From symmetric tilings of 2D hyperbolic space to 3D euclidean crystalline patterns: EPINET

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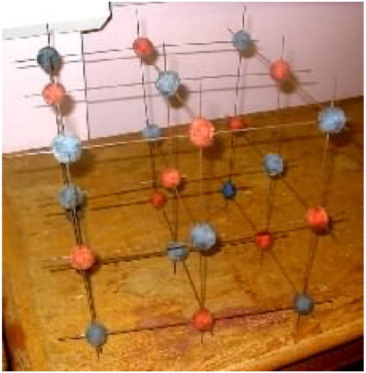
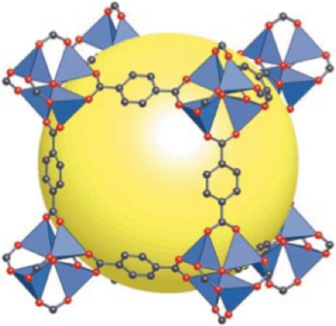
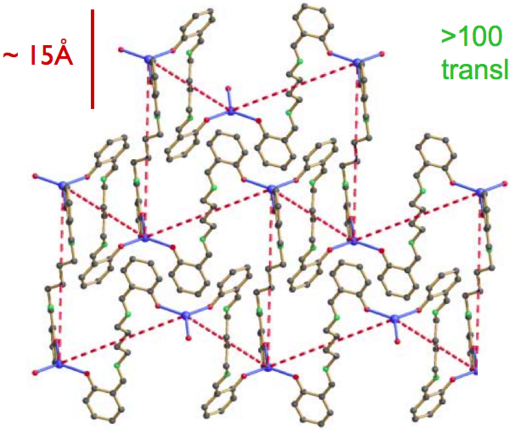
Nets model structure



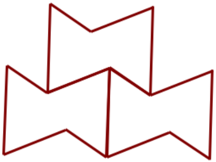
Crum-Brown (1883) NaCl

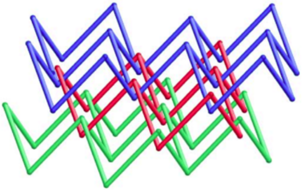


Yaghi, O'Keeffe (2003)
metal-organic frameworks



>100 atoms per translational cell





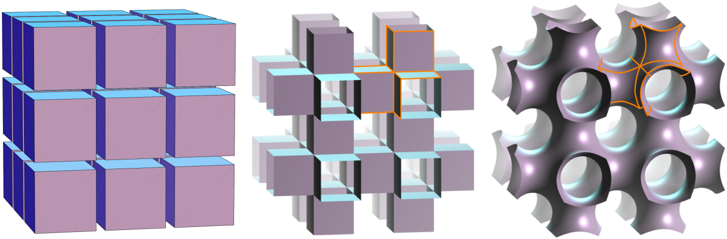
$[Ag_2(H_2L)_3](NO_3)_2$ and $[Ag_2(H_2L)_3](ClO_4)_2$
H_2L = N, N'-bis(salicylidene)-1,4-di-aminobutane

Chen et al, 1999

Triply Periodic Minimal Surfaces.

We use TPMS as scaffolds for 3-periodic nets.

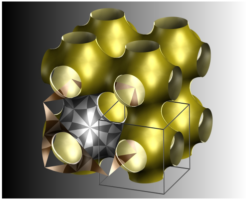
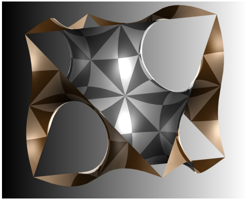
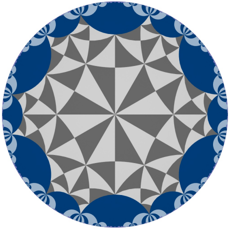
Example: the primitive cubic net is carried by



