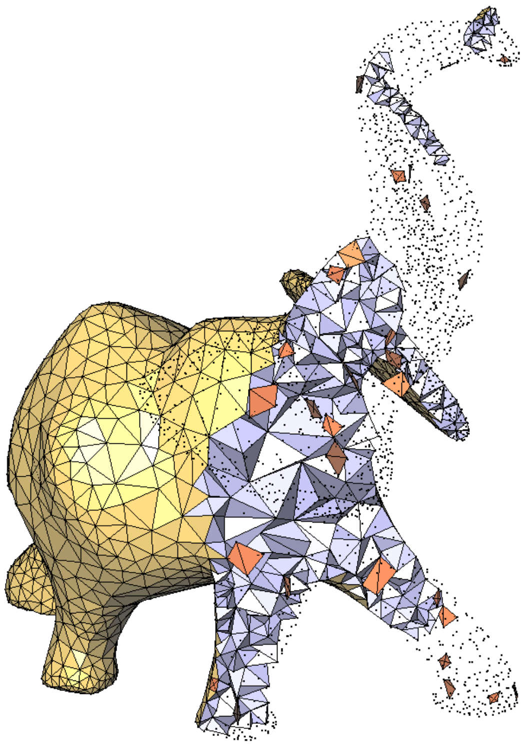


Sliver



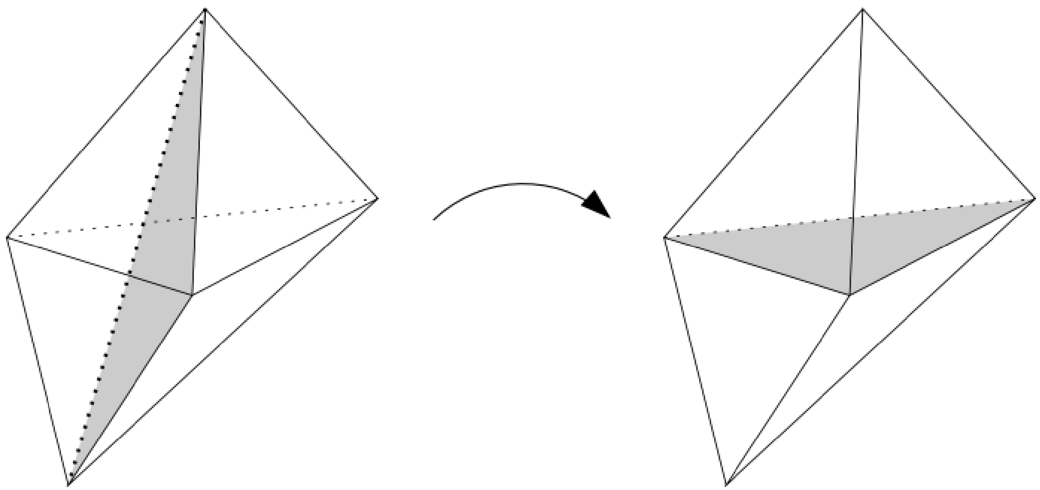
4 well-spaced vertices near the equator of their circumsphere

