An introduction to orbifolds

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Subdivide and Tile:

Triangulating spaces for understanding the world

Lorentz Center

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- $\Gamma < \mathrm{Isom}(\mathbf{R}^{\mathrm{n}})$ or $\mathbf{H}^n$ discrete and acts properly discontinuously (e.g. a group of symmetries of a tessellation).

  - If Γ has no fixed points $\Rightarrow \Gamma\backslash\mathbf{R}^n$ is a manifold.
  - If Γ has fixed points $\Rightarrow \Gamma\backslash\mathbf{R}^n$ is an orbifold.

Motivation

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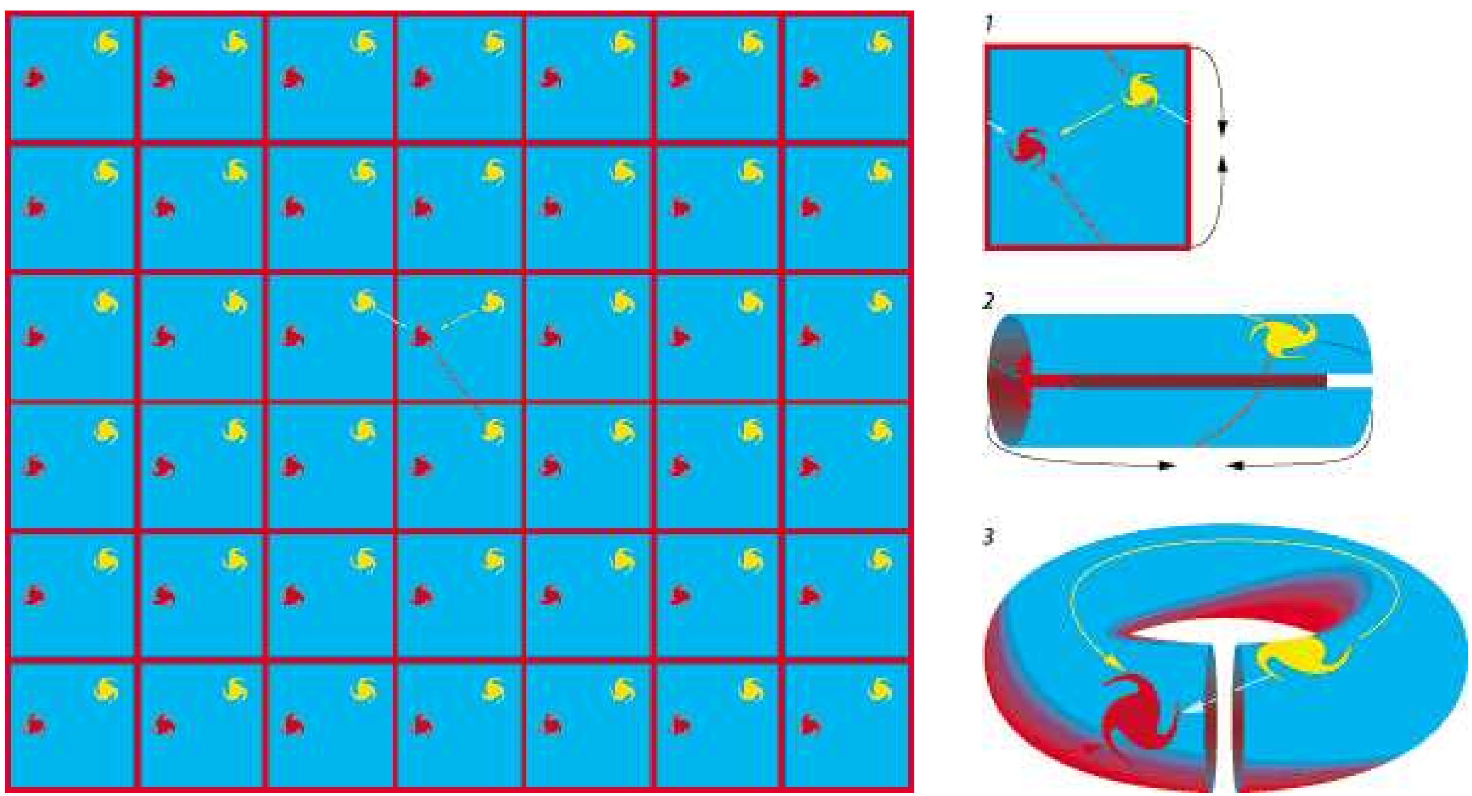
$\cdots$ (there are other notions of orbifold in algebraic geometry, string theory or using grupoids)