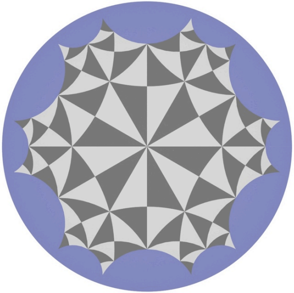
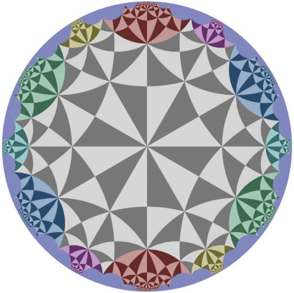
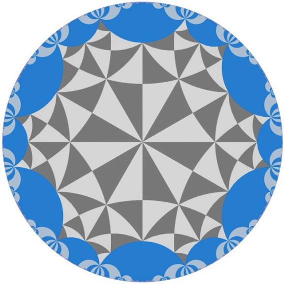
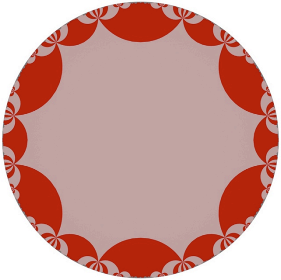
- A 3D tiling by cubes.
- The edges of an infinite polyhedron.
- A 2D tiling of Schwarz's Primitive (P) surface.

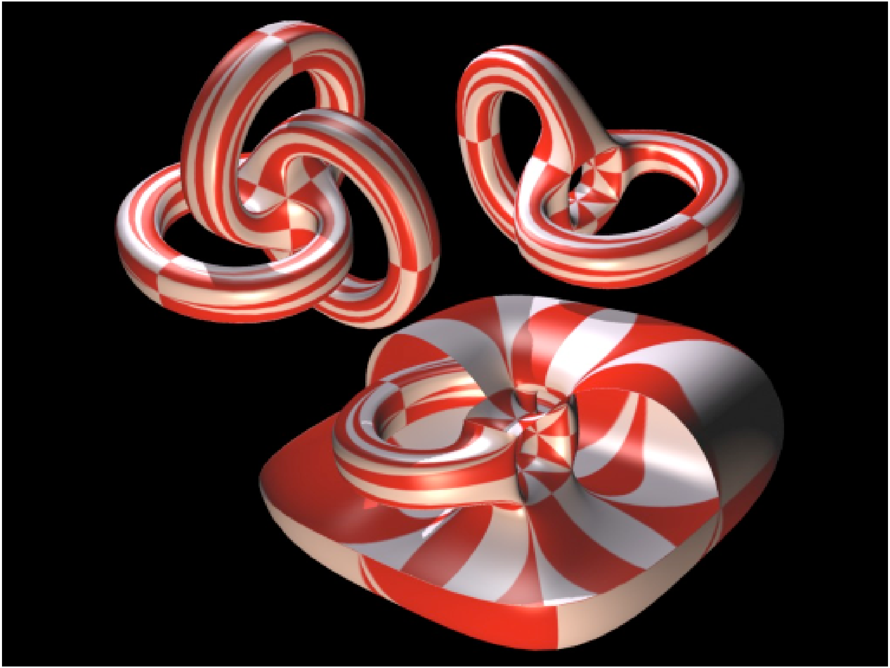
Hyperbolic geometry

The intrinsic geometry of a TPMS is hyperbolic.

The asymmetric unit of the P surface is a hyperbolic triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}$. The primitive translational unit cell is a dodecagon. With opposite sides identified, the dodecagon glues up into a genus-3 surface.