Slivers



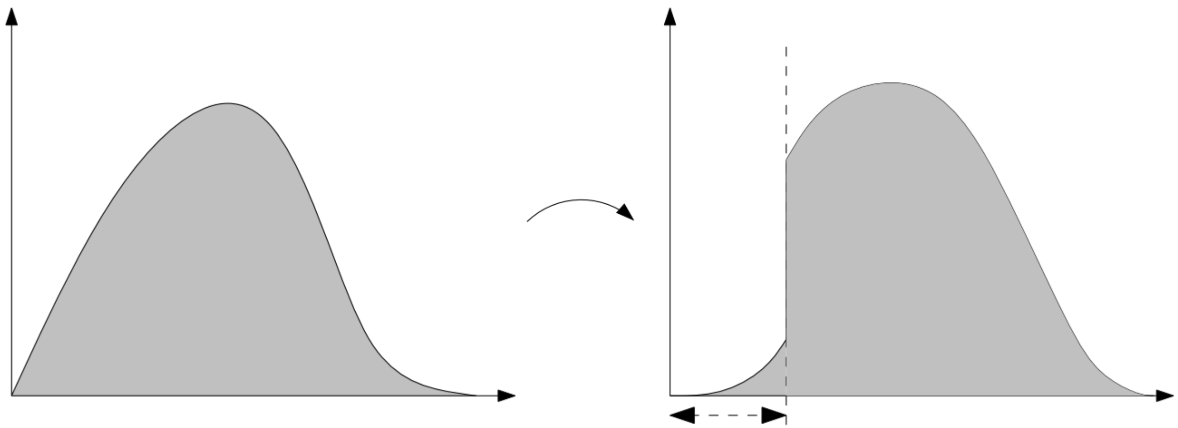
Sliver Exudation [Edelsbrunner-Guoy]

- Delaunay triangulation turned into a regular triangulation with null weights.
- Small increase of weights triggers edge-facets flips to remove slivers.

