The Power Cell Algorithm for Local Homology Vineyards

I The Problem

II Local Homology

III Complex Pairs

IV Computation

with P. Bendich, D. Cohen-Steiner, J. Harer, D. Morozov

I The Problem

I.1 Problem Definition

Input: finite set $U \subseteq \mathbb{R}^n$, $\varepsilon > 0$

Output: simplest stratified space $\mathbb{X} \subseteq \mathbb{R}^n$ with Hausdorff distance $d_H(\mathbb{X}, U) \leq \varepsilon$.

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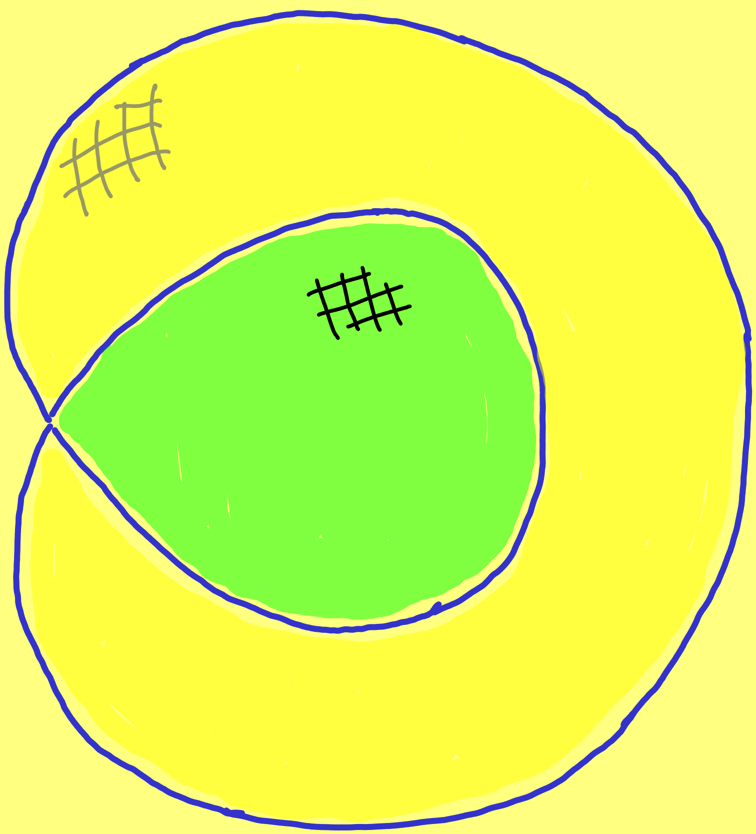
Today: homology of $\mathbb{X}$ at a point z.

I.2 Stratified Spaces



$\emptyset = \mathbb{X}_{-1} \subseteq \mathbb{X}_0 \subseteq \ldots \subseteq \mathbb{X}_m = \mathbb{X}$

$\mathbb{X}_i - \mathbb{X}_{i-1}$ is i-stratum,

an i-dim. manifold

components are i-dim. pieces

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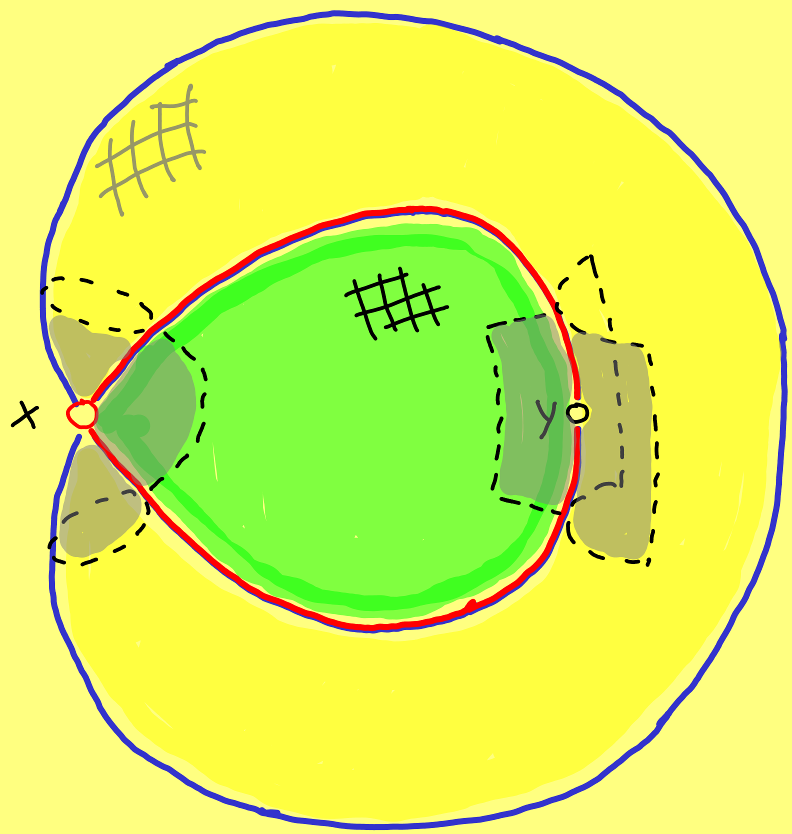

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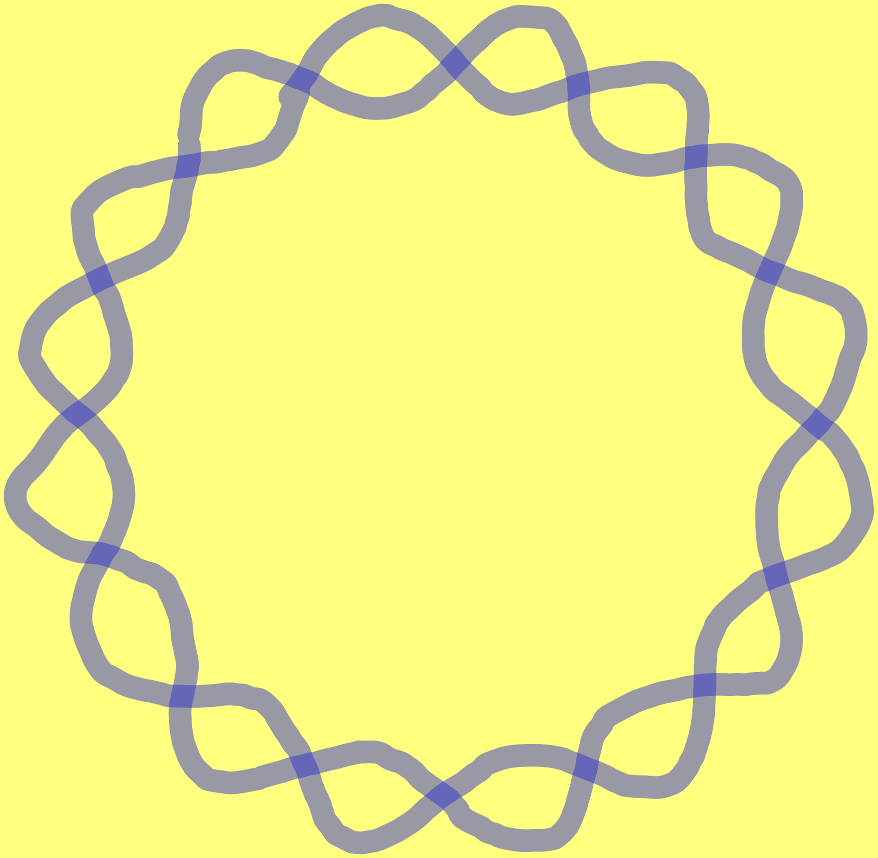
I.3 SCALES



a cycle of 1-cycles

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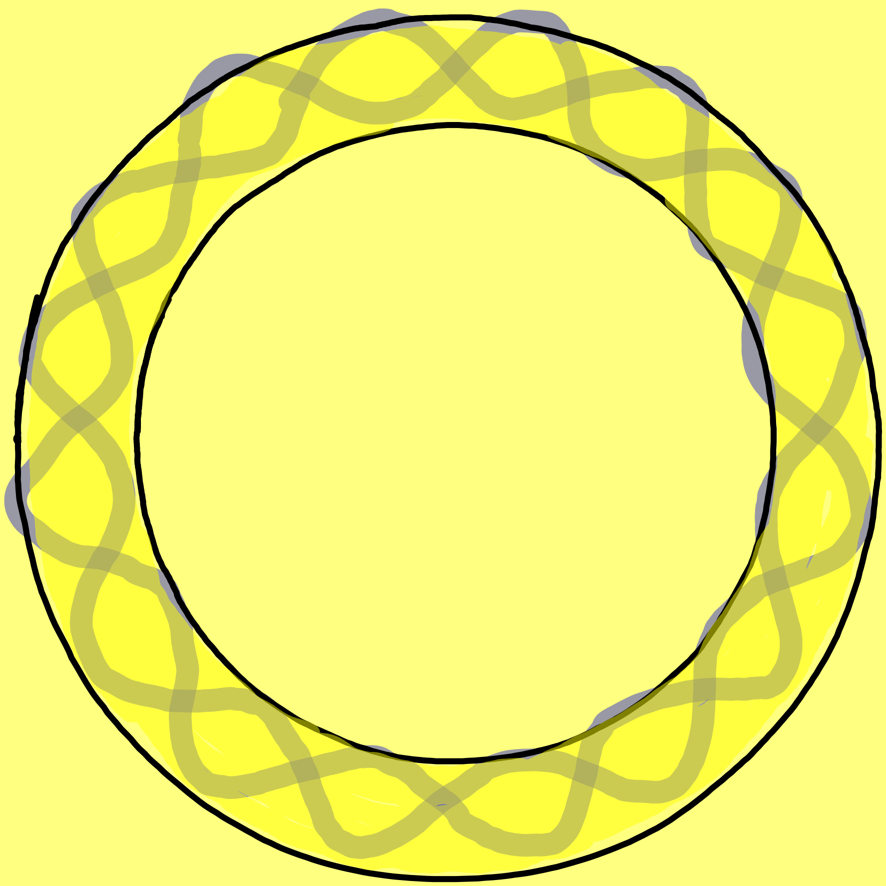


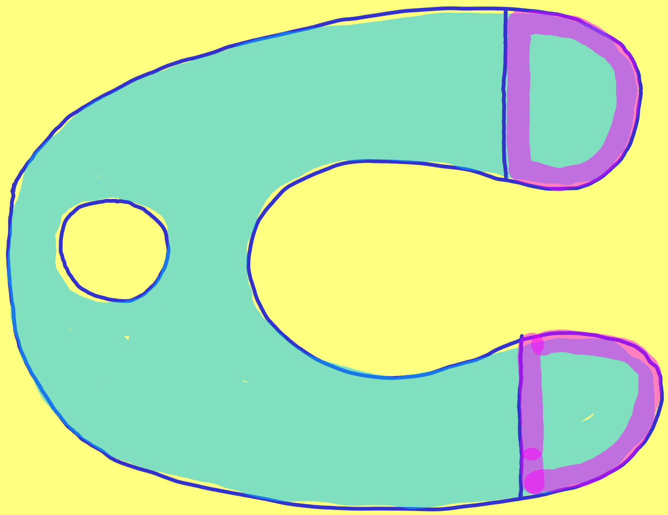
a cycle of 1-cycles

or

a 1-cycle

II Local Homology

II.1 Relative Homology



$Y_0 \subseteq Y$

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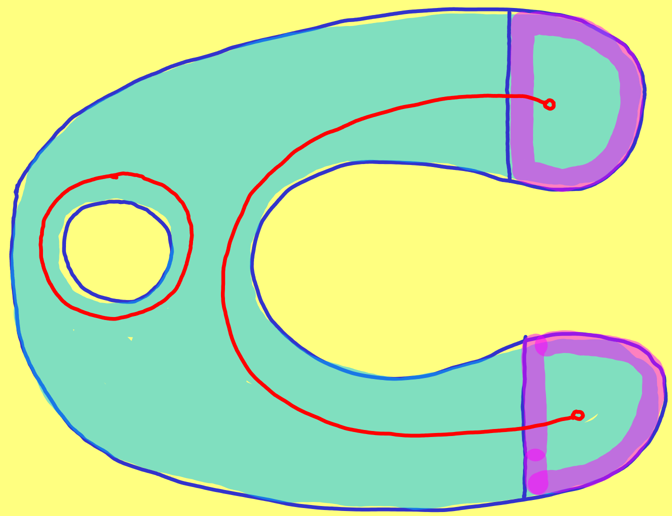

$Y_0 \subseteq Y$

$\operatorname{rk} H_1(Y, Y_0) = 2$

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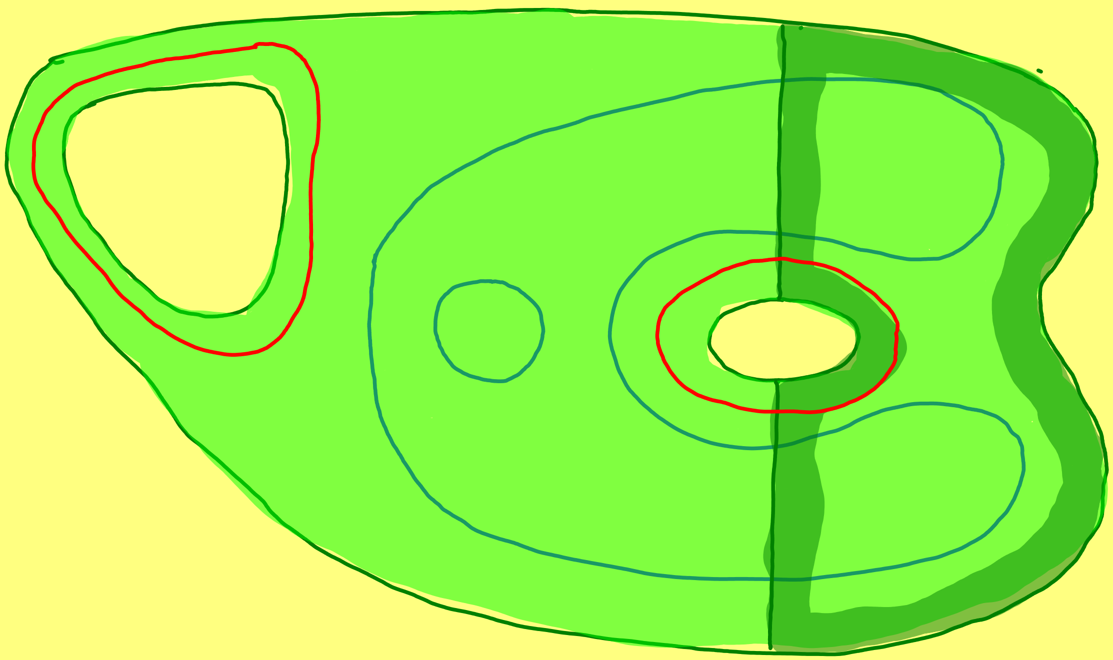

$X_0 \subseteq X$

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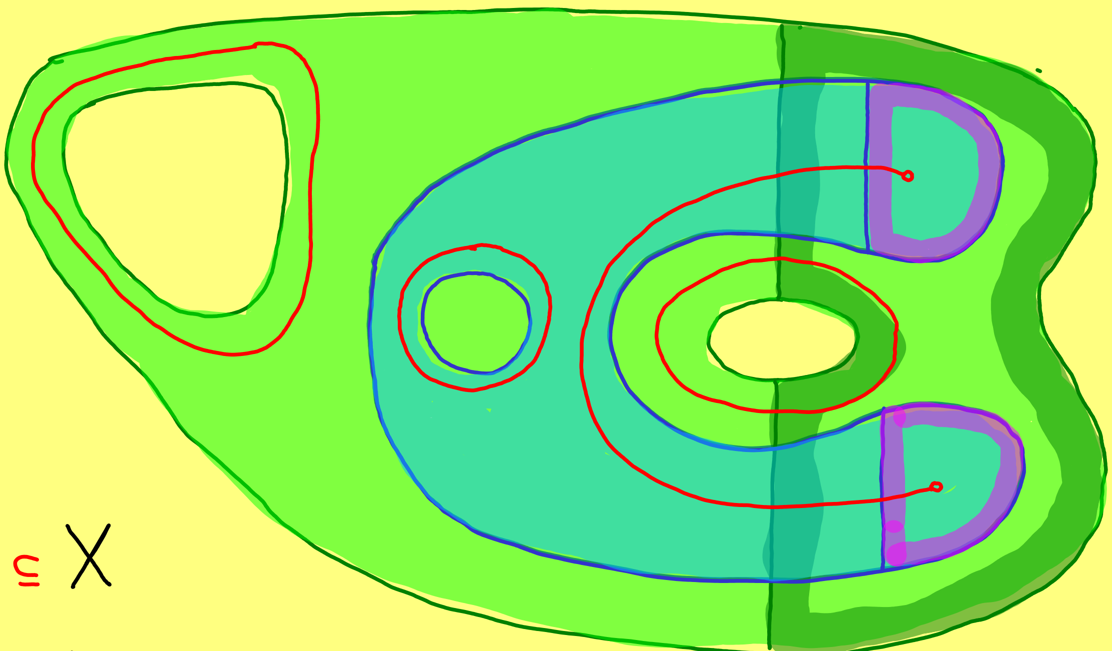