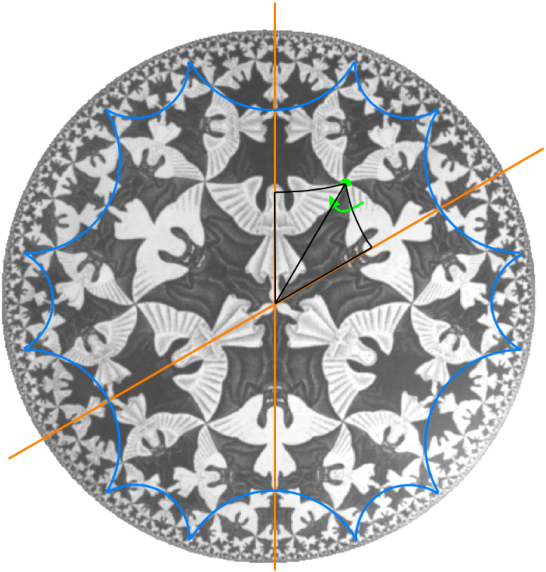







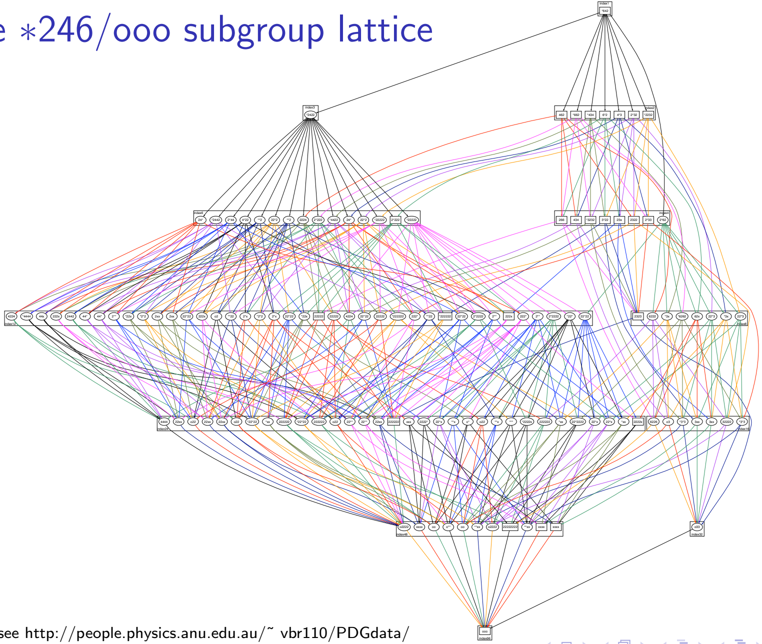

... play Stu's Escher animation

Symmetries of the hyperbolic plane

Reflection
Rotation
Translation



The ∗246/ooo subgroup lattice



see http://people.physics.anu.edu.au/˜ vbr110/PDGdata/

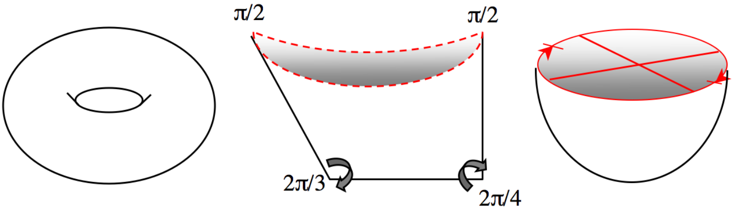
131 subgroups of $*246$ that preserve the dodecagon (ooo)

ooo	o22	22*x	3xx	*2*2	44x	2*222	434
22222222	22xx	22*2222	222x	*22x	2xx	*22222	2*33
xxxx	22**	22*x	2222	**22	4224	*22222	2322
xxxx	o22	***	*222222	222*	4224	*2442	*3232
xxxx	22**	**x	2xx	2*x	2**	2*222	266
o**	2222*	*xx	2*2222	2**	22*22	**2	23x
o2222	o22	*xx	222x	44*	2*2222	4*22	2*62
oo	*22*22	22*x	2**	22*22	*22*	*4422	*2422
**xx	22xx	o22	**22	22222	2323	2*44	*2232
**xx	222222	222222	o2	22222	6222	24*	*434
o2222	*22*22	*xx	44*	2442	62x	24*	6*2
oo	22xx	3xx	2*x	222x	*3x	22*2	*662
o33	o*	*3*3	*4444	*2*2	*6262	**2	4*3
222222	xxx	*3*3	222*	22*22	22*3	22*2	2*32
2222x	22xx	32222	22*22	*22x	*3x	2224	462
222222	**x	6226	22*22	*222222	22*3	3*22	*642
4444	*2222x	o3					