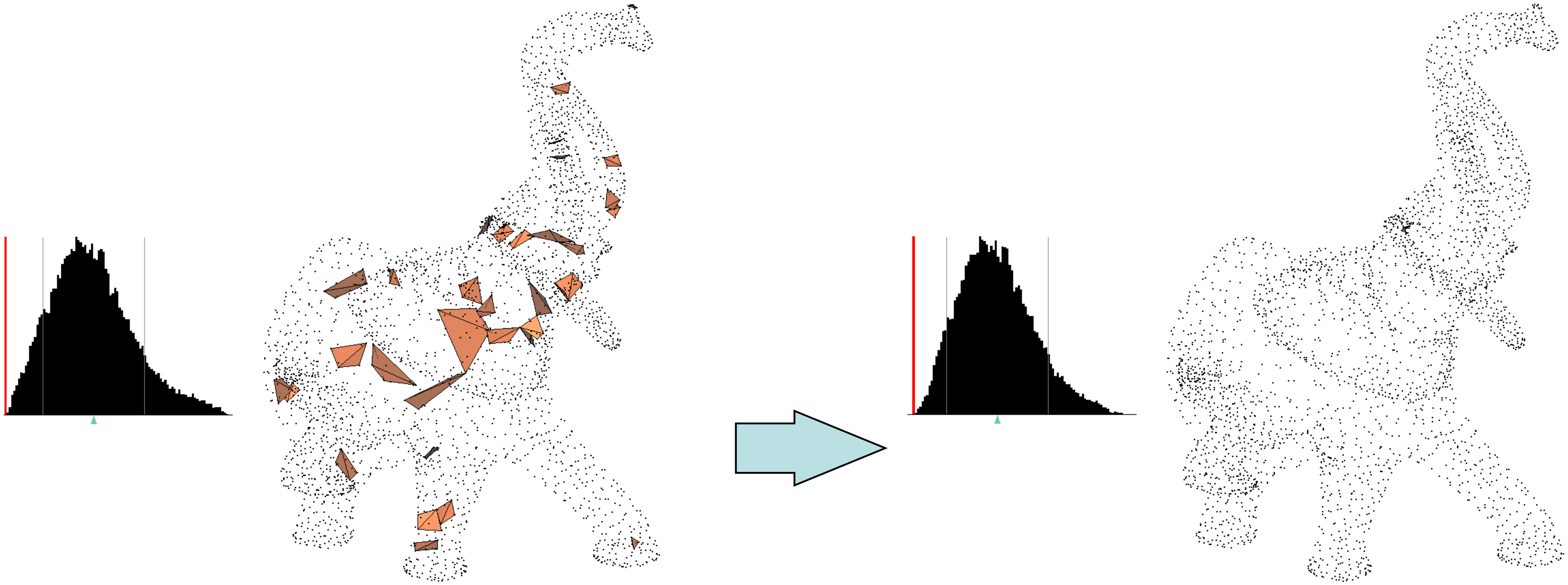


Sliver Exudation Process

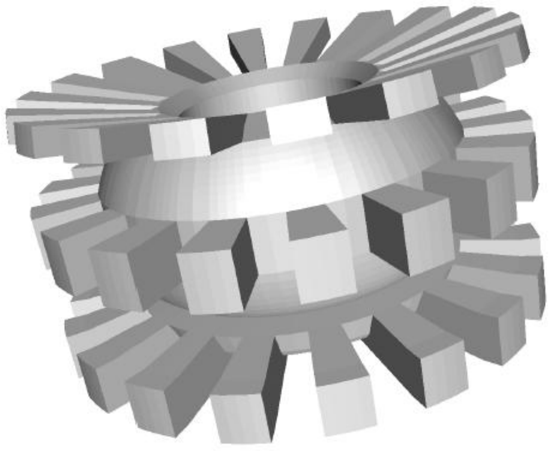
- Try improving all tetrahedra with an aspect ratio lower than a given bound
- Never flips a boundary facet