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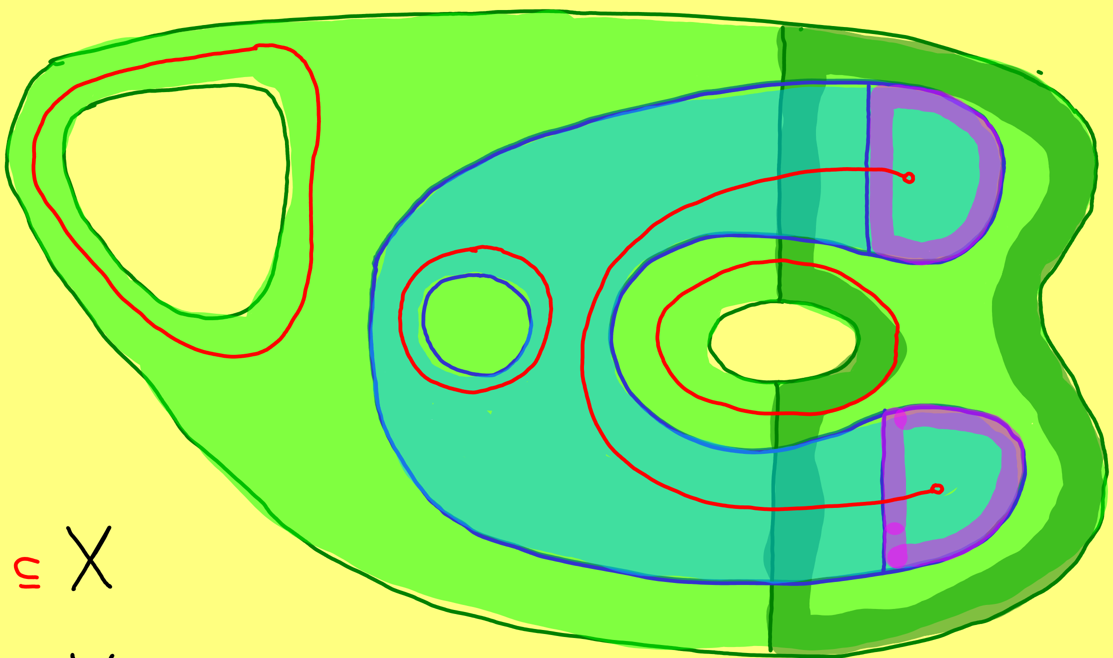

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$Y \subseteq X$

$Y_0 \subseteq X_0$

$f : H_1(Y, Y_0) \to H_1(X, X_0)$

$\operatorname{rk} \operatorname{im} f = 1$

II.2 Local Homology

$\mathbb{X} \subseteq \mathbb{R}^n$, $\quad z \in \mathbb{R}^n$,

$d_z : \mathbb{R}^n \to \mathbb{R}$ defined by $d_z(x) = \| x - z \|$

$B_r = d_z^{-1}[0, r]$, $\quad B^r = d_z^{-1}[r, \infty)$

II.2 LOCAL HOMOLOGY

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$$B_r = d_z^{-1}[0, r], \quad B^r = d_z^{-1}[r, \infty)$$

local homology of $\mathbb{X}$ at z is

$$H(\mathbb{X}, \mathbb{X} - z) = \lim_{r \to 0} H(\mathbb{X}, \mathbb{X} - \operatorname{int} B_r)$$

$$= \lim_{r \to 0} H(\mathbb{X}, \mathbb{X} \cap B^r)$$

II.3 Multi-Scale Version

$$0 \to \cdots \to H(\mathbb{X}, \mathbb{X} \cap B^r) \to \cdots \to H(\mathbb{X}, \mathbb{X} \cap B^s) \to \cdots \to H(\mathbb{X})$$

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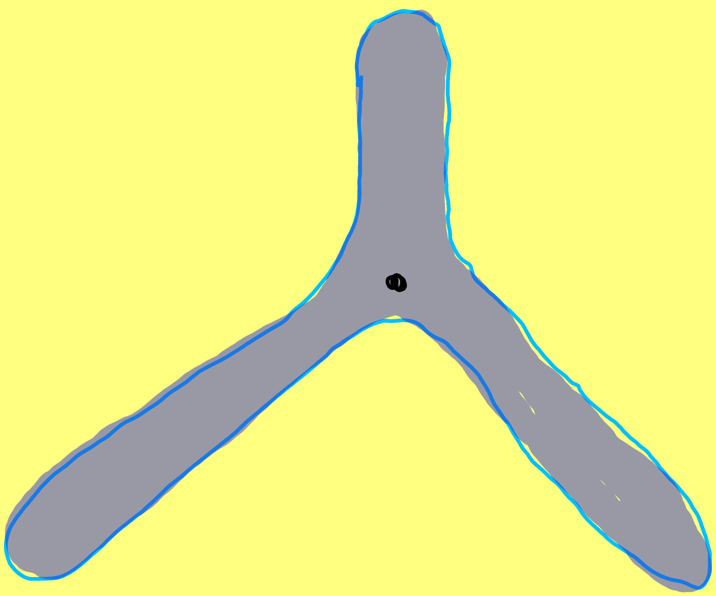

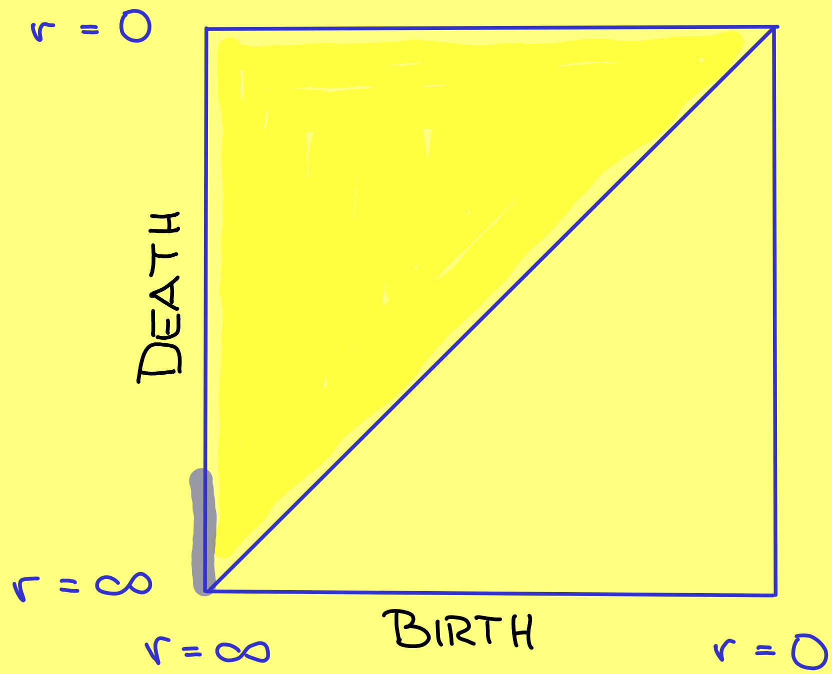

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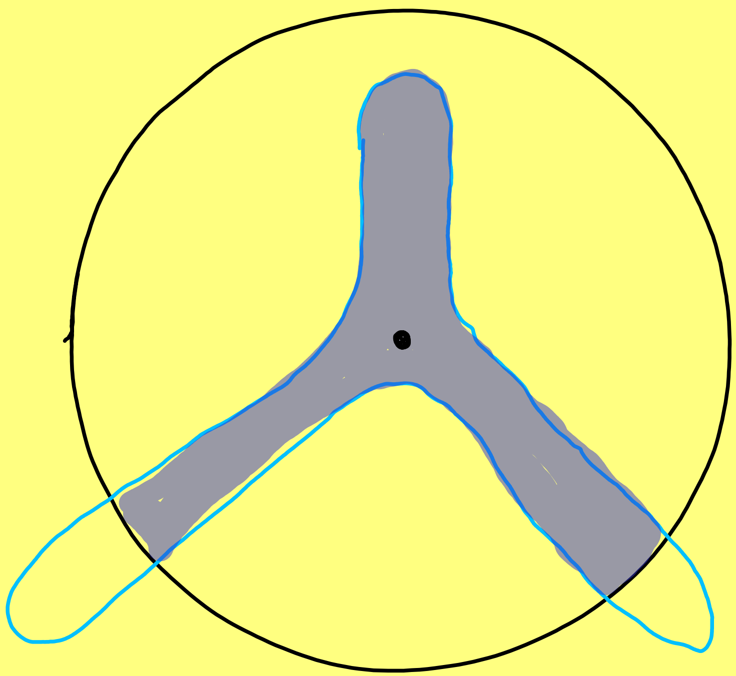

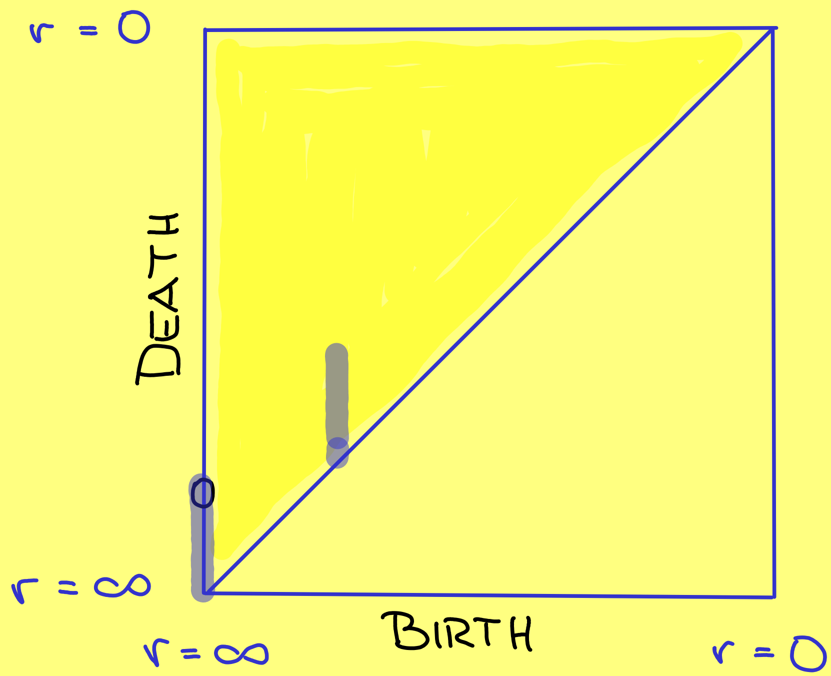

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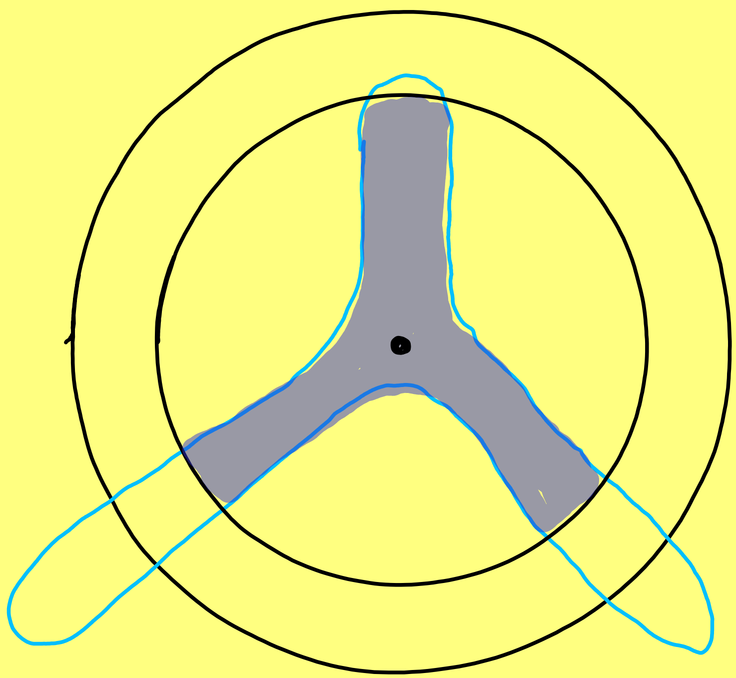

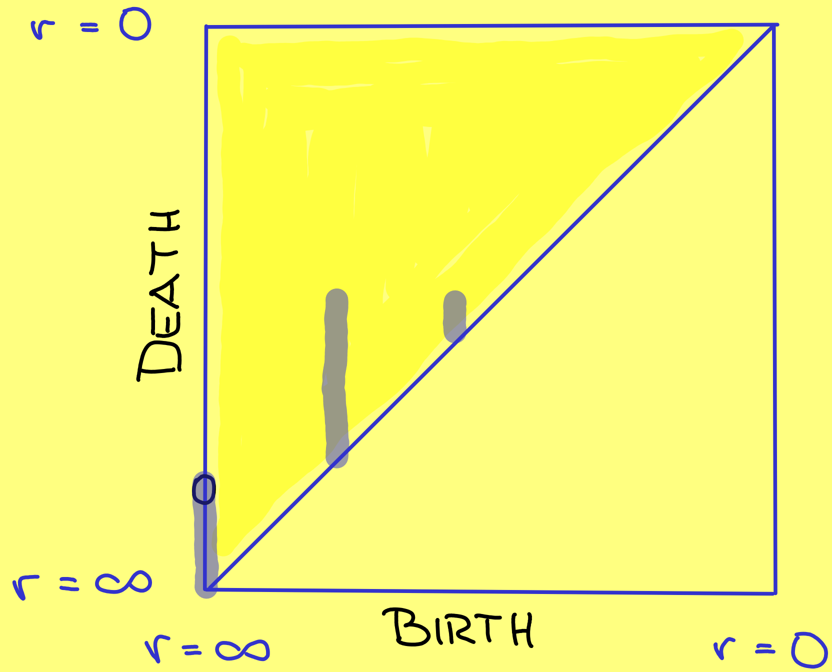

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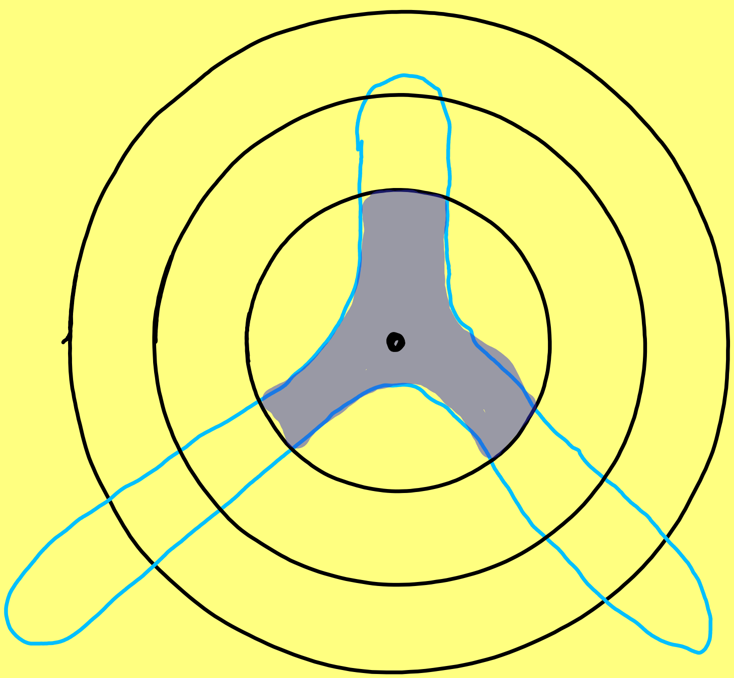