Examples: tessellations of Euclidean plane

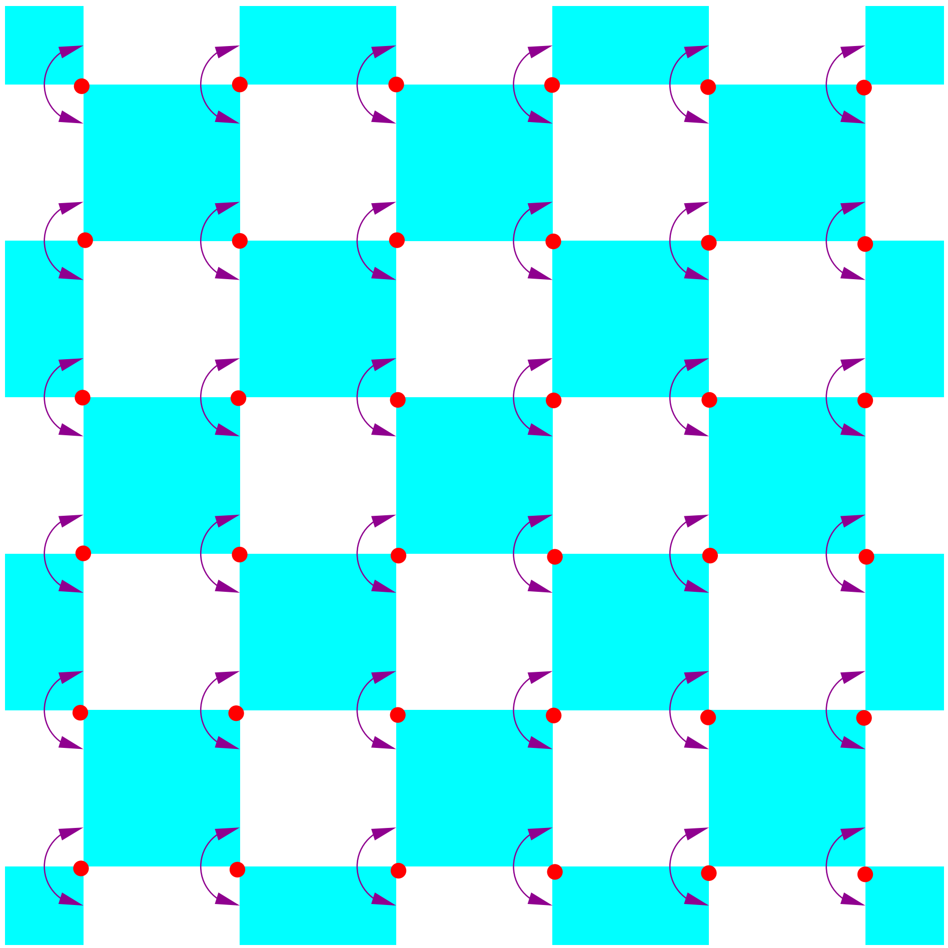
$$\Gamma = \langle (x,y) \to (x+1,y), (x,y) \to (x,y+1) \rangle \cong \mathbf{Z}^2$$

$$\Gamma \backslash \mathbf{R}^2 \cong T^2 = S^1 \times S^1$$



Examples: tessellations of Euclidean plane

Rotations of angle π around red points (order 2)