Orbifold symbols

An orbifold is the quotient of a manifold by a discrete group acting on it. Orbifolds derived from 2D manifolds of constant curvature (the sphere, Euclidean plane, and hyperbolic plane) have a canonical symbol encoding their topology and orders of rotational points. Conway's notation for this is:

$$\mathrm{ooo}\ldots c_1 c_2 c_3 \ldots * m_1 m_2 \ldots [*m_4 m_5 \ldots] \ldots \times \times\times$$



o

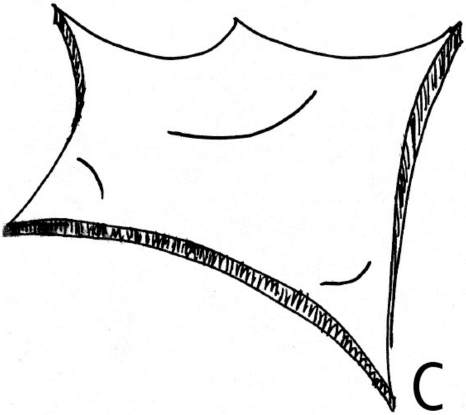
$34 * 22$

$\times$

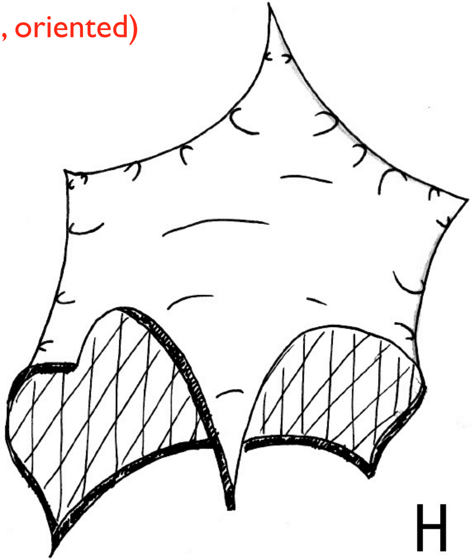
Divide orbifolds into 8 classes:

*	Coxeter	*simply connected*
	Hat	
-	Stellate	
x	Projective	
**	Annulus	*multiply connected*
*x	Möbius	
o	Torus	
xx	Klein	

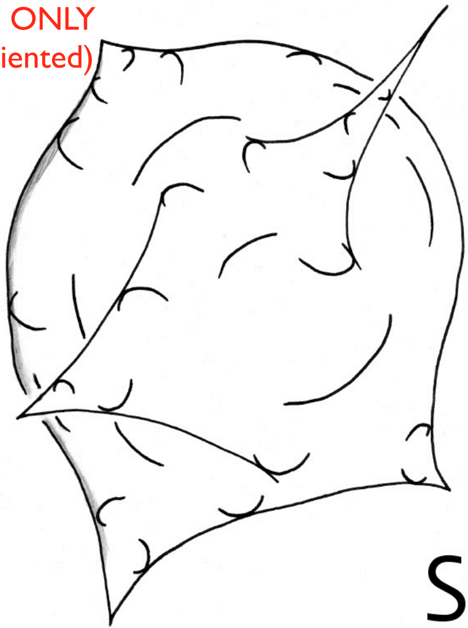
MIRRORS ONLY
(simple, bounded, oriented)



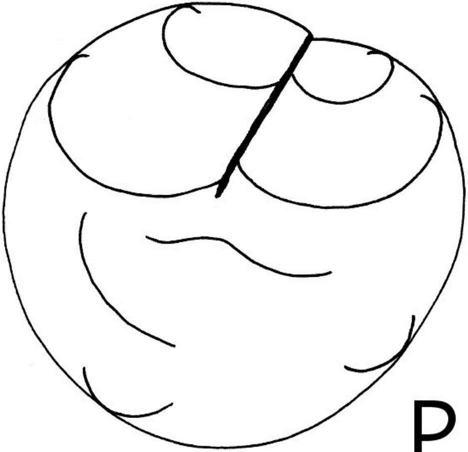
MIRRORS, ROTATION CENTRES
(simple, bounded, oriented)



ROTATION CENTRES ONLY
(simple, unbounded, oriented)

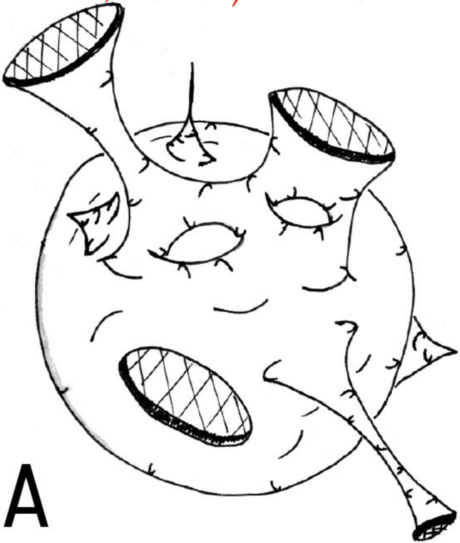


SINGLE CROSS-CAP
(simple, unbounded, non-oriented)

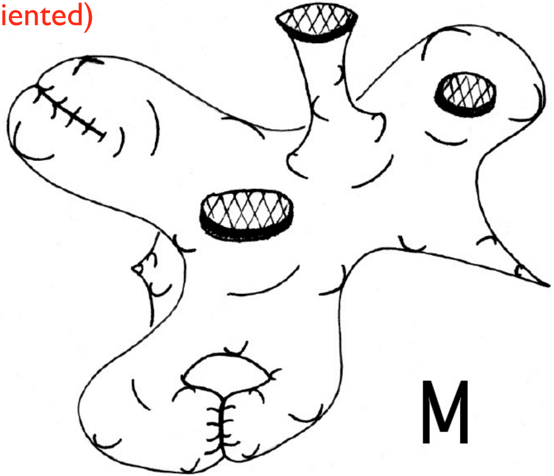


P

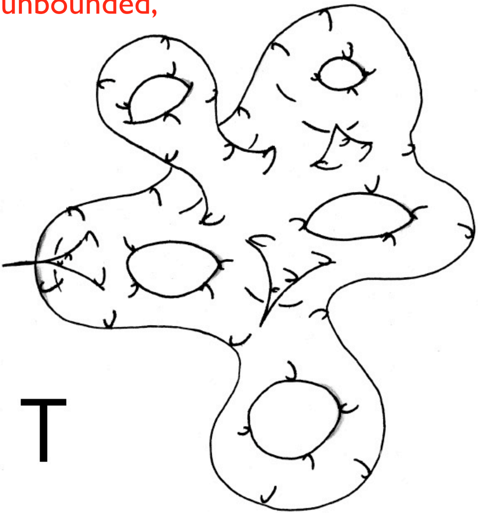
DISJOINT MIRRORS
(multiply-connected, bounded, oriented)



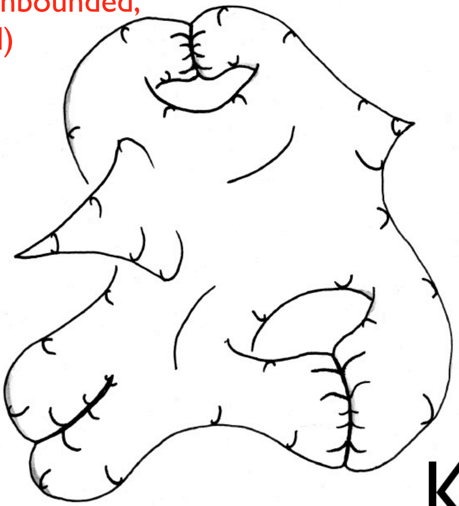
MIRRORS, CROSS-CAPS (multiply-connected, bounded, non-oriented)



TRANSLATIONS ONLY (multiply-connected, unbounded, oriented)



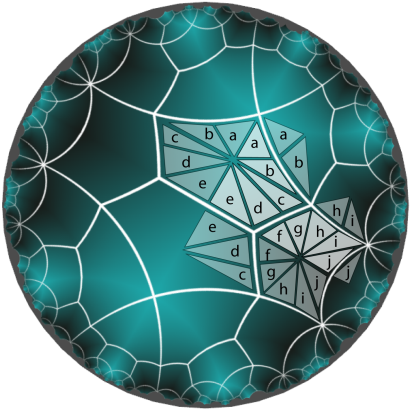
MULTIPLE CROSS-CAPS (multiply-connected, unbounded, non-oriented)



K

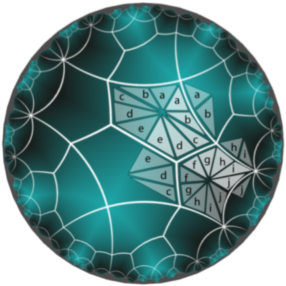
Delaney-Dress tiling theory

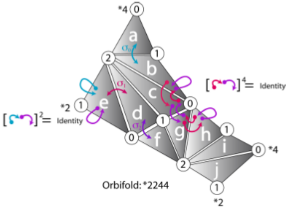
An algorithmic approach to encoding the symmetries and topology of periodic tilings, using triangulations of orbifolds. Developed by Dress, Huson, Delgado-Friedrichs, mid 1980's to mid 1990's.

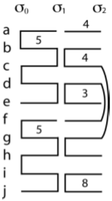


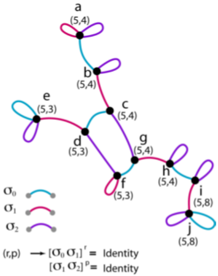
See O.D.-F. (2003) Theoret. Comp. Sci. 303:431-445

Delaney-Dress symbols

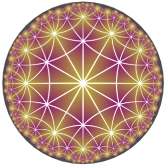


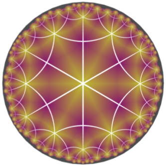


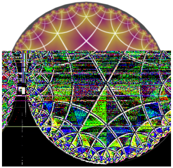


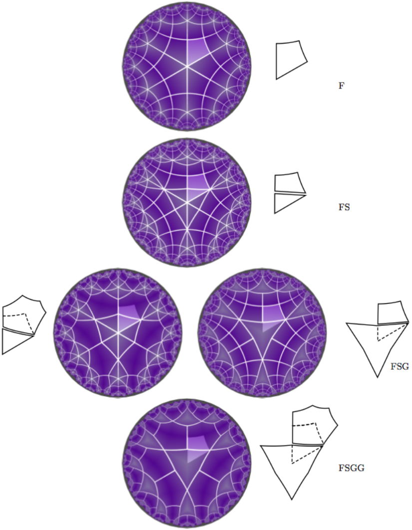


Enumeration of D-symbols via splits and glues



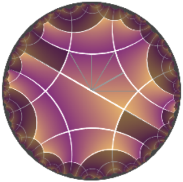




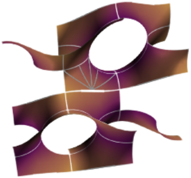


Embedding D-symbols

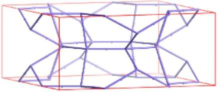
To get a tiling of $\mathbb{H}^2$ that is compatible with the surface covering map we must match the combinatorics of the D-symbol to the geometry of a specific group of isometries. There may be more than one way to do this. e.g. Two distinct $*2^5$ subgroups of $*246$:



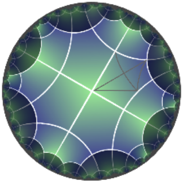
UQC73



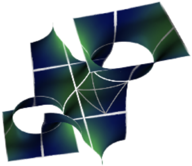
EPC73



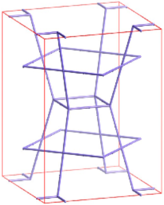
sqc3695



UQC72



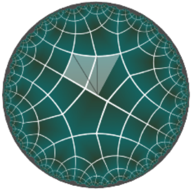
EPC72



sqc3865

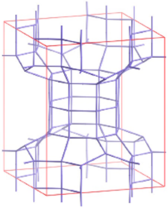
Embedding D-symbols

Automorphisms of the D-symbol that are not $*246$ isometries. e.g. sides of different length in $*2224$:

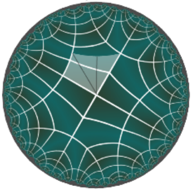


UQC5737

EPC5737

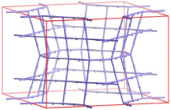


sqc11593



UQC5736

EPC5736



sqc11592

EPINET — http://epinet.anu.edu.au

Results from our enumeration of tilings and nets derived from Coxeter orbifolds are available online.

EPINET

Home | Glossary

Euclidean Patterns in Non-Euclidean Tilings

Structures | Search

Welcome to the EPINET project

The EPINET project explores 2D hyperbolic (H^2) *tilings* as a source of crystalline frameworks (or *networks*) in 3D euclidean (E^3) space. Our aim is to enumerate networks with a broad spectrum of properties that are of possible interest to geometers, structural chemists, and statistical physicists. The guiding principal is one of *hyperbolic surface tiling*, where the 3D crystallinity of an underlying surface induces 3–periodic networks. The extraordinary wealth of hyperbolic tilings allows us to enumerate networks and their spatial realisations ("embeddings") with greater breadth than conventional approaches.

Search the databases

- Hyperbolic Subgroup Tilings
- U-Tilings
- Hyperbolic Nets
- Systre Nets

Explore the databases

- Structure Taxonomy

EPINET — http://epinet.anu.edu.au

2706 Hyperbolic tilings with 2451 net topologies
6095 Surface-compatible tilings generate 14532 3-periodic nets.

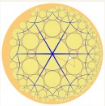
EPINET Home | Glossary

Euclidean Patterns in Non-Euclidean Tilings — Structures | Search

Overview of EPINET Structures

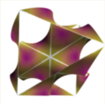
Structure Types

Tilings



Hyberbolic Subgroup Tilings:

- show all
- list by orbifold symbol



Surface-compatible U-Tilings:

- show all
- list by subgroup

Nets

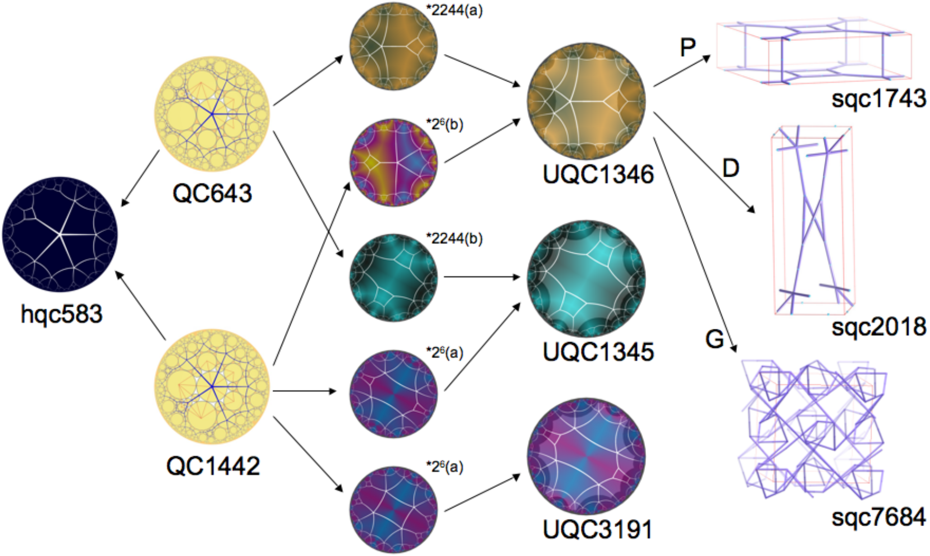
2D Hyperbolic nets (h-nets):

- show all
- list by orbifold symbol

3D Systre nets (s-nets):

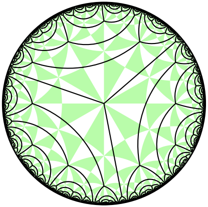
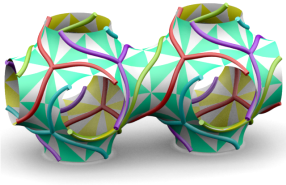
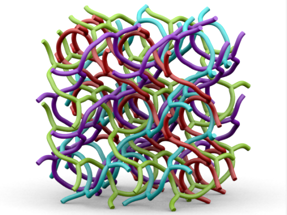
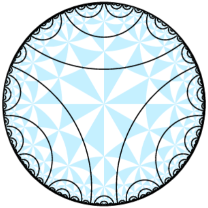
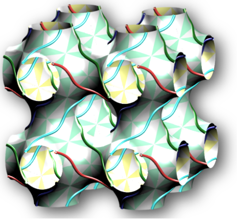
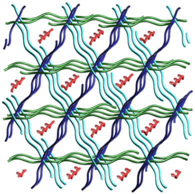
- show all
- list by spacegroup

Structure relationships in Epinet



Free tilings

Surface reticulations derived from packings of trees or lines in $\mathbb{H}^2$ give multi-component interwoven nets and (helical) rod packings.

Free tilings

We extend Delaney-Dress tiling theory to a quadrangulation decomposition of the infinite-sided polygons. These new quadrangulations can be enumerated via regular D-symbols.