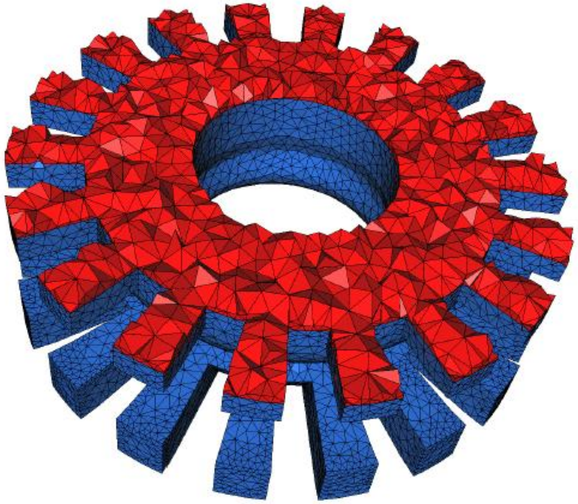


Example Sliver Exudation

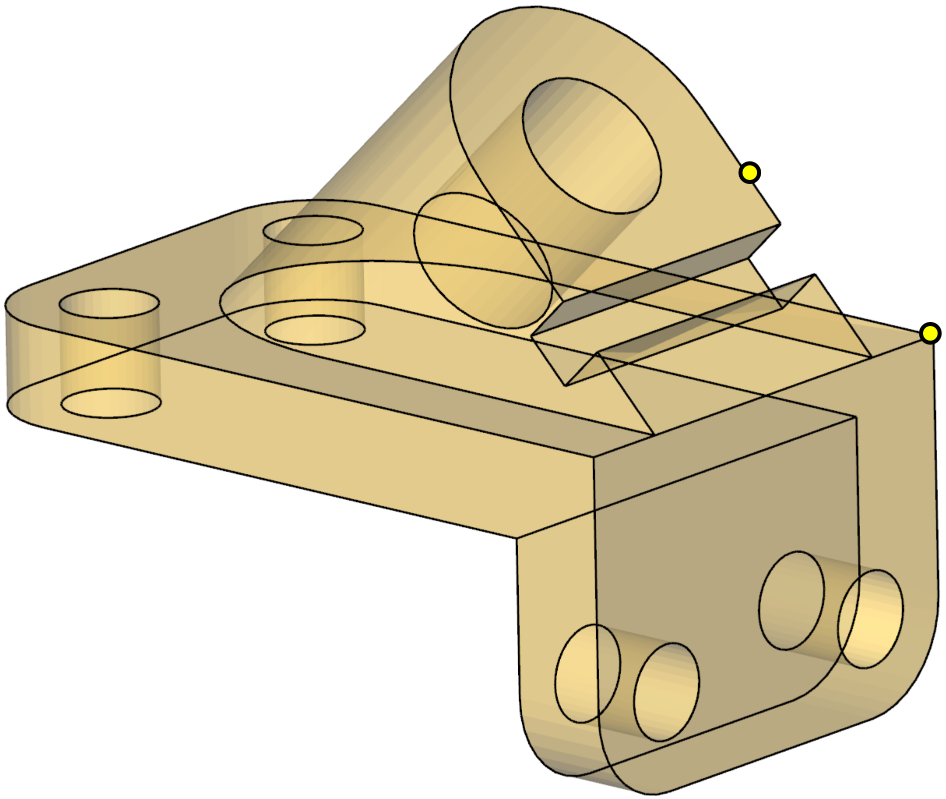


Piecewise Smooth Surfaces



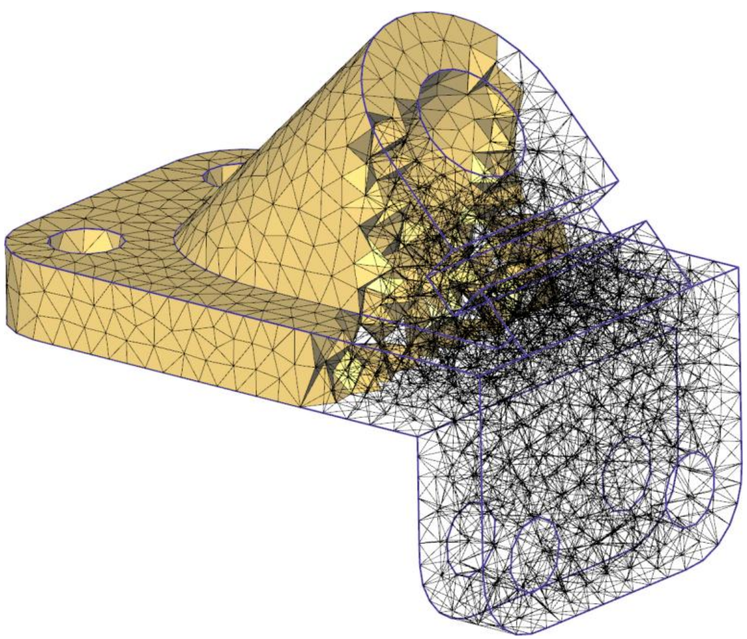


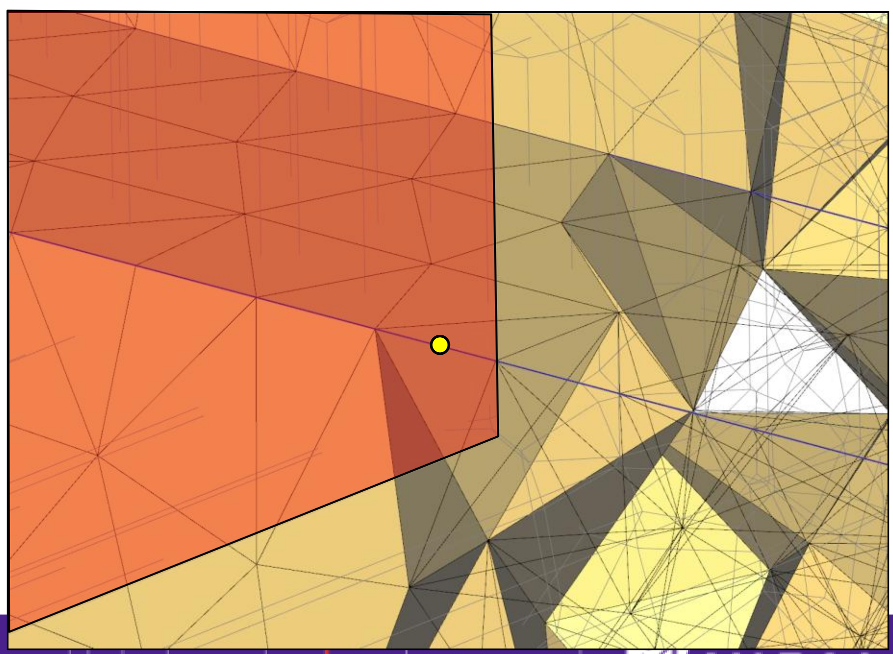
Input: Piecewise smooth complex



More Delaunay Filtering

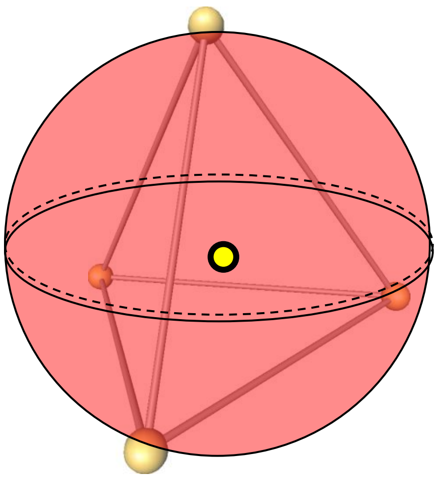
primitive	dual of	test	against
Voronoi vertex	tetrahedron	inside	domain
Voronoi edge	facet	intersect	domain boundary