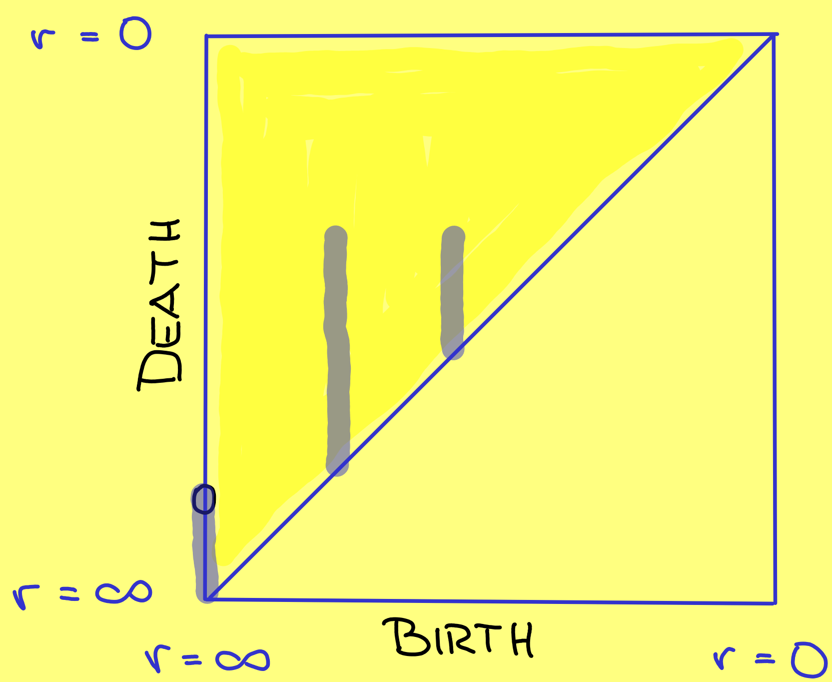

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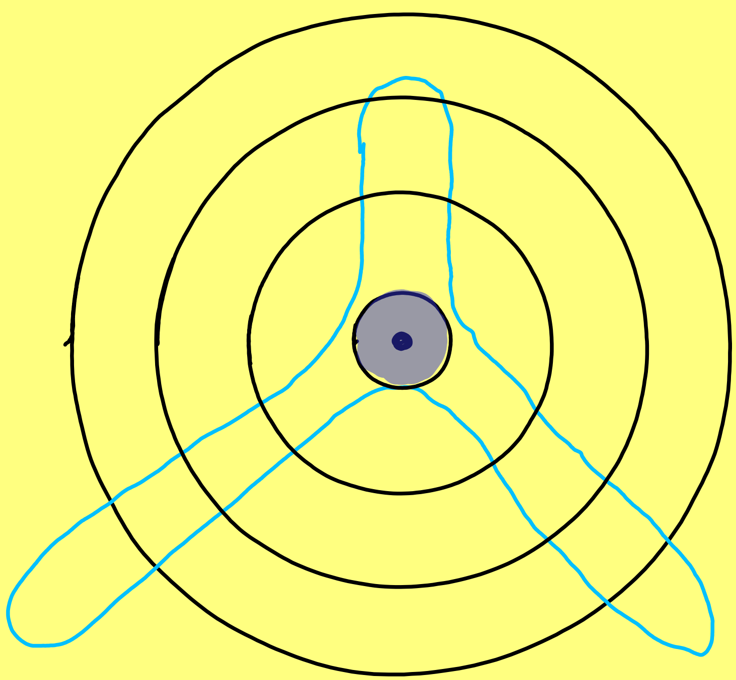

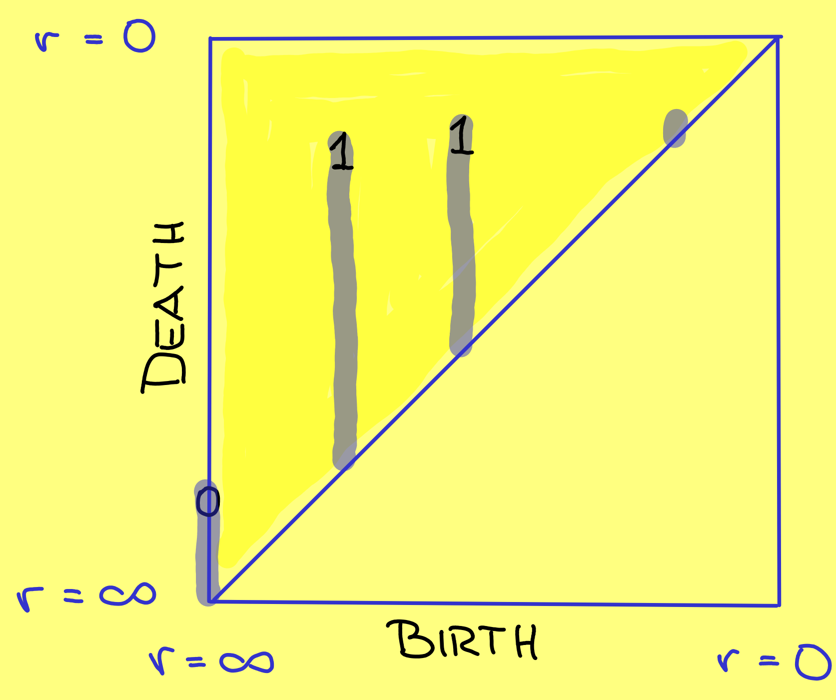

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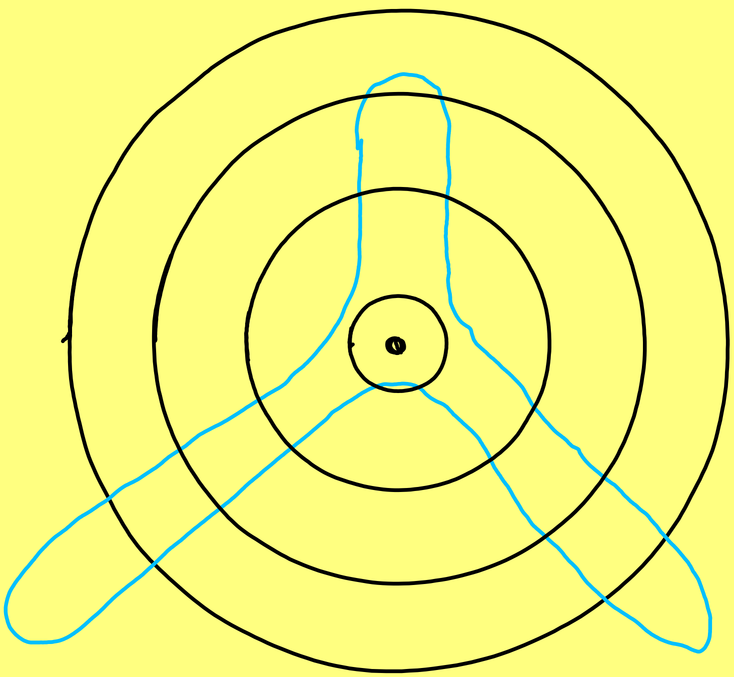

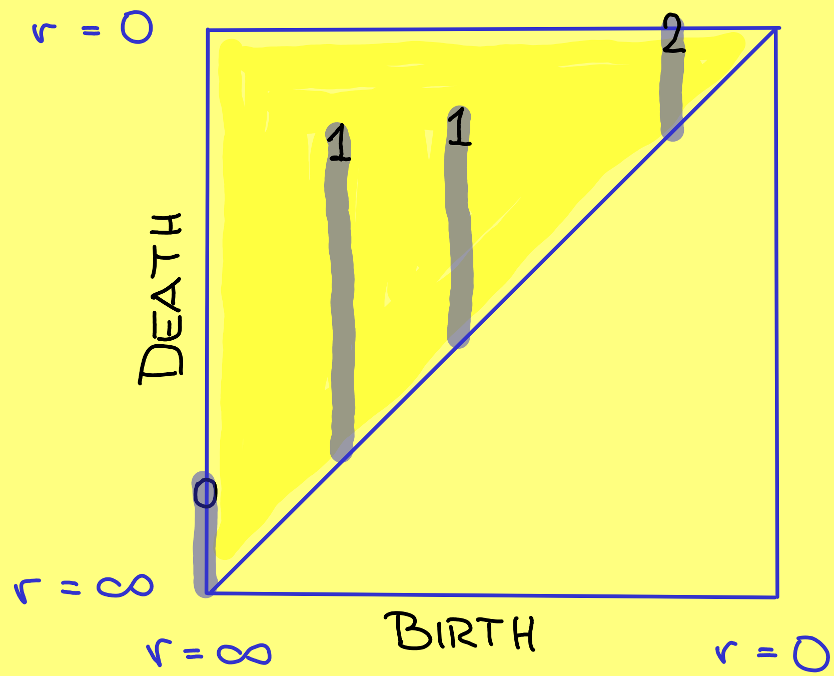

II.3 Multi-Scale Version

$\mathrm{Dgm}(d_z \mid \mathbb{X}_\alpha)$ changes discontinuously with α

II.4 $(\alpha|r)$-Diagrams

Instead of shrinking r grow α.

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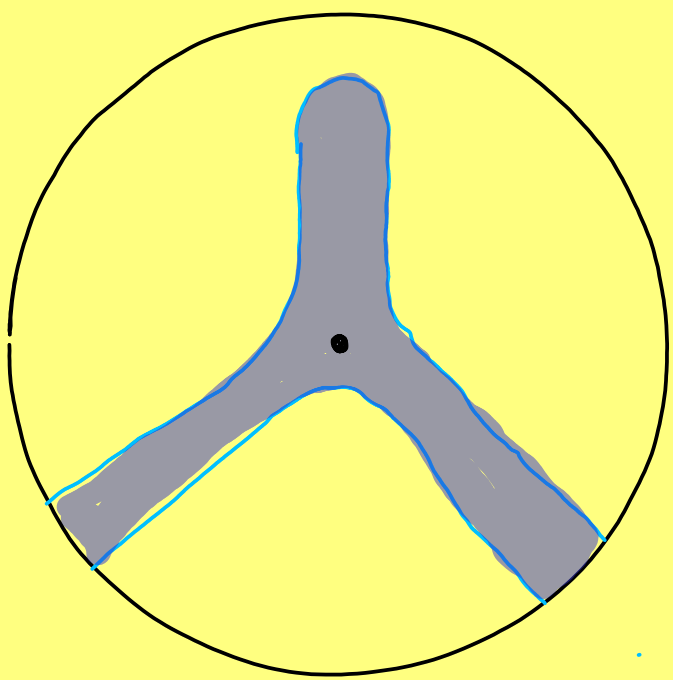

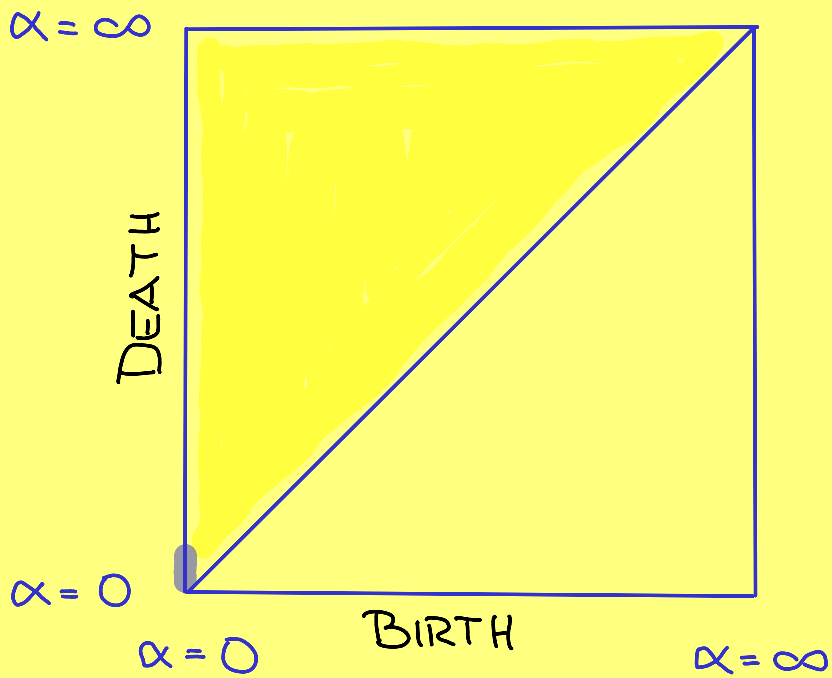

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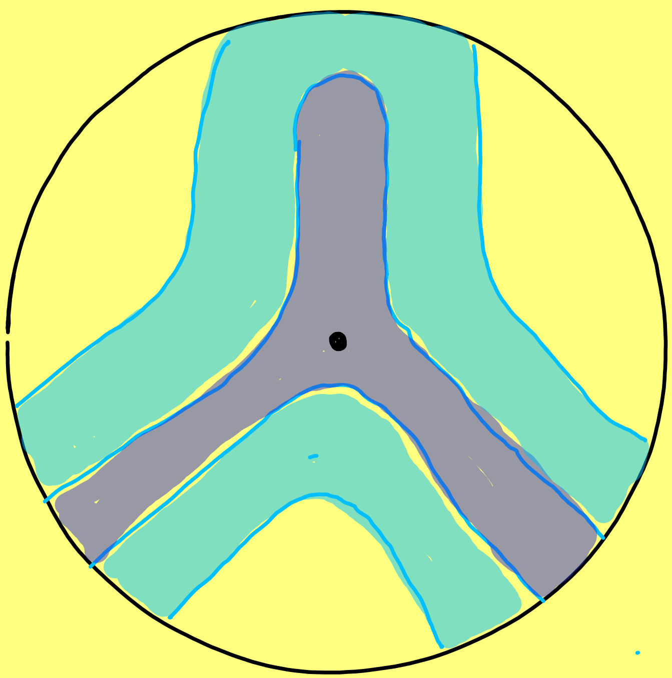

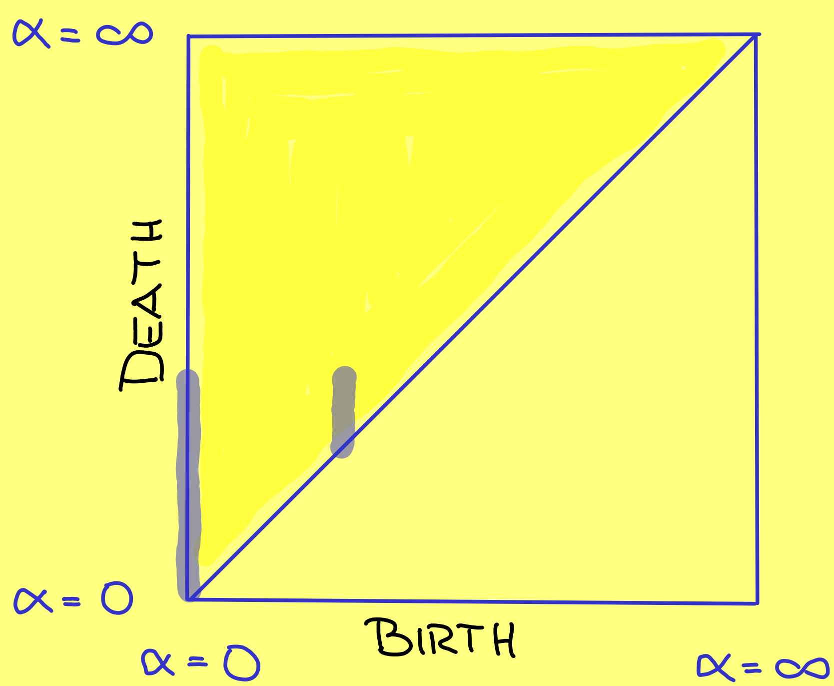

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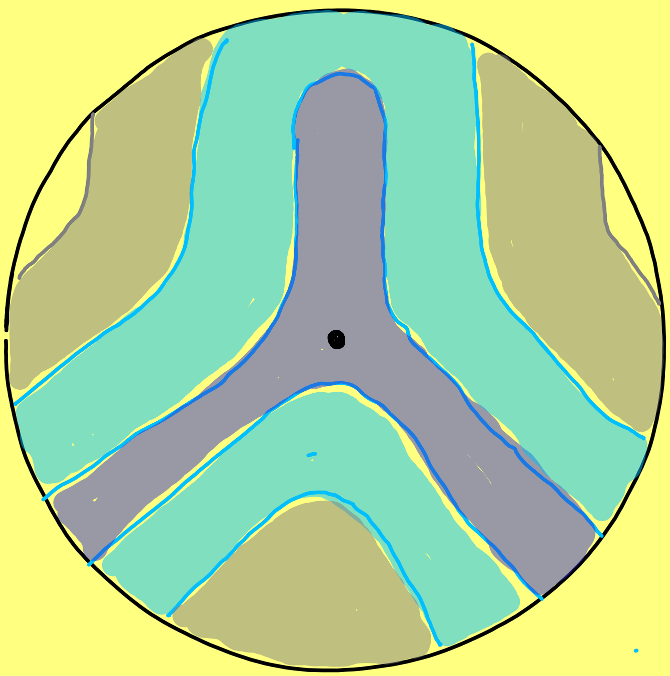