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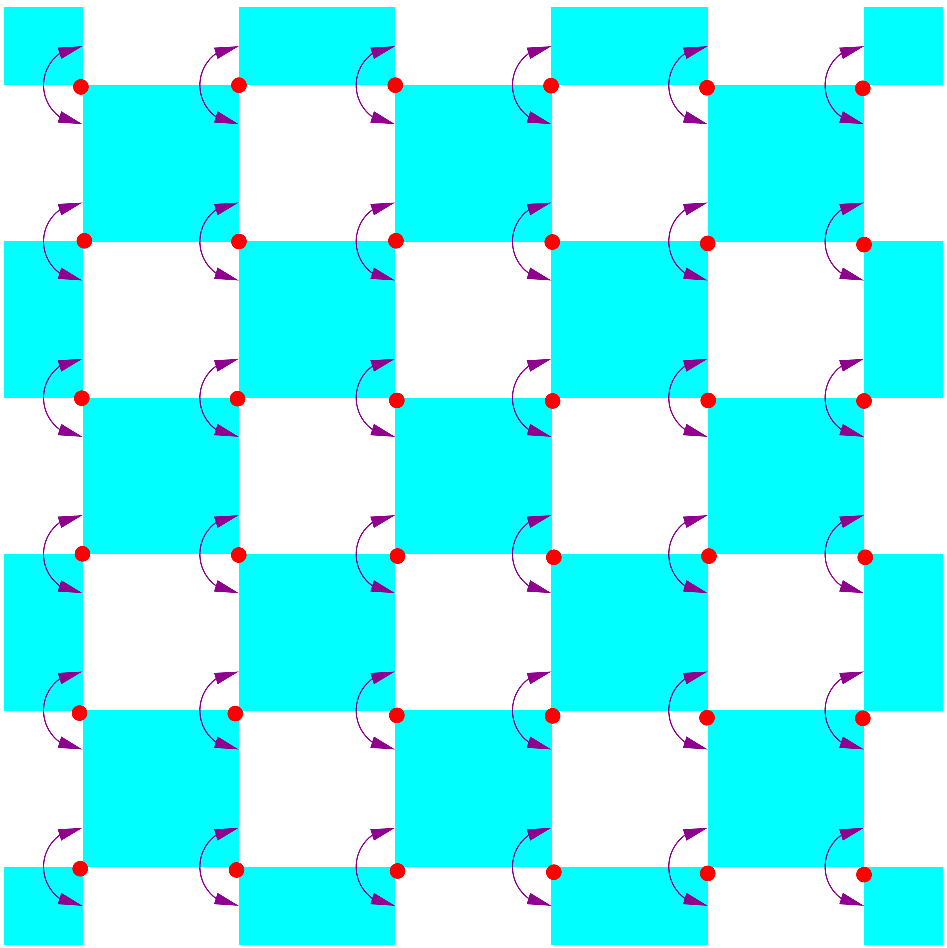

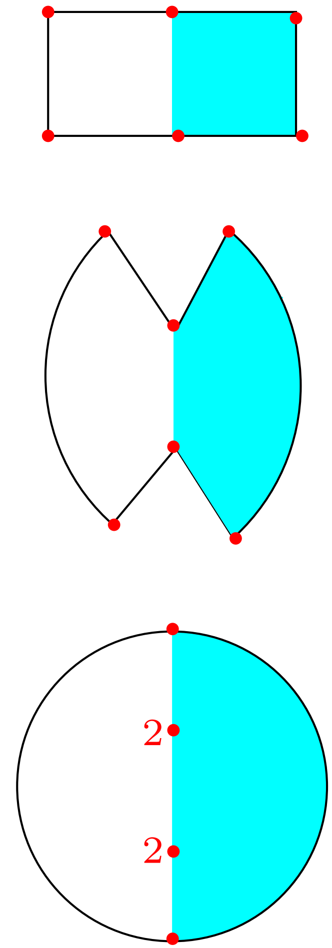

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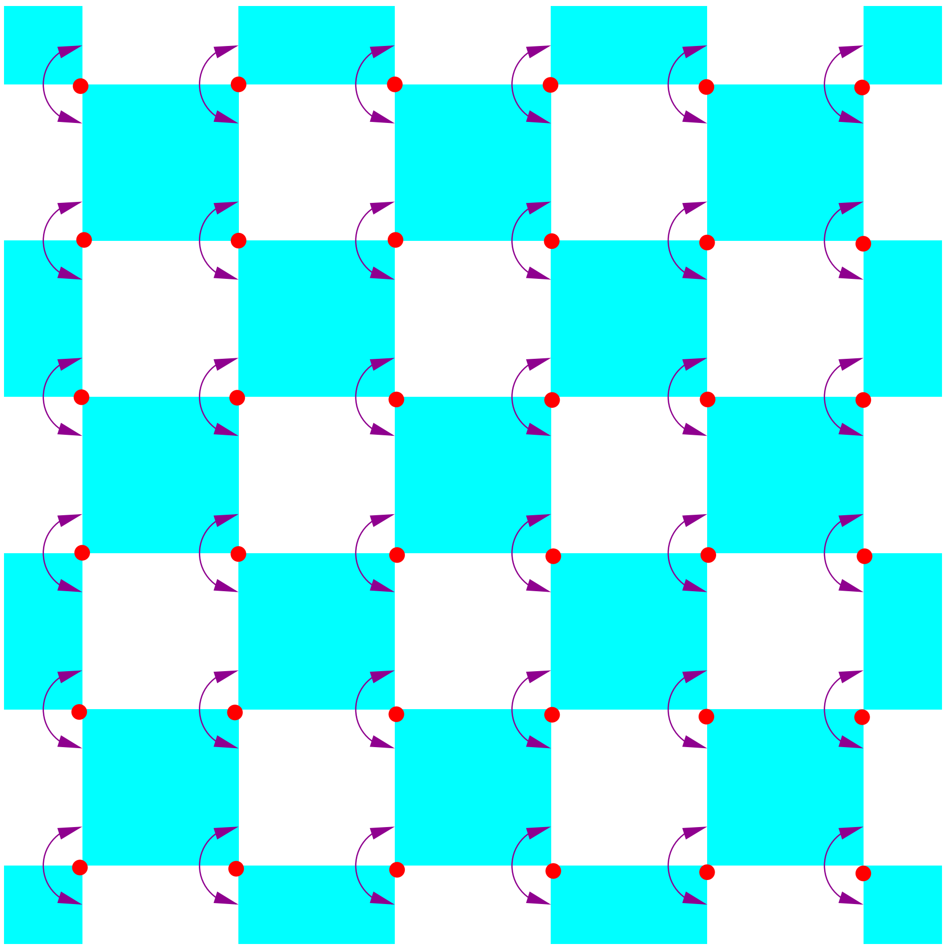
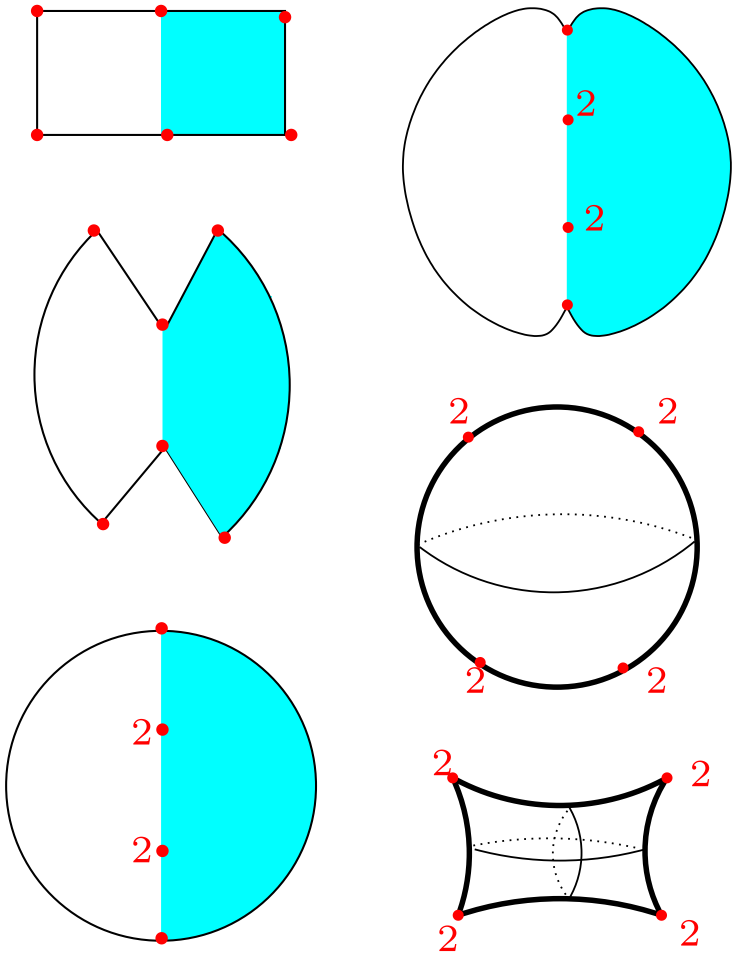
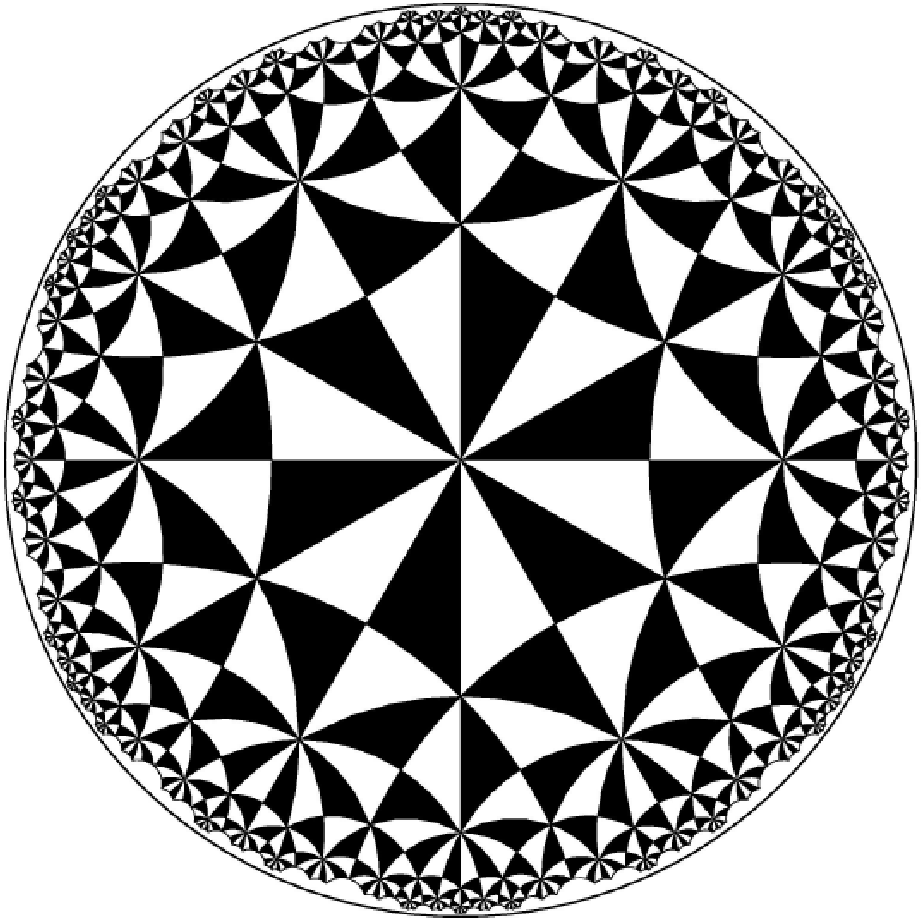




Example: tessellations of hyperbolic plane

Rotations of angle π, $\pi/2$ and $\pi/3$ around vertices (order 2, 4, and 6)



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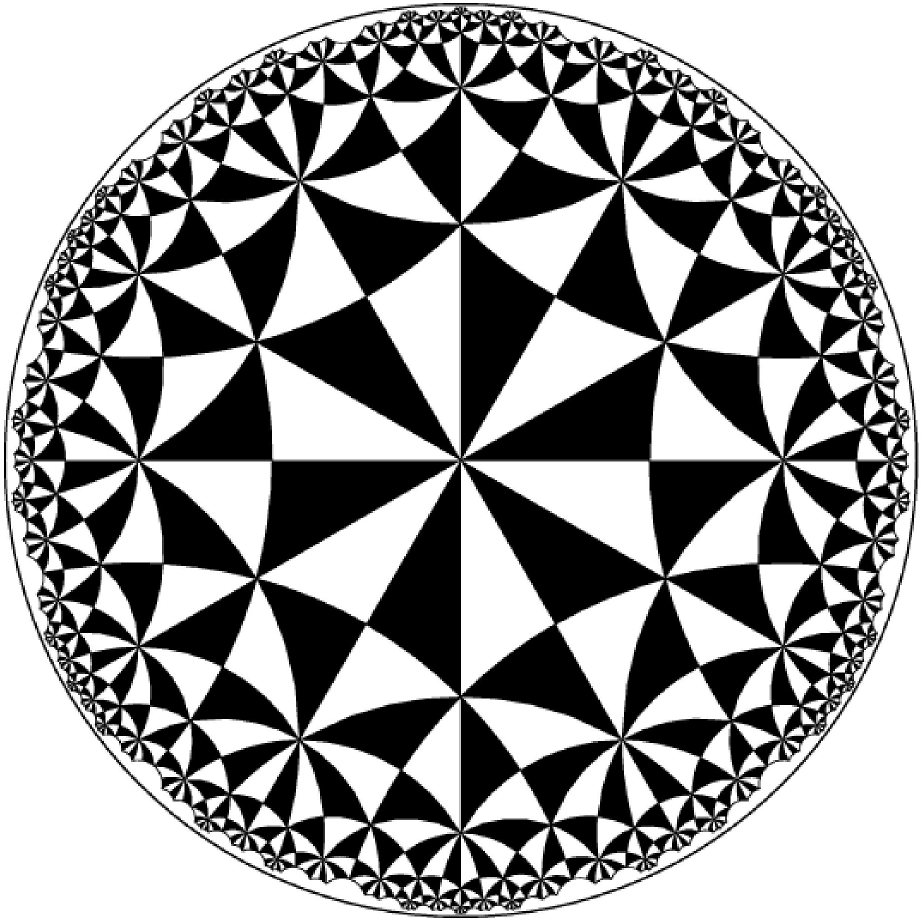

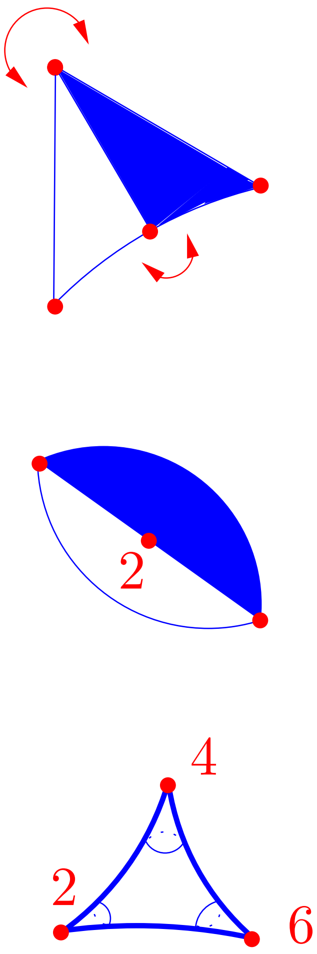

Informal Definition

- An orbifold $\mathcal{O}$ is a metrizable topological space equipped with an atlas modelled on $\mathbf{R}^n/\Gamma$, $\Gamma < O(n)$ finite, with some compatibility condition.

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- Singular (or branching) locus: Σ= points modelled on $Fix(\Gamma)/\Gamma$.
- Γ_x (the minimal Γ): isotropy group of a point $x \in \mathcal{O}$.
- $|\mathcal{O}|$ underlying topological space (possibly not a manifold).

Formal Definition

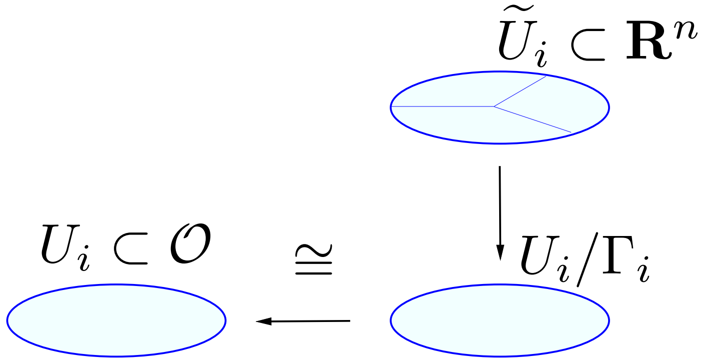
- An orbifold $\mathcal{O}$ is a metrizable top. space with a (maximal) atlas

$$\{U_i, \widetilde{U}_i, \Gamma_i, \phi_i\}$$

$\bigcup U_i = \mathcal{O}$, $\quad \Gamma_i < O(n)$

$\widetilde{U}_i \subset \mathbf{R}^n$ is Γ_i-invariant

$\phi_i : U_i \cong \widetilde{U}_i/\Gamma_i$ homeo



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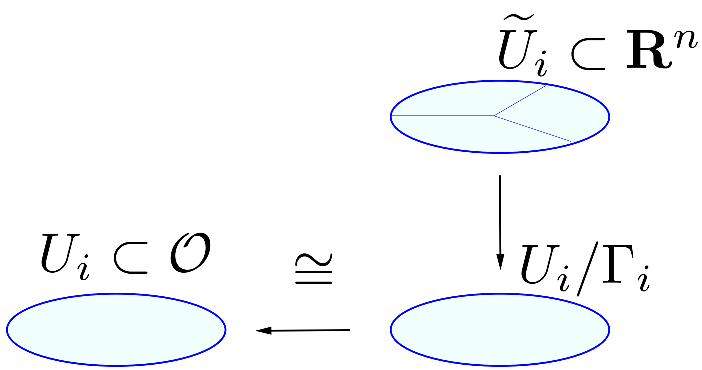


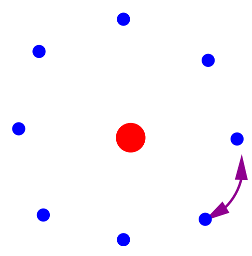
If $y \in U_i \cap U_j$, then there is U_k s.t. $y \in U_k \subset U_i \cap U_j$ and

- $i_* : \Gamma_k \hookrightarrow \Gamma_i$
- $i : \widetilde{U}_k \hookrightarrow \widetilde{U}_i$, diffeo with the image,
- $i(\gamma \cdot x) = i_*(\gamma) \cdot i(x)$

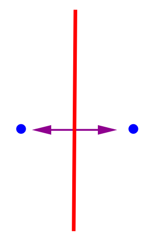
$\Gamma_x = \bigcap_{x \in U_i} \Gamma_i$ isotropy group of a point $\quad \Sigma_{\mathcal{O}} = \{\, x \mid \Gamma_x \neq 1 \}$

Dimension 2

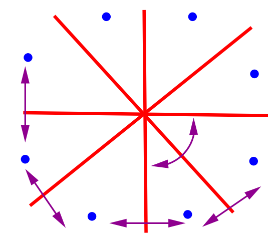
Finite subgroups of $O(2)$:



cyclic group of rotations
with n elements

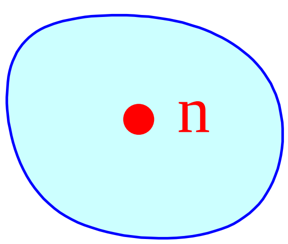


reflexion along a line

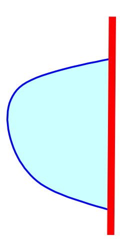


dihedral group
rots. and reflex.

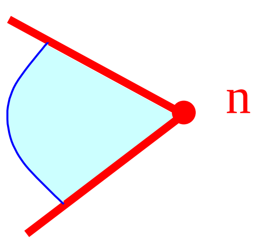
Local picture of 2-orbifolds:



Γ_x cyclic order n



Γ_x has 2 elements



Γ_x dihedral

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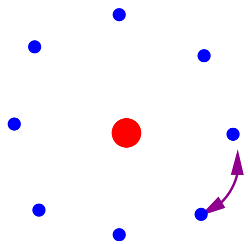

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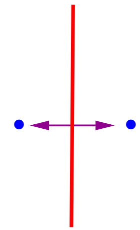

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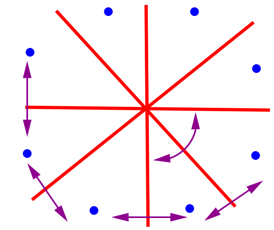

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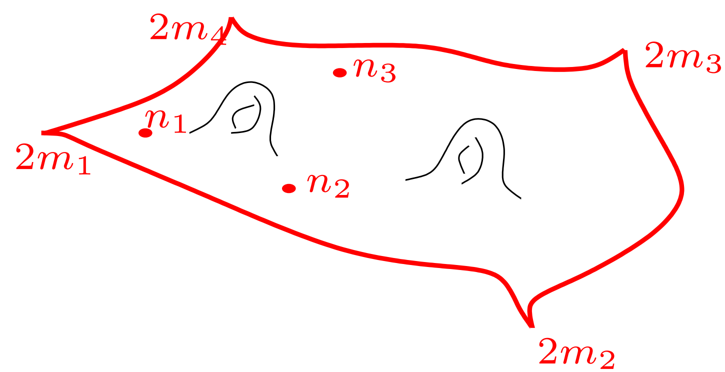
General picture in dim 2

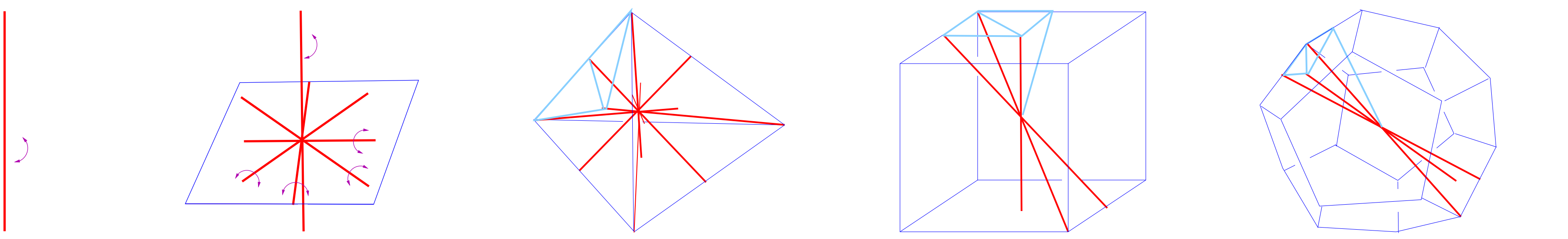


- $|O^2|$ surface,

  $\Sigma_{\mathcal{O}^2} = \partial|O^2|$ and points

Dimension 3 (loc.orientable)

Finite subgroups of $SO(3)$ (all elements are rotations):



C_n	D_{2n}	T_{12}	O_{24}	I_{60}
cyclic	dihedral	tetrahedral	octahedral	icosahedral

- $\mathcal{O}$ dim 3 and orientable: $|\mathcal{O}| =$ manifold, $\Sigma_{\mathcal{O}}$ trivalent graph

$\Sigma_{\mathcal{O}}$ locally:

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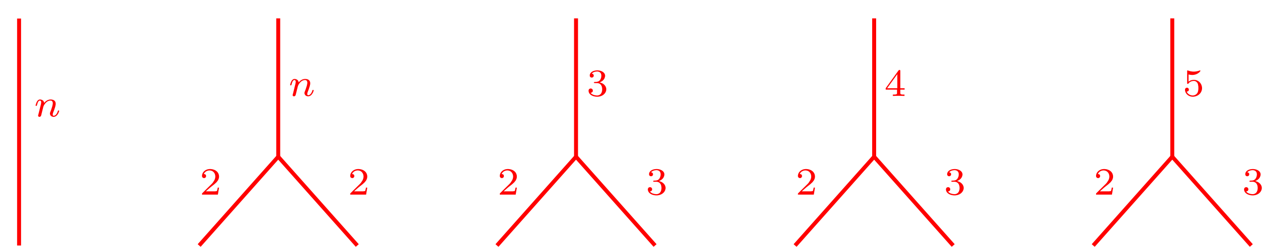

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