Voronoi face	edge	intersect	crease

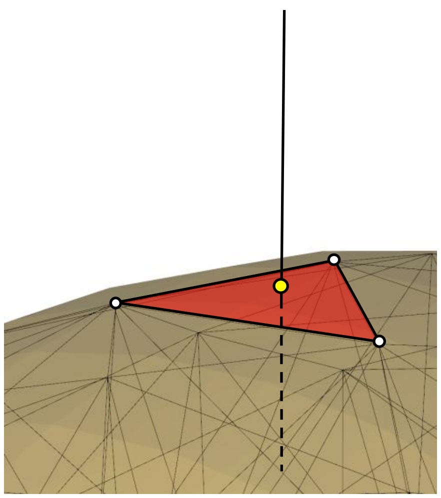


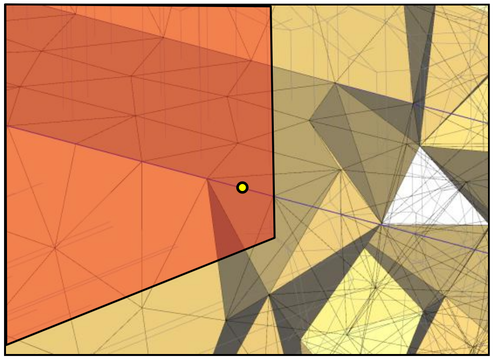


Delaunay Refinement

- Steiner points







Enrich Set of Rules...

Delaunay refinement

Apply the following rules with priority order

Rule 1+2: While there is an edge e in some $\mathrm{Del}_{|L_j}(\mathcal{P})$
with vertices $\notin L_j$
While there is a bad edge e in some $\mathrm{Del}_{|L_j}(\mathcal{P})$
refine_edge(()e)