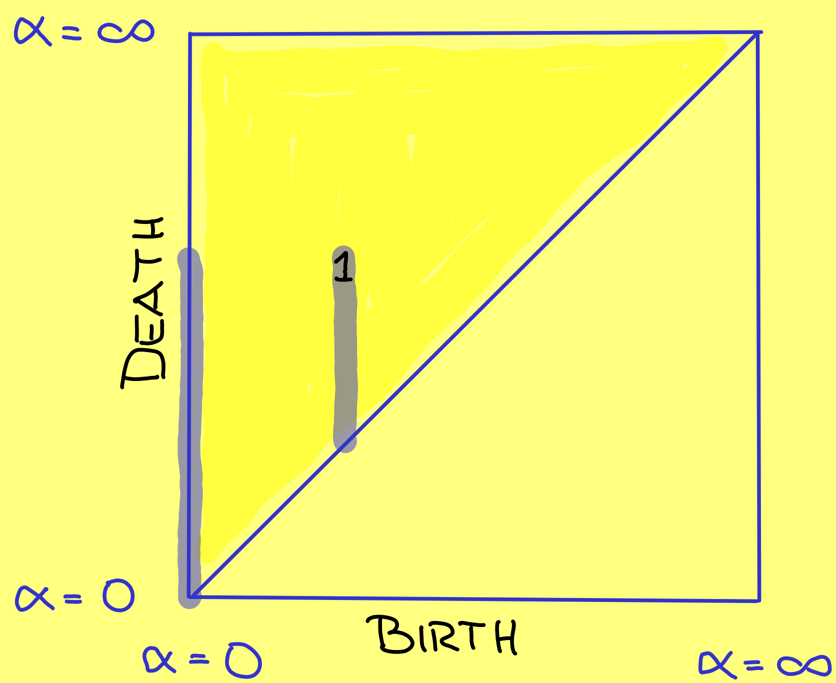

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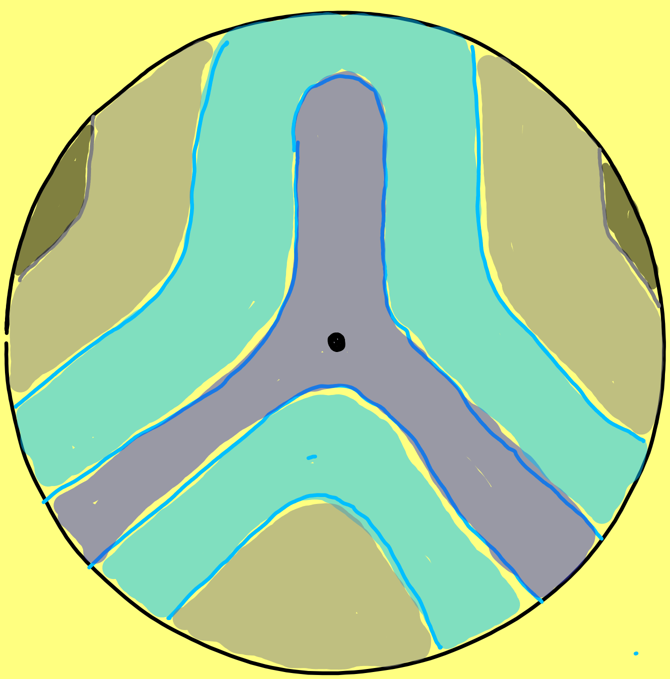
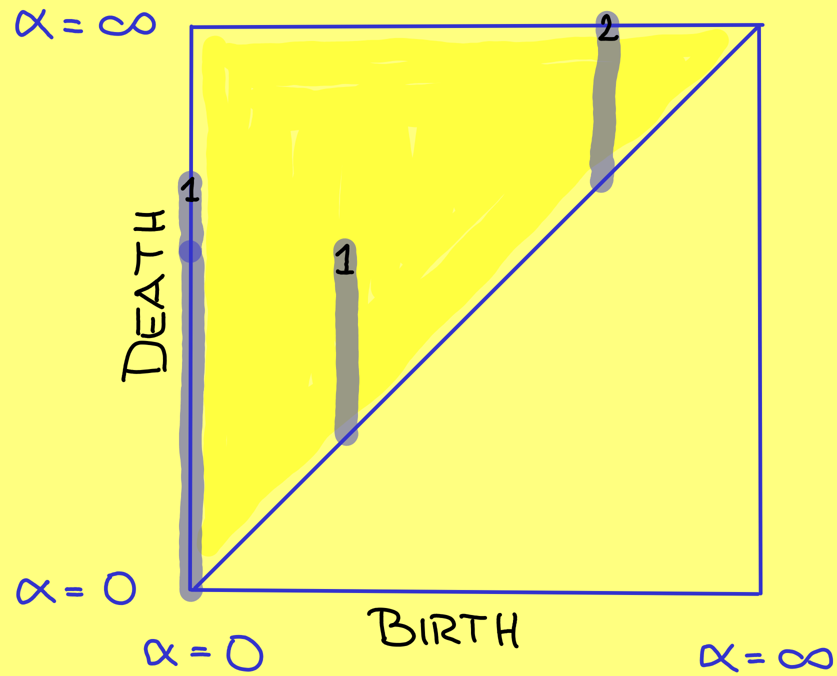




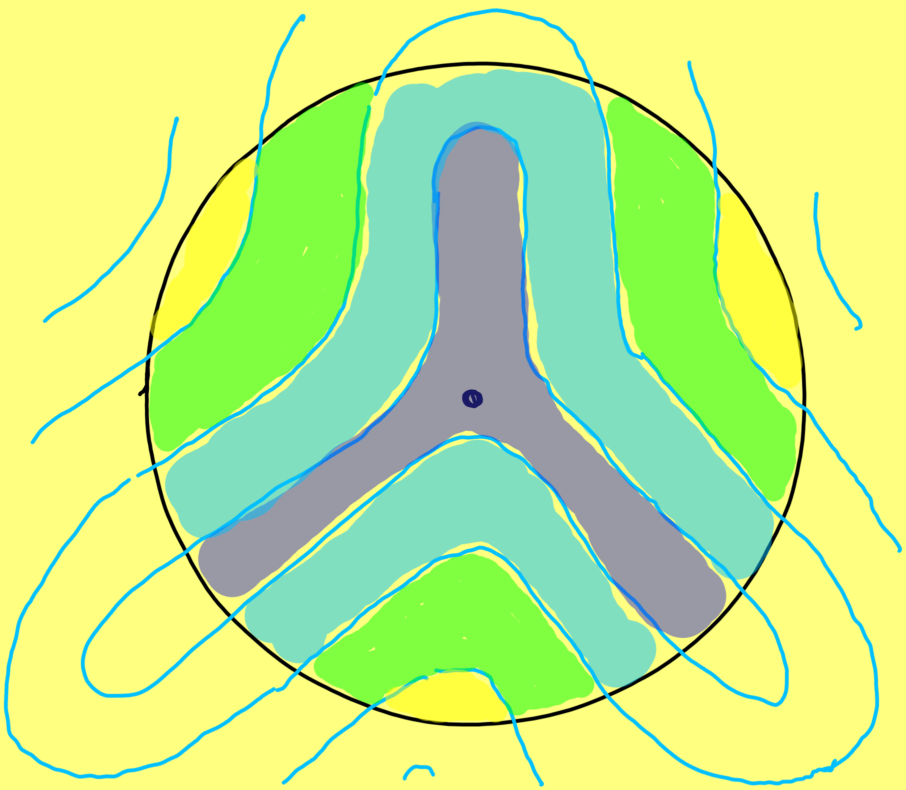
II.5 $(\alpha | r)$-VINEYARDS

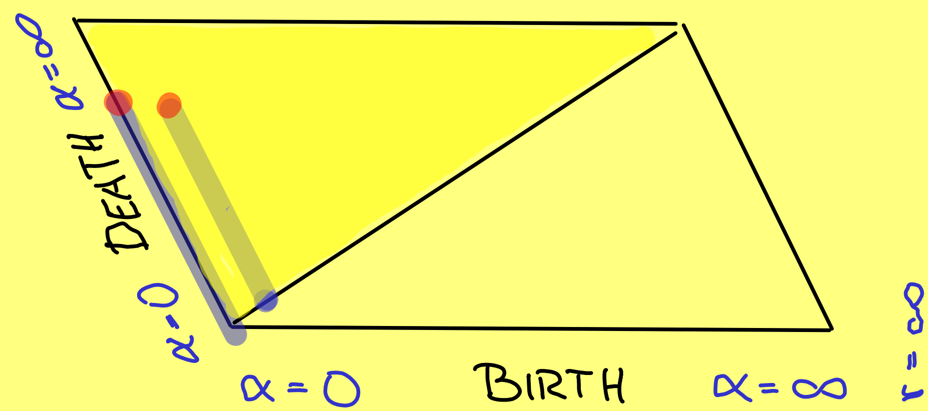
STABILITY LEMMA.

The bottleneck distance between the persistence diagrams of d_X restricted to open balls of radii $r < s$ is

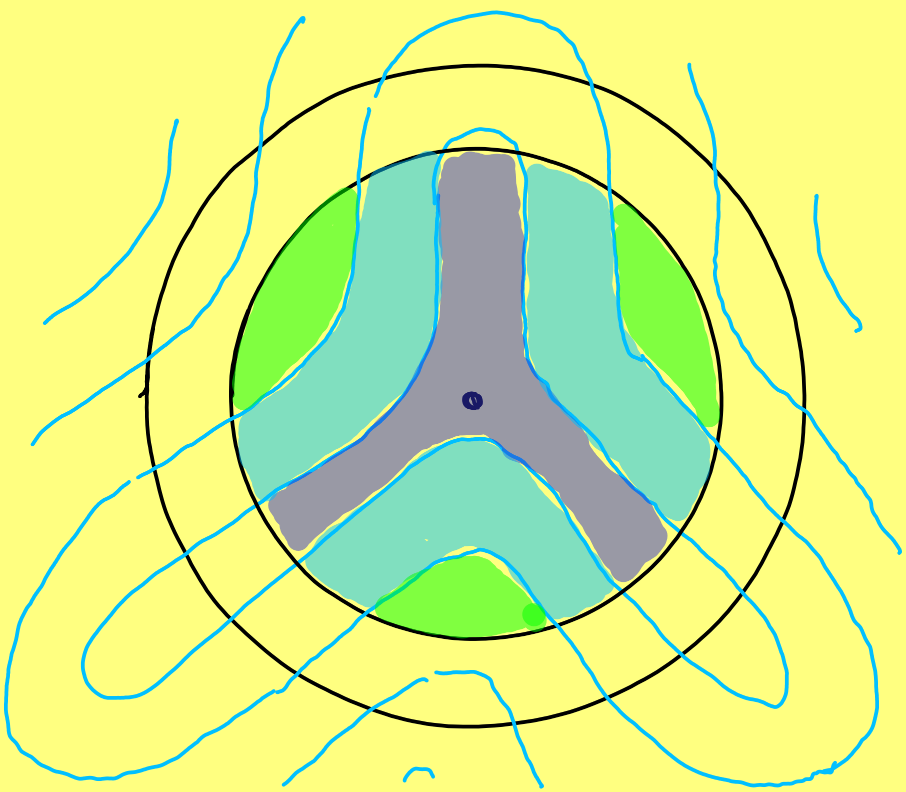
$$d_B\big(\mathrm{Dgm}(d_X|(B_r, \partial B_r)),\ \mathrm{Dgm}(d_X|(B_s, \partial B_s))\big) \leq s - r.$$

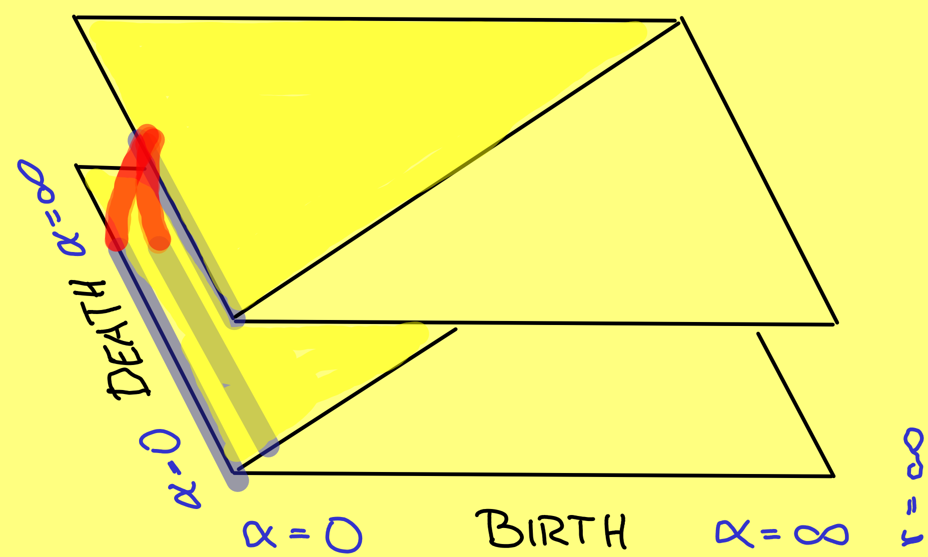
II.5 $(\alpha|r)$-Vineyards





II.5 (α|r)-VINEYARDS





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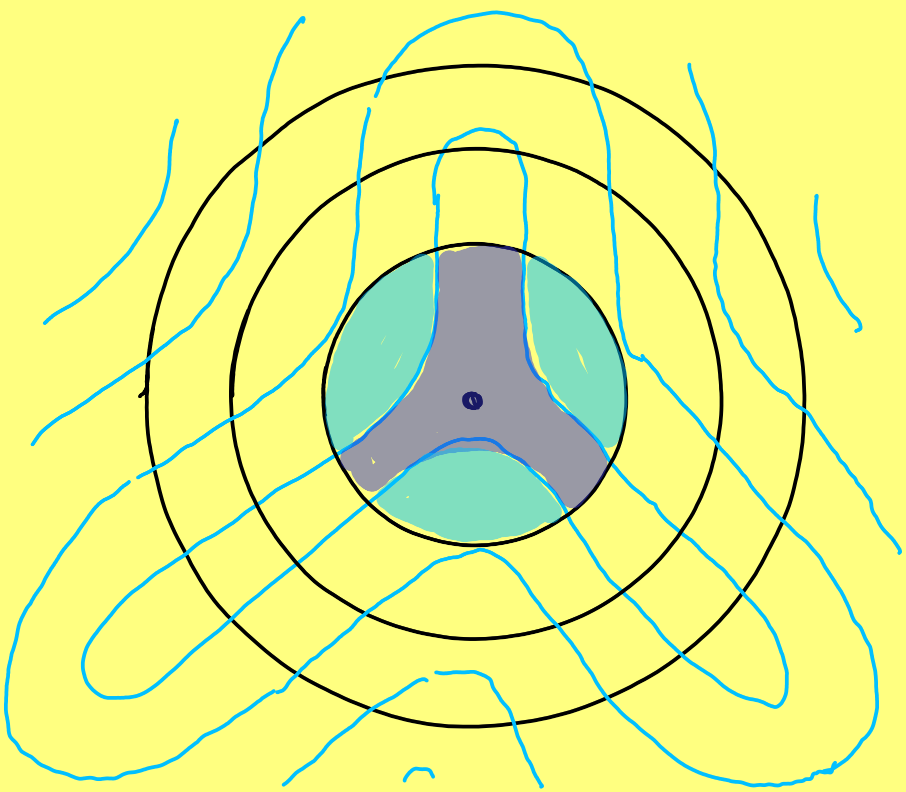

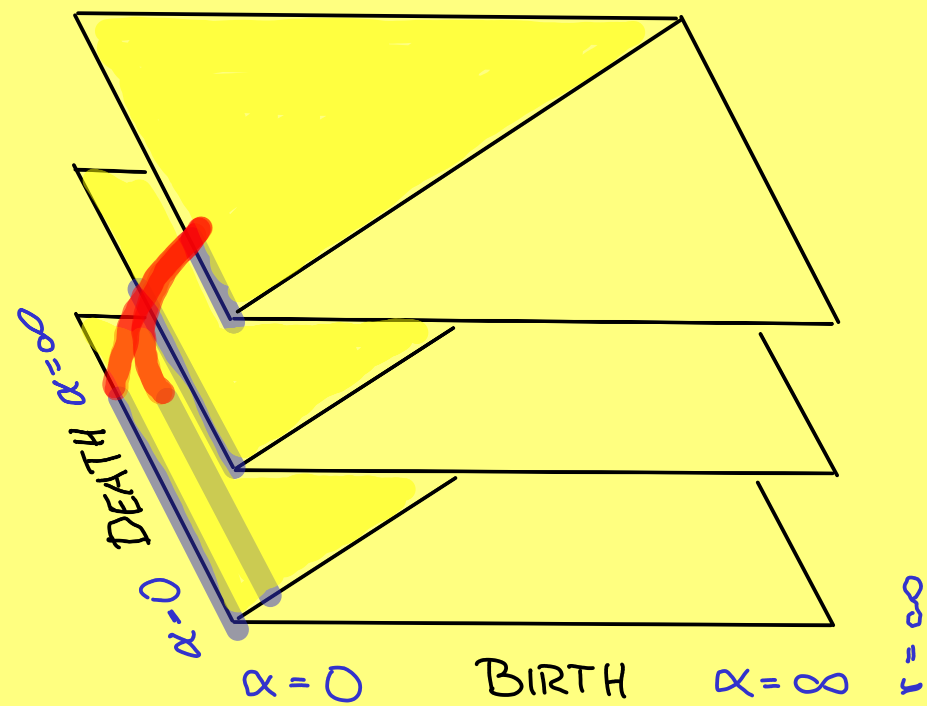

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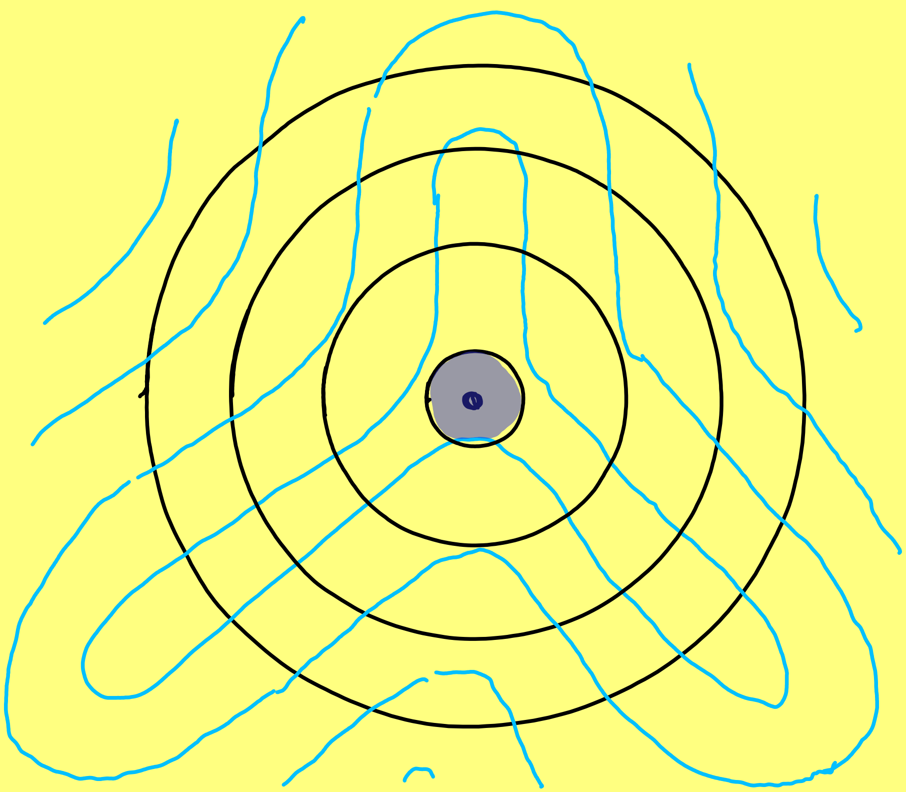
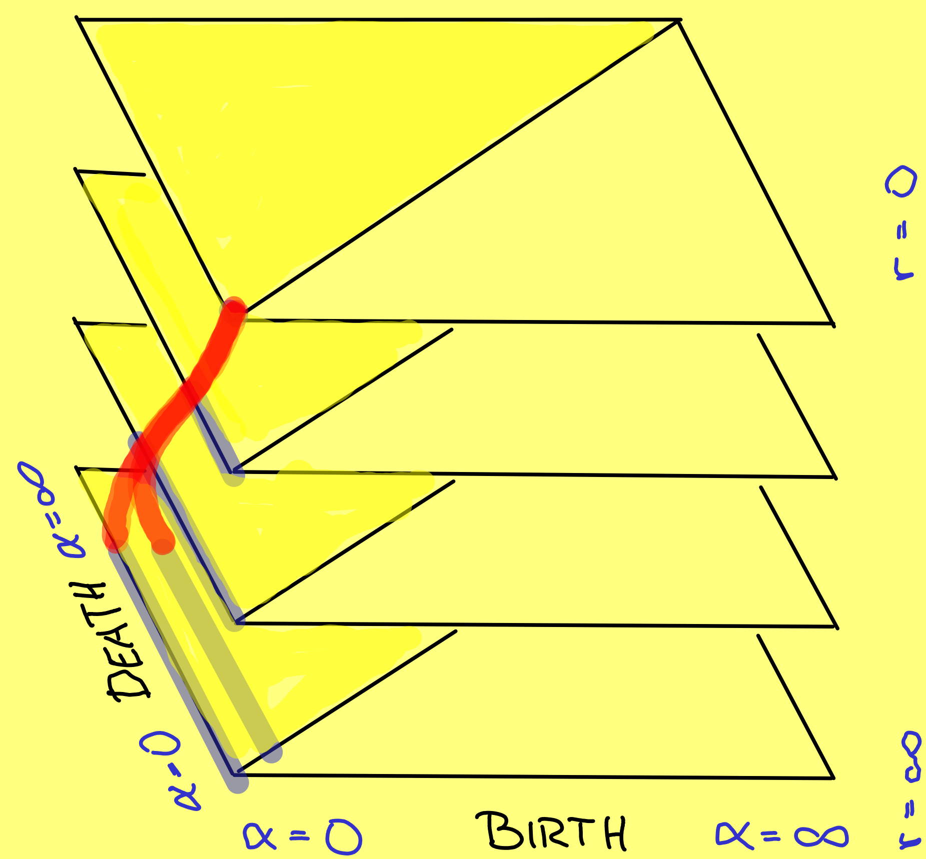
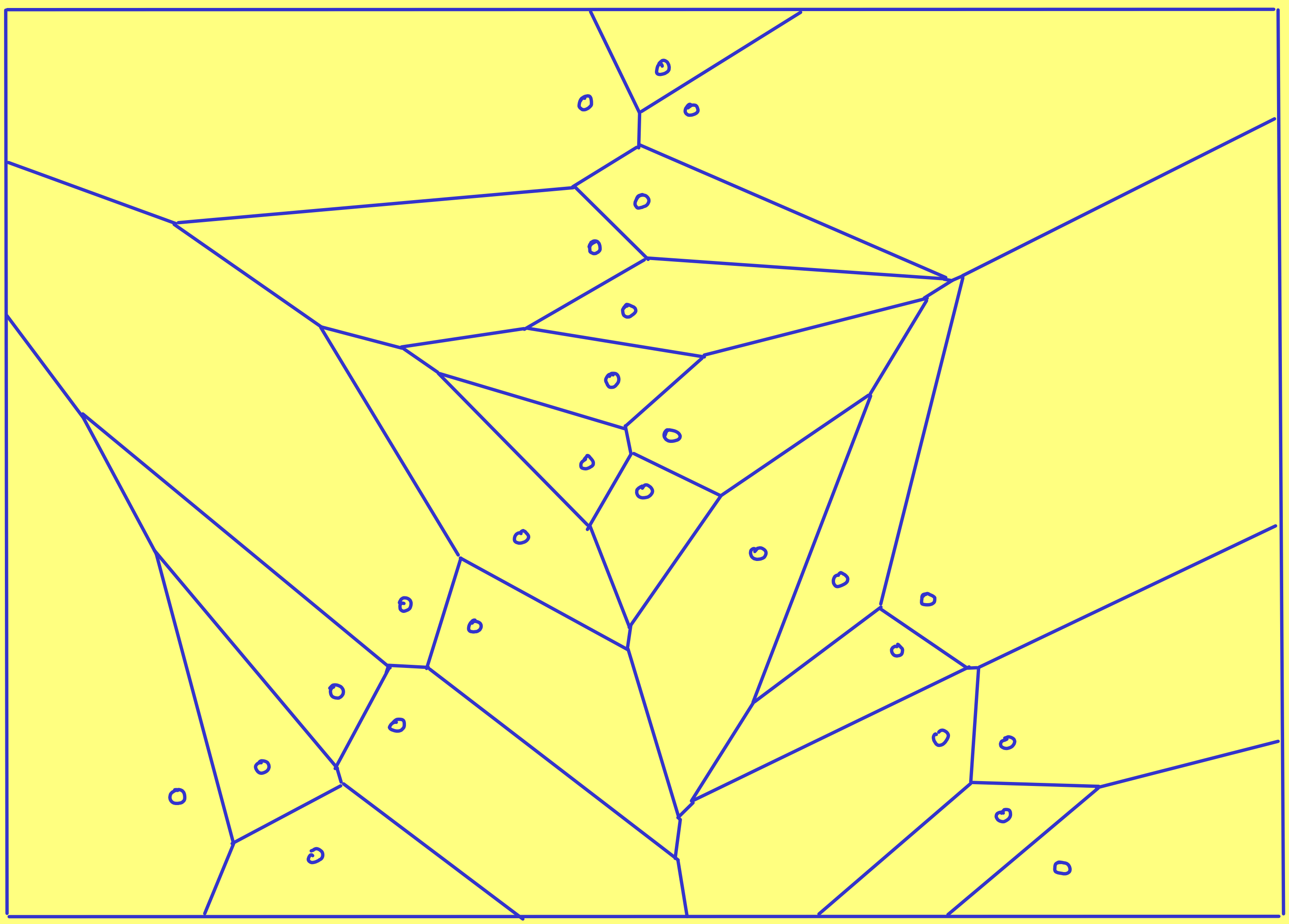




III Complex Pairs

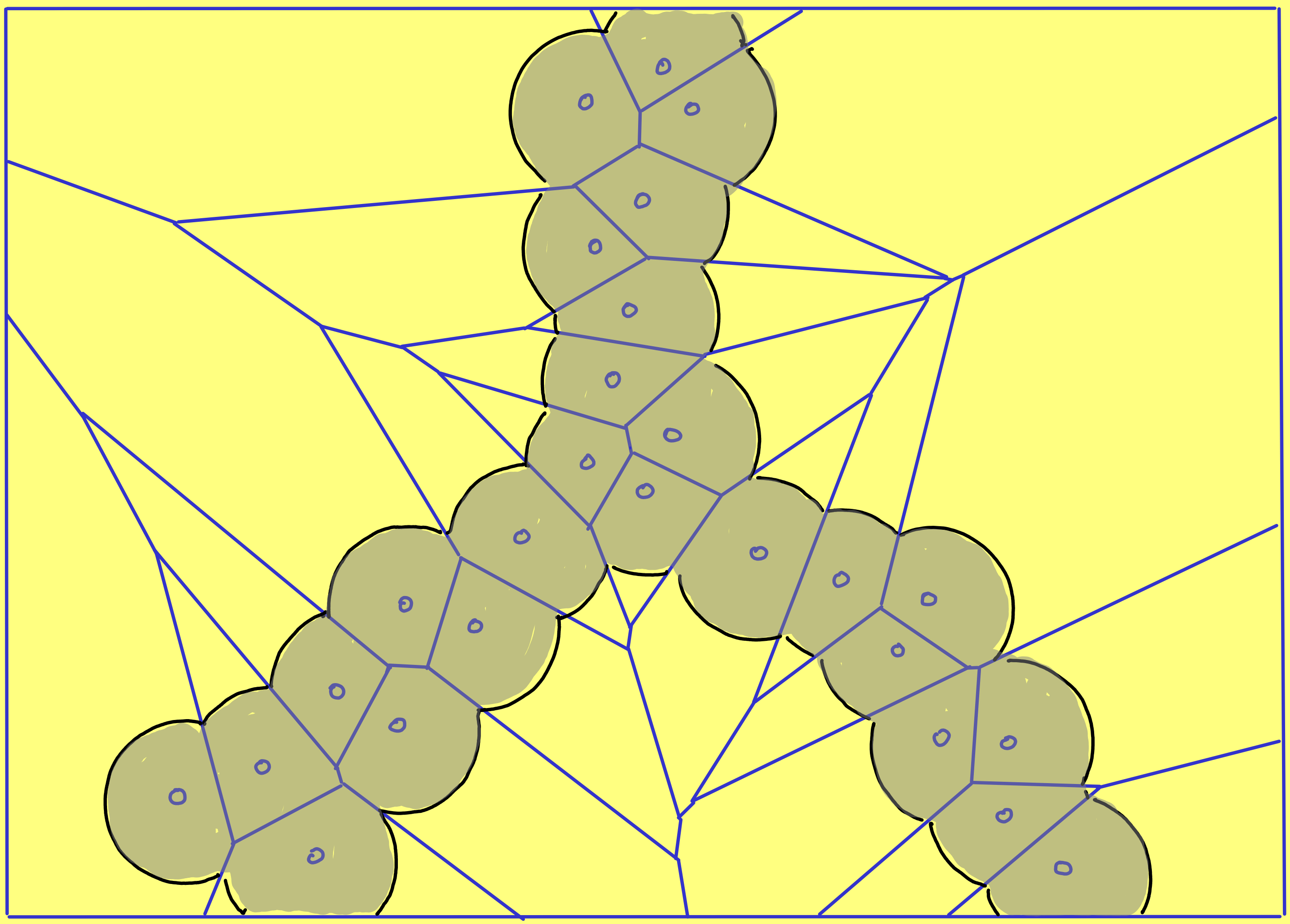
III.1 Voronoi Decomposition

$\mathrm{Vor}(U \mid \mathbb{R}^n)$



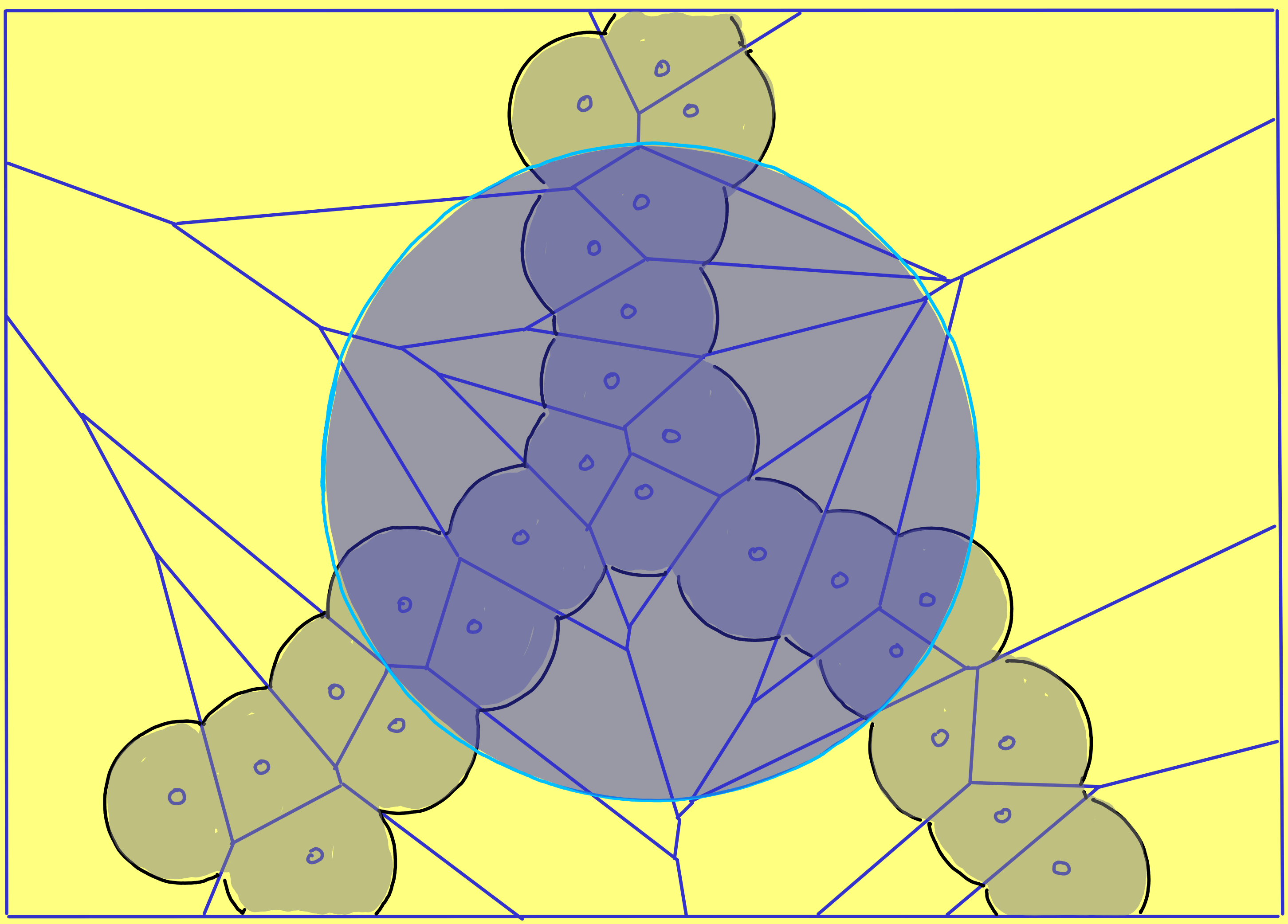
III.1 Voronoi Decomposition

$\mathrm{Vor}(U \mid \mathbb{R}^n)$, U_α

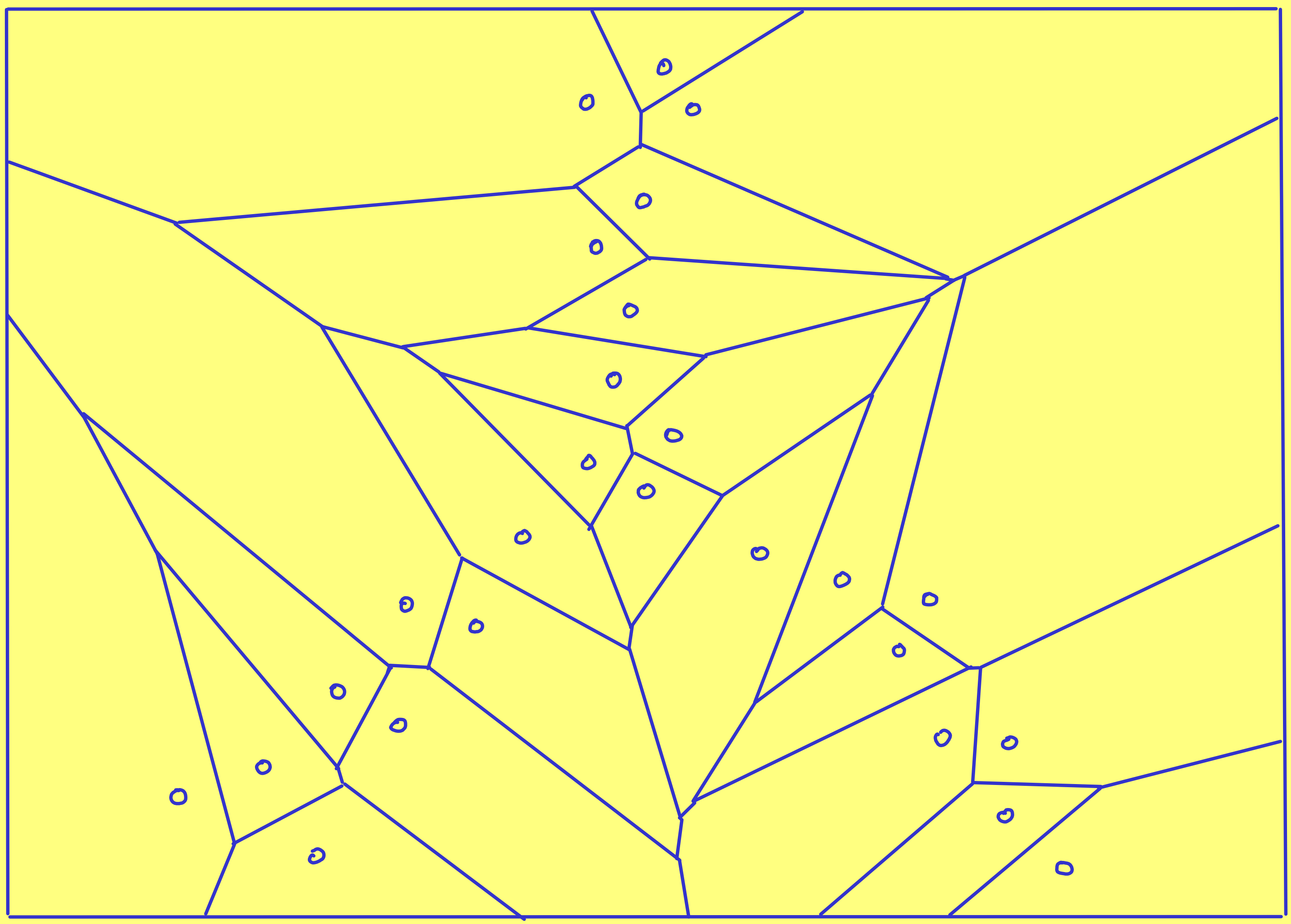


III.1 Voronoi Decomposition

$\mathrm{Vor}(U \mid \mathbb{R}^n)$, U_α, $(B_r, \partial B_r)$

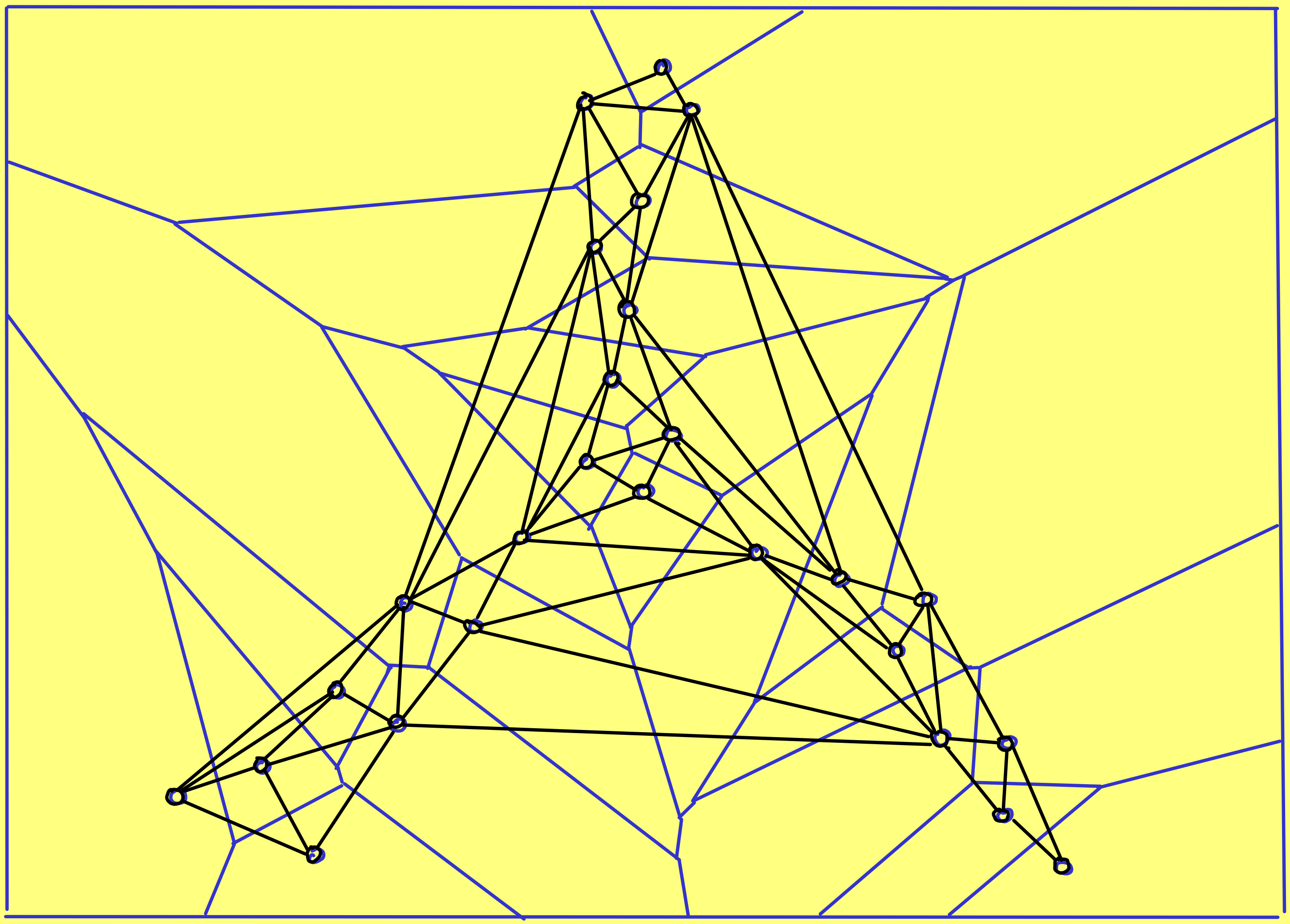


$\mathrm{Vor}(U \mid \mathbb{R}^n)$,

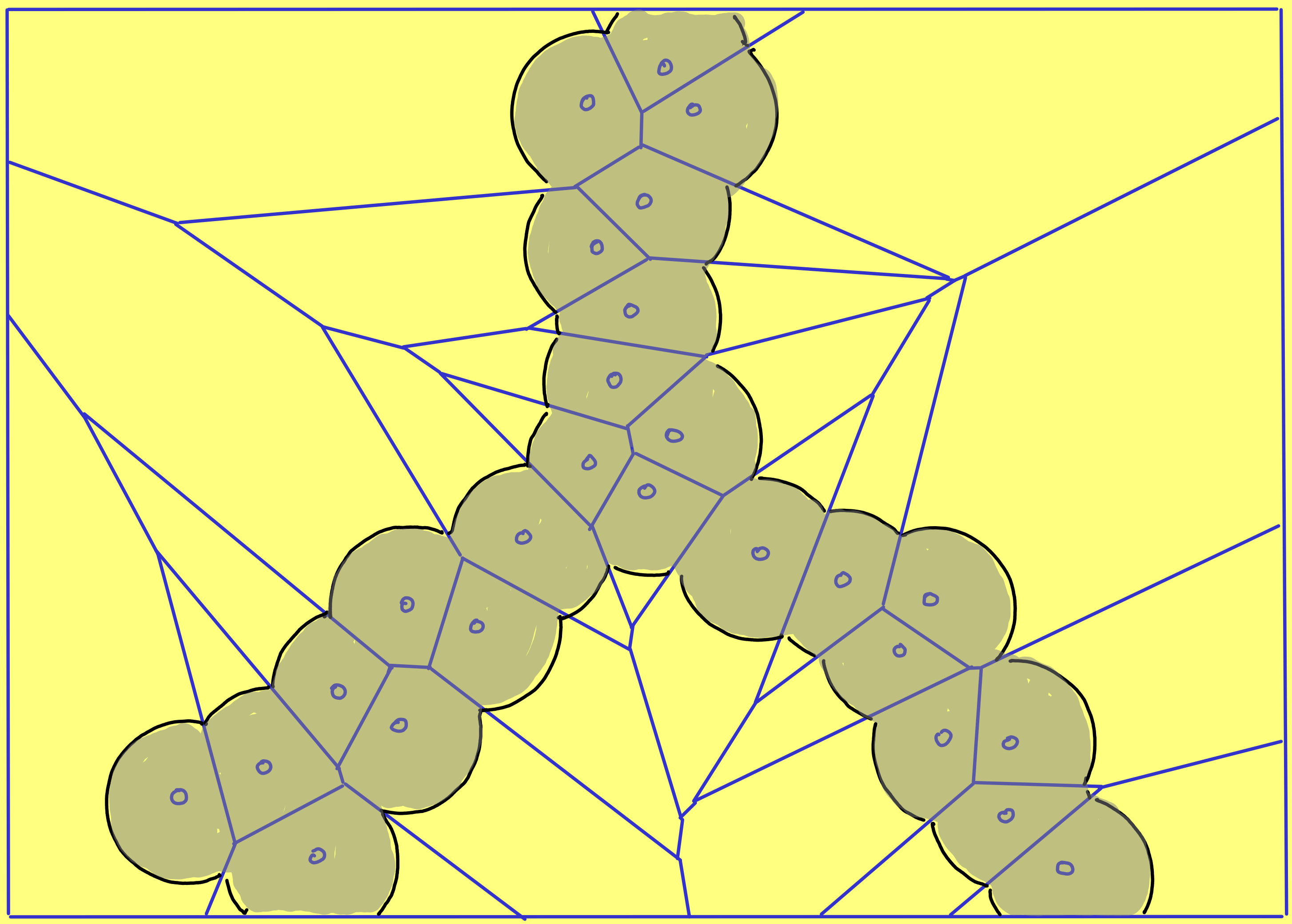


III.2 Delaunay Triangulations

$\mathrm{Vor}(U \mid \mathbb{R}^n)$, $\mathrm{Del}(U \mid \mathbb{R}^n)$

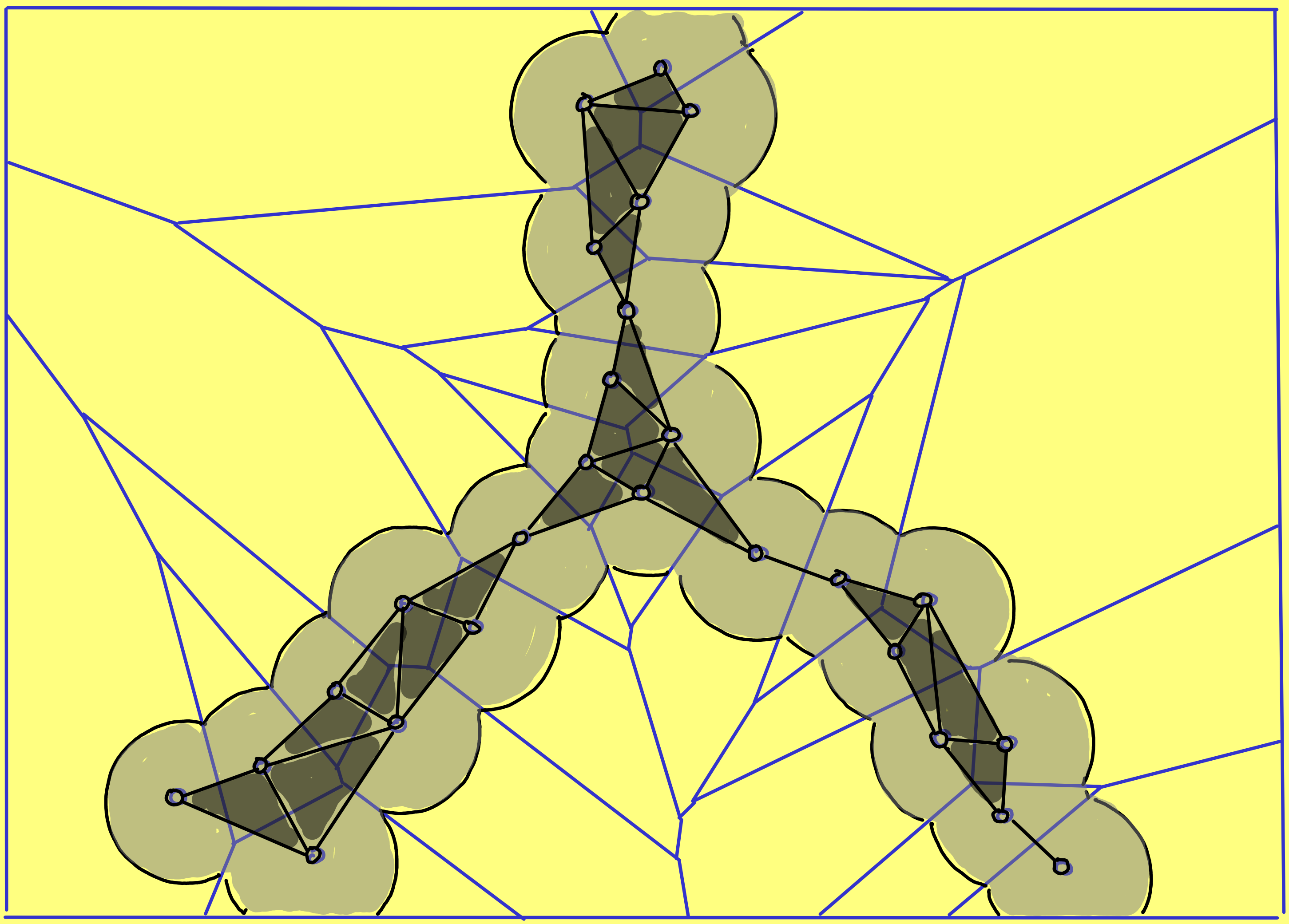


$\mathrm{Vor}(U \mid \mathbb{R}^n)$, U_α

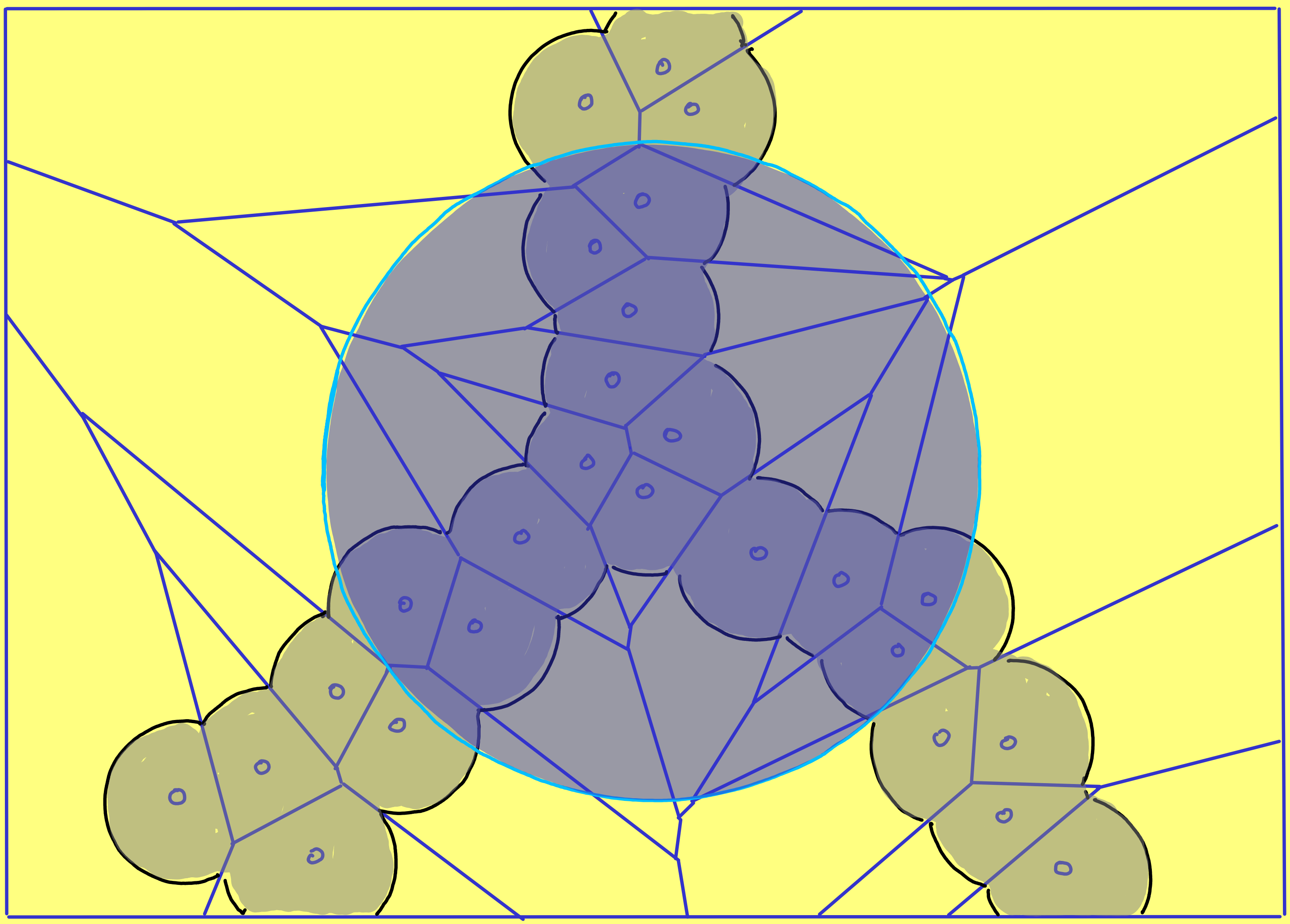


III.2 DELAUNAY TRIANGULATIONS

$\mathrm{Vor}(U \mid \mathbb{R}^n)$, U_α, $\mathrm{Del}(U \mid U_\alpha)$

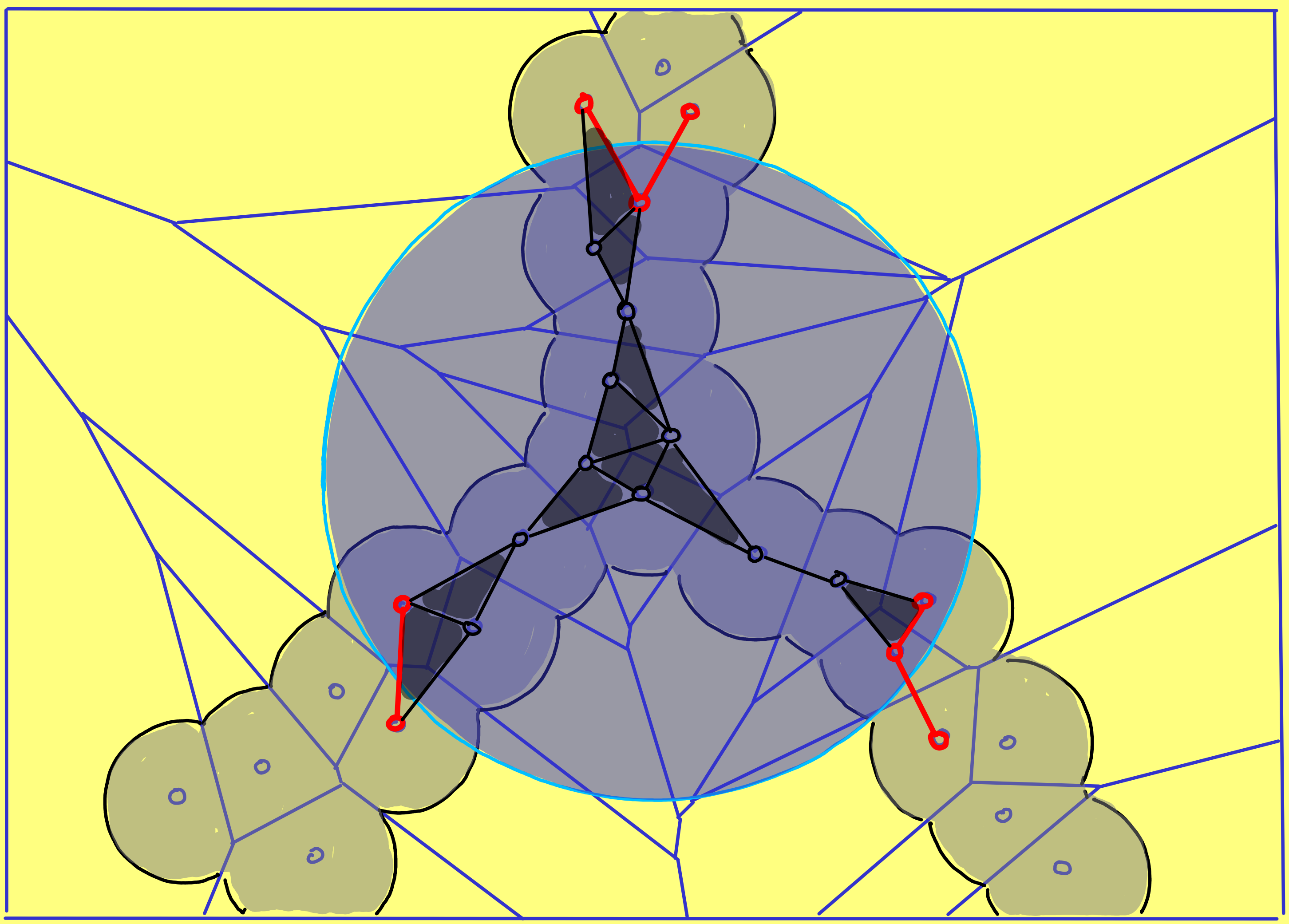


$\mathrm{Vor}(U \mid \mathbb{R}^n),\quad U_\alpha,\quad (B_r, \partial B_r)$



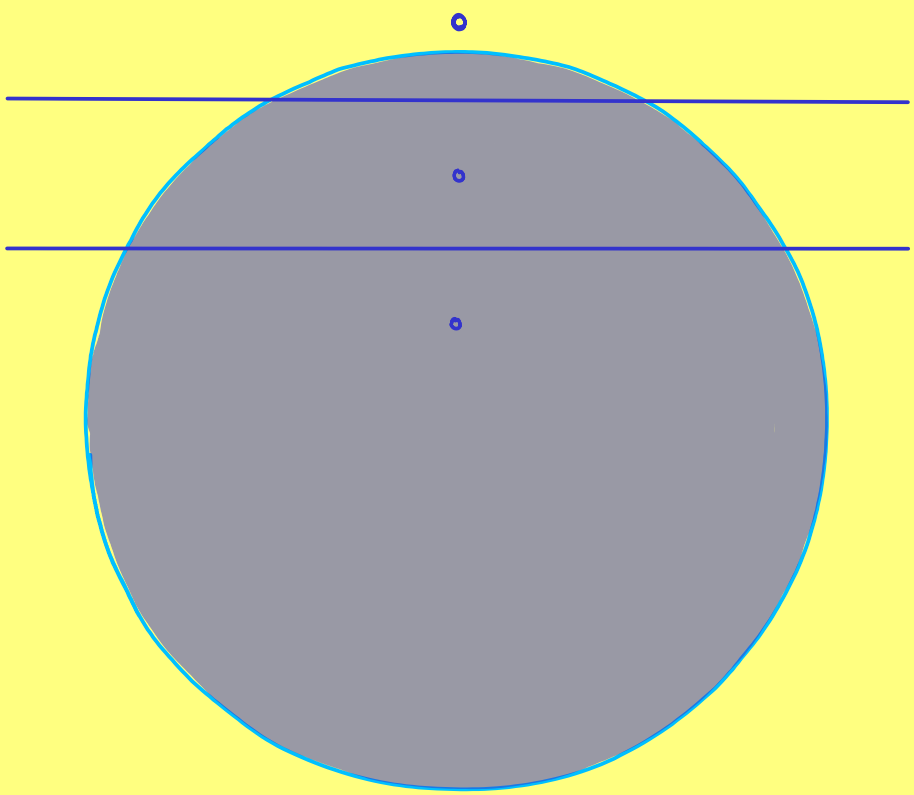
III.2 Delaunay Triangulations

$\mathrm{Del}(U \mid U_\alpha \cap B_r)$, $\mathrm{Del}(U \mid U_\alpha \cap \partial B_r)$



III.2 Delaunay Triangulations

$V(u) \cap \partial B_r$ may be non-simple



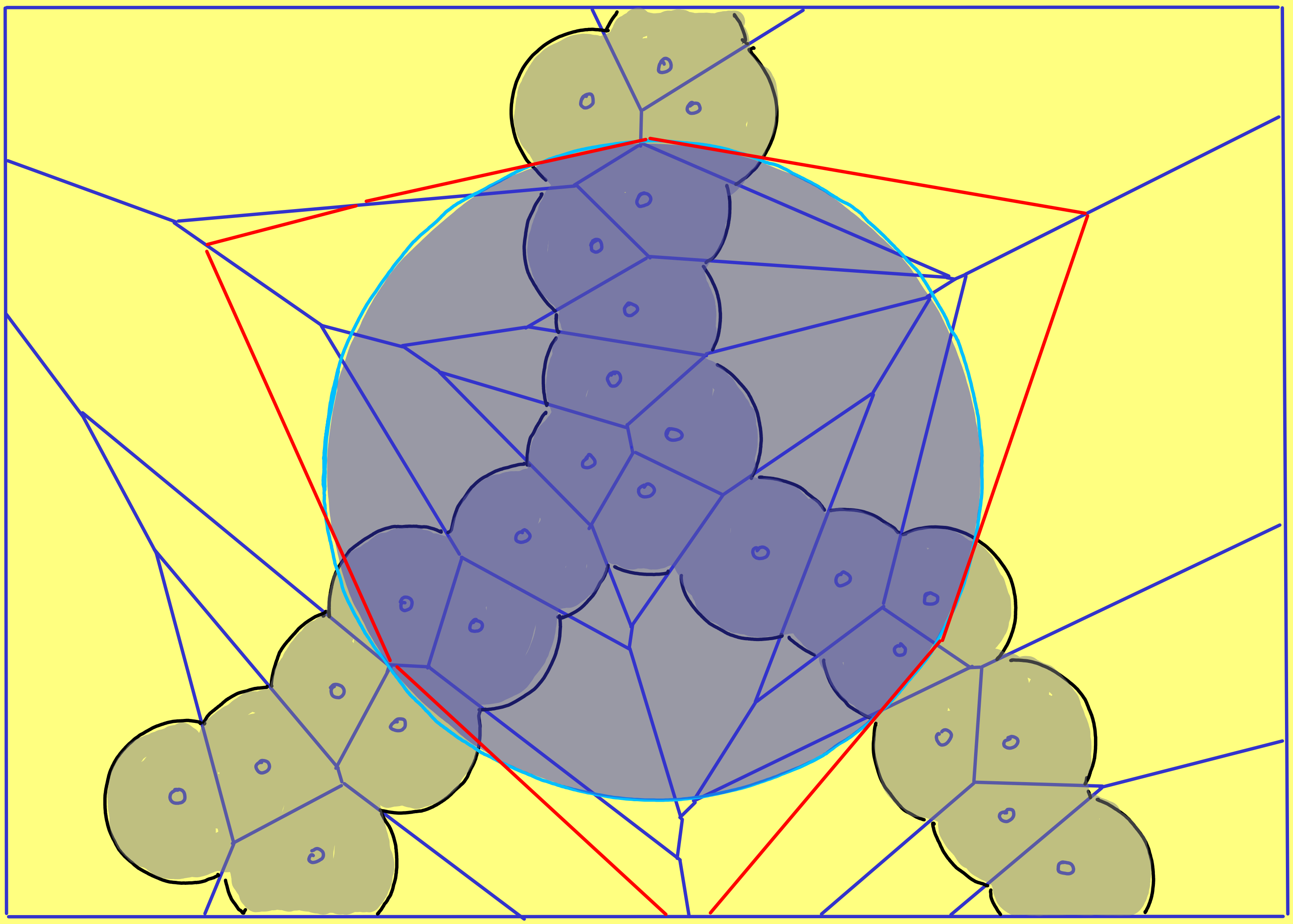
III.3 POWER CELL

$u \in U$ with weight α^2, $\quad z$ with weight r^2

power cell $Z(\alpha) = \{x \mid \|x-z\|^2 - r^2 \leq \|x-u\|^2 - \alpha^2, \ \forall u\}$

III.3 Power Cell

$\mathrm{Vor}(U \mid \mathbb{R}^n),\ U_\alpha,\ (B_r, \partial B_r),\qquad \partial Z(\alpha)$



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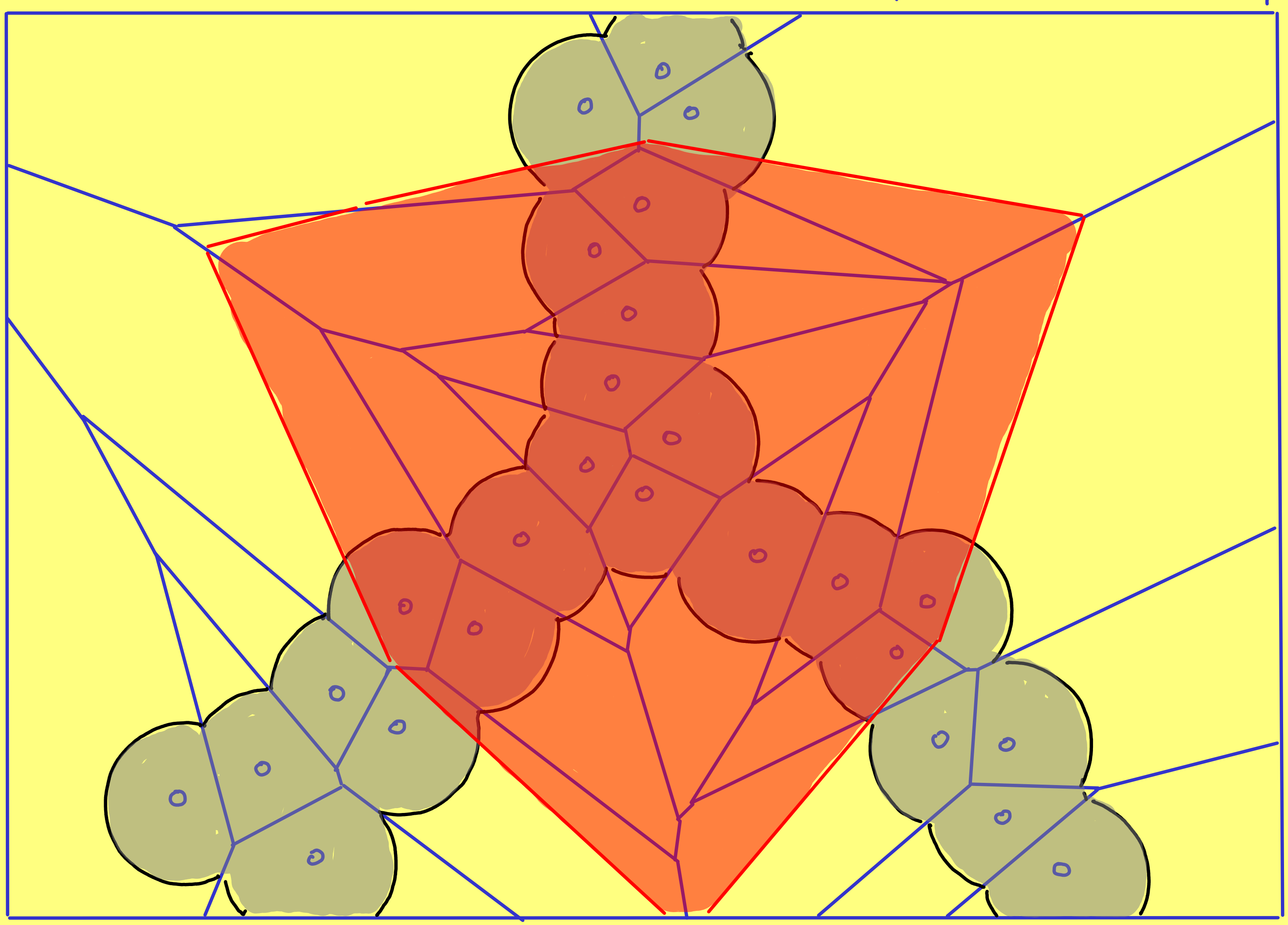
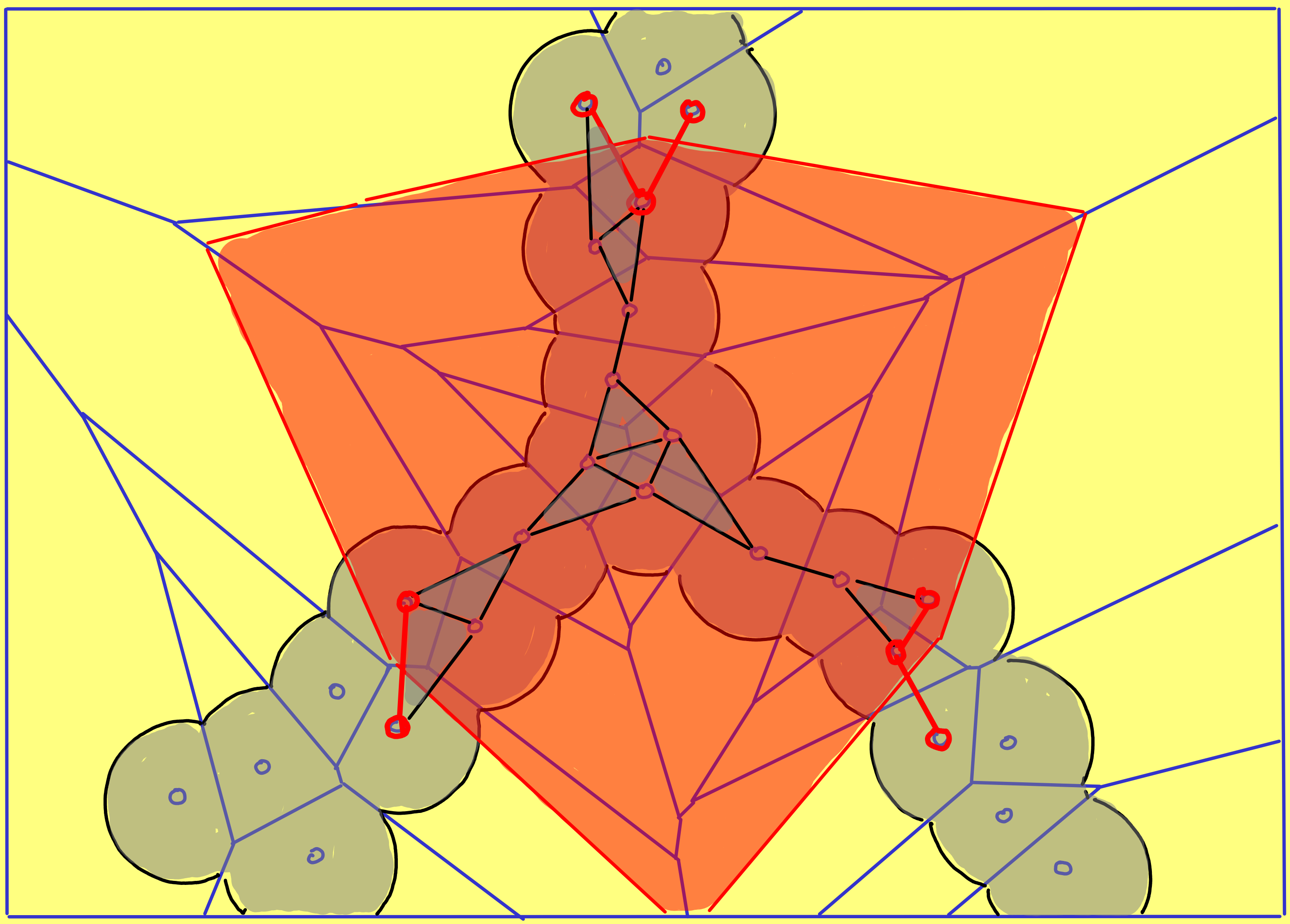


III.3 Power Cell

$\mathrm{Vor}(U \mid \mathbb{R}^n),\ U_\alpha,\ (K(\alpha), L(\alpha)),\ (Z(\alpha), \partial Z(\alpha))$



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III.3 Power Cell

Power Cell Lemma.

$B_r \cap \operatorname{int} Z(\alpha) \neq \emptyset$. Then

$$(U_\alpha \cap B_r,\ U_\alpha \cap \partial B_r) \simeq (U_\alpha \cap Z(\alpha),\ U_\alpha \cap \partial Z(\alpha)).$$

IV Computation

IV.1 Union Complexes

$K(\alpha)$ and $L(\alpha)$ change non-monotonically

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union complexes change monotonically:

$$K[0,\alpha] = \bigcup_{0 \le t \le \alpha} K(t)$$

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UNION COMPLEX LEMMA.

$$(K[0,\alpha],\ L[0,\alpha]) \simeq (K(\alpha),\ L(\alpha))$$

IV.2 PERSISTENCE DIAGRAMS

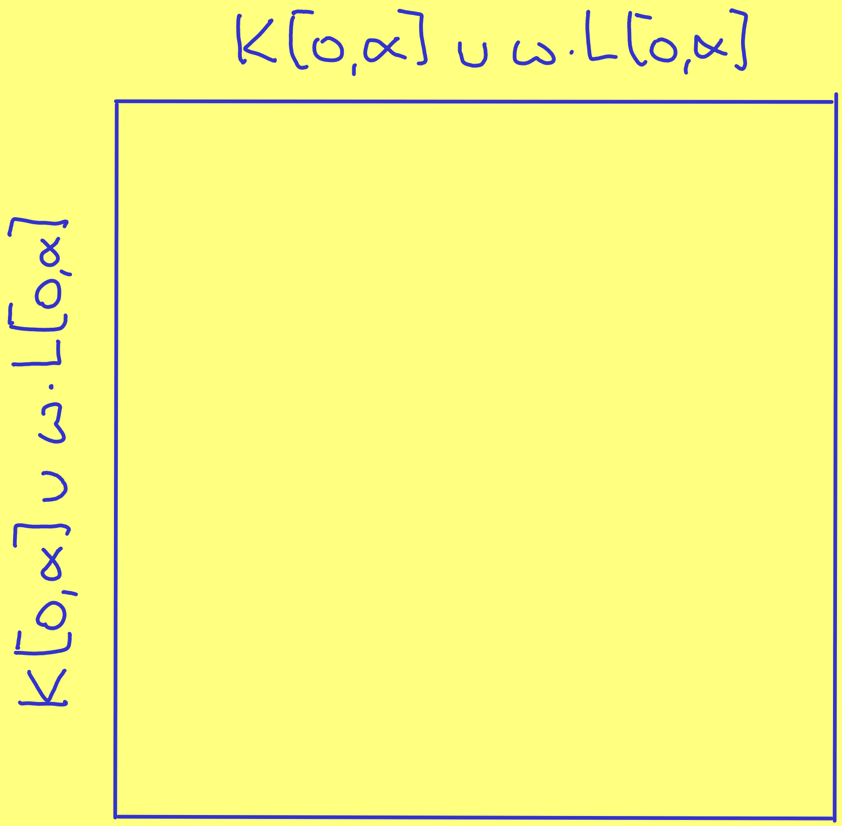
$\ldots \to H(K[0,\alpha], L[0,\alpha]) \to \ldots$ for $0 \le \alpha \le \infty$.

compatible ordering of the simplices

IV.2 Persistence Diagrams

$$\ldots \to H(K[0,\alpha], L[0,\alpha]) \to \ldots \quad \text{for } 0 \le \alpha \le \infty.$$

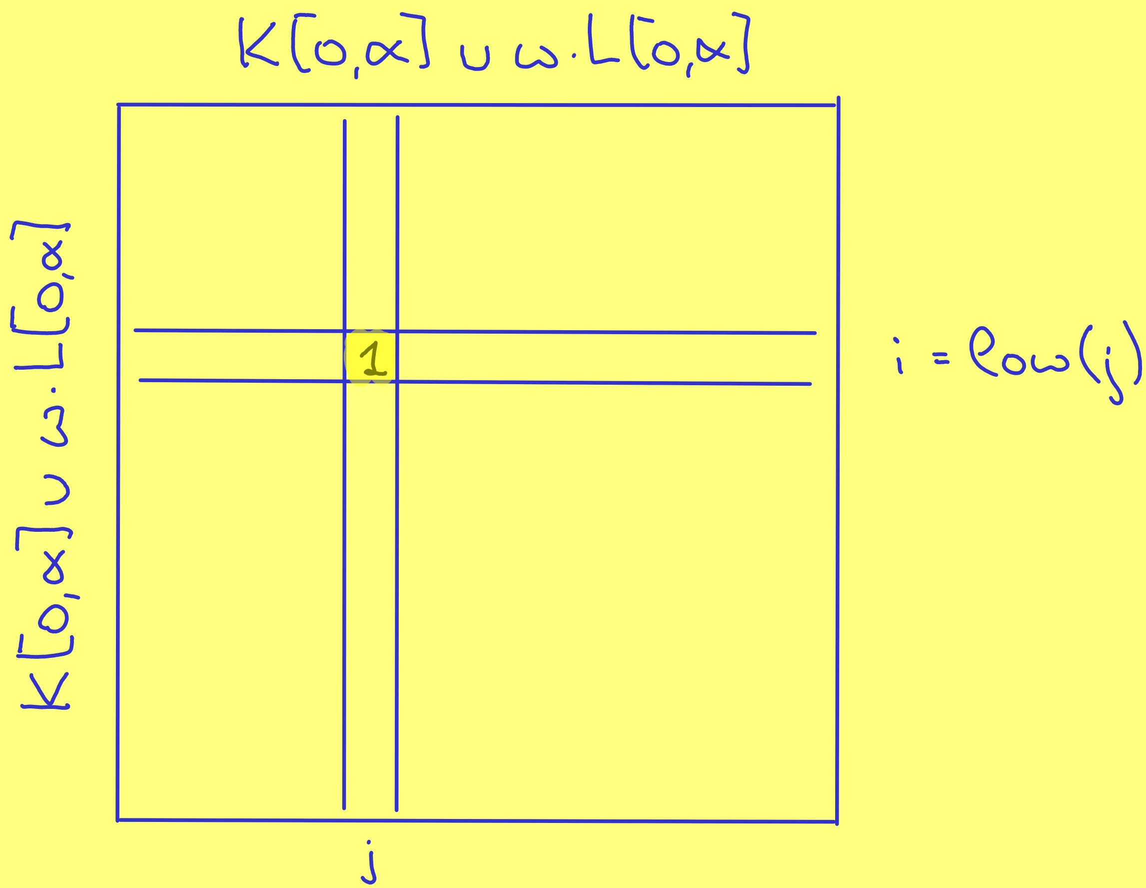
compatible ordering of the simplices



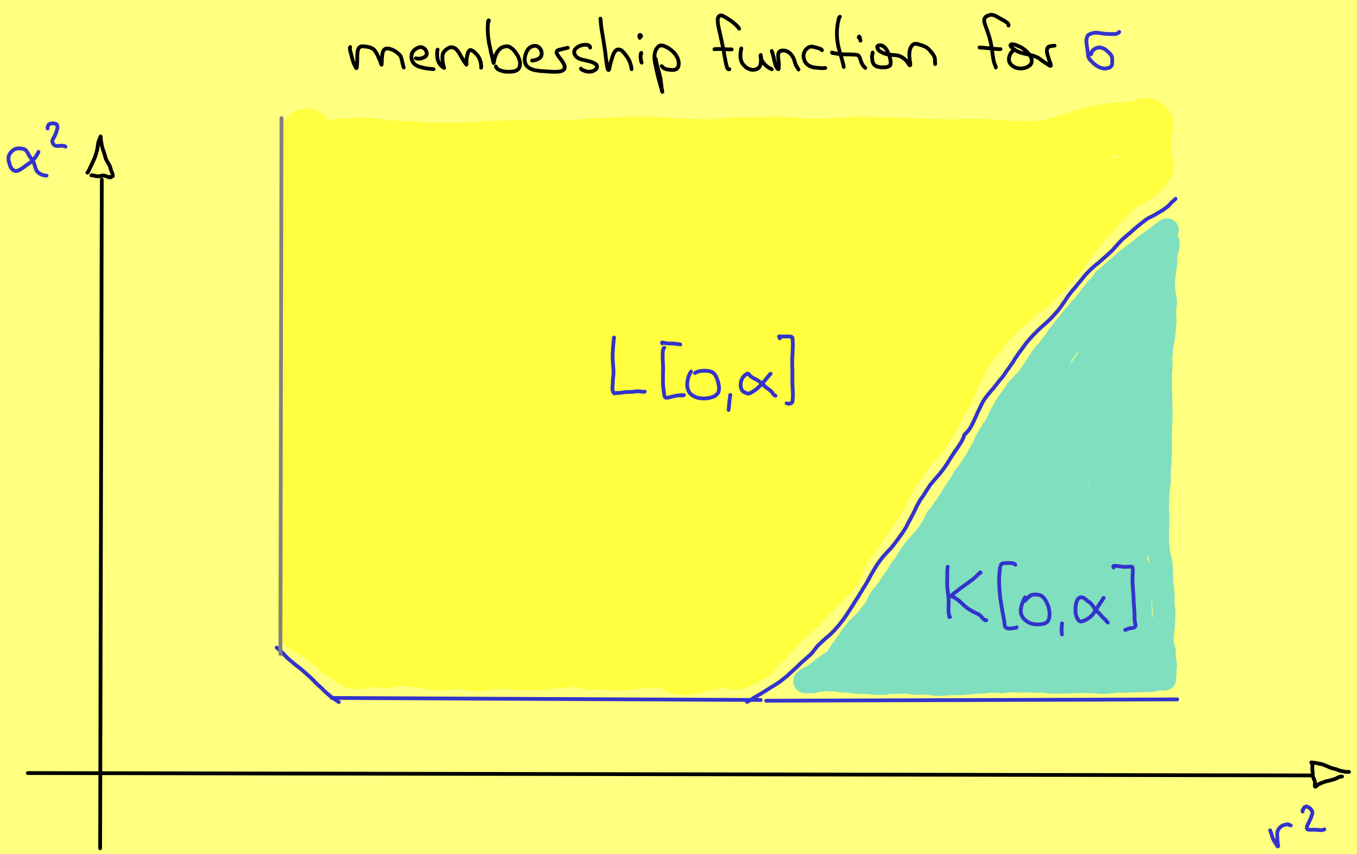
IV.2 Persistence Diagrams

$\ldots \to H(K[0,\alpha], L[0,\alpha]) \to \ldots$ for $0 \le \alpha \le \infty$.

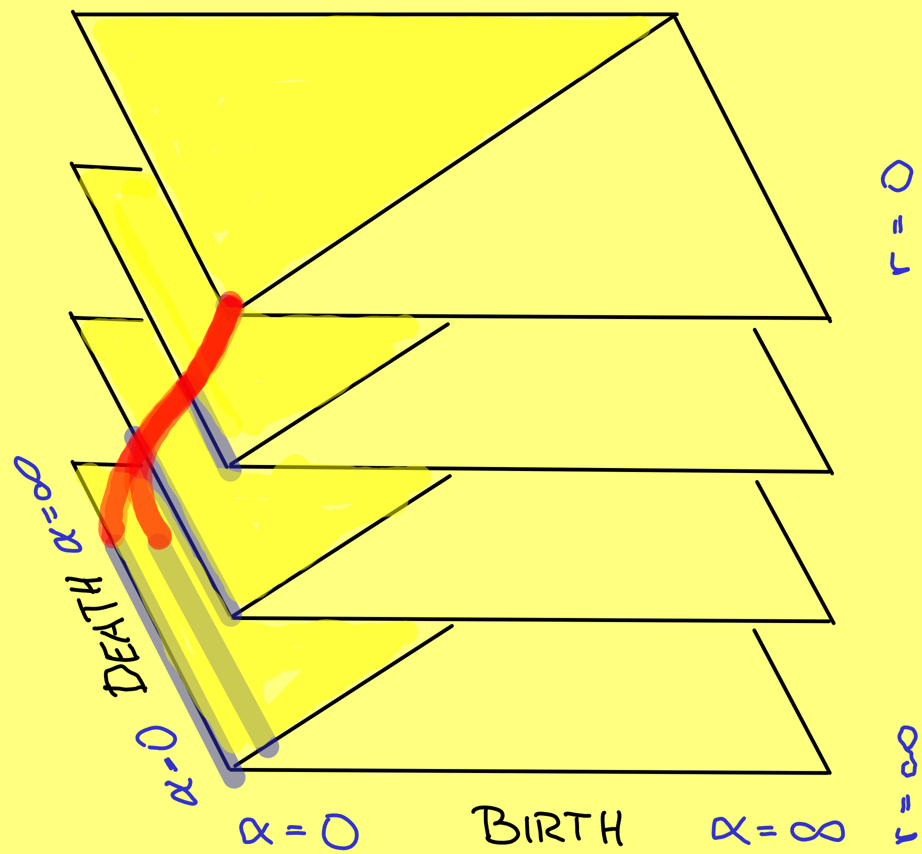
compatible ordering of the simplices



IV.3 VINEYARD



IV.3 VINEYARD



const $\cdot N^2$ transpositions

$O(N^3)$ time for $\mathrm{Vnrd}(d_U \mid d_Z)$.

Thank You