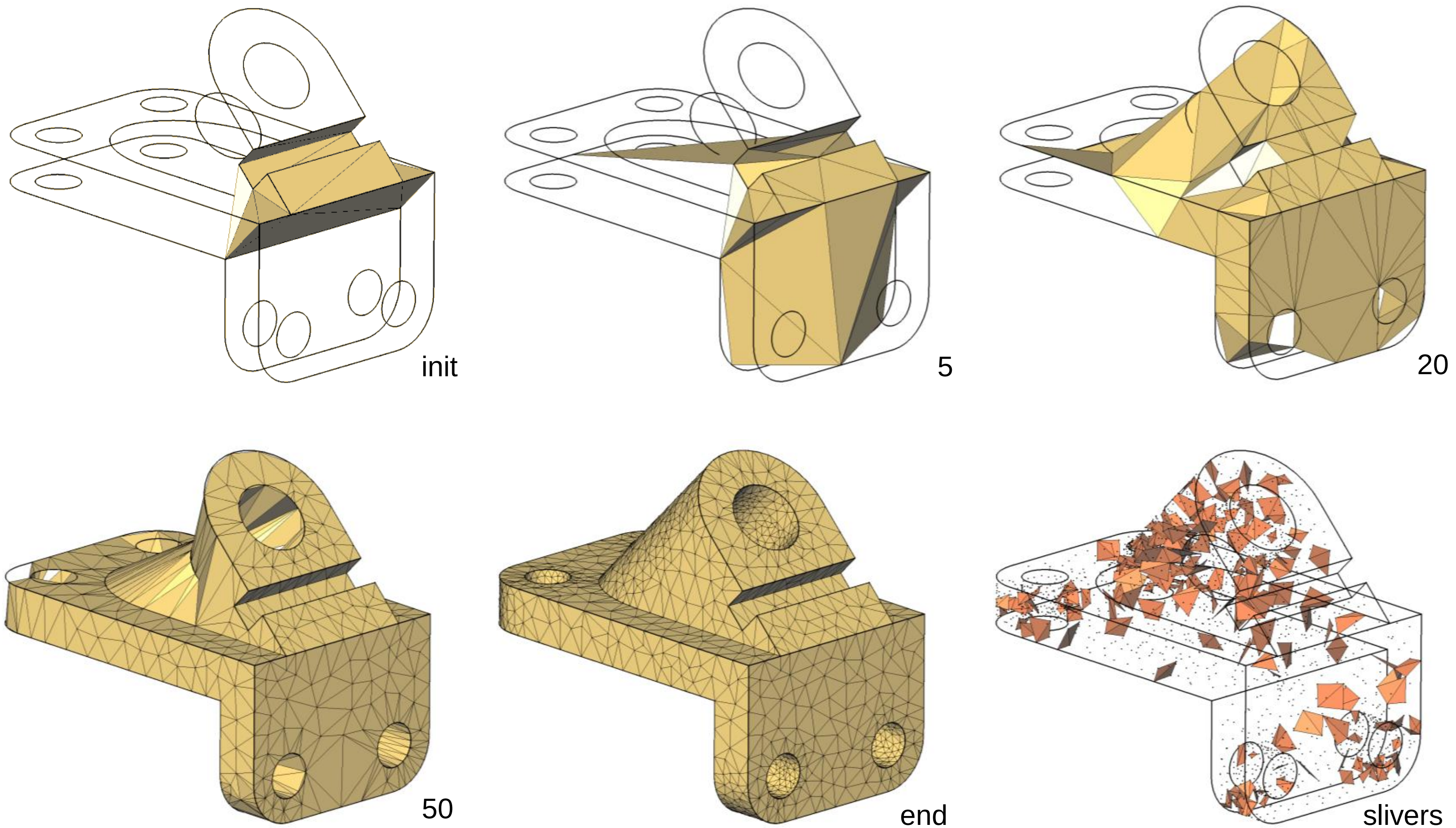
Rule 3+4: While there is a facet f in some $\mathrm{Del}_{|S_i}(\mathcal{P})$
with vertices $\notin S_i$
While there is a bad facet f in $\mathrm{Del}_{|\mathrm{bd}\,O}(\mathcal{P})$
refine_facet_or_edge(f)

Rule 5: While there is a bad tetrahedron t in $\mathrm{Del}_{|O}(\mathcal{P})$
refine_tetrahedron_or_facet_or_edge(t)

Sliver exudation

Delaunay refinement is followed by a sliver exudation phase

Example



Summary

- Meshes
  - Definition, variety
  - Background
    - Voronoi
    - Delaunay
    - constrained Delaunay
    - restricted Delaunay
  - Generation
    - 2D, 3D, Delaunay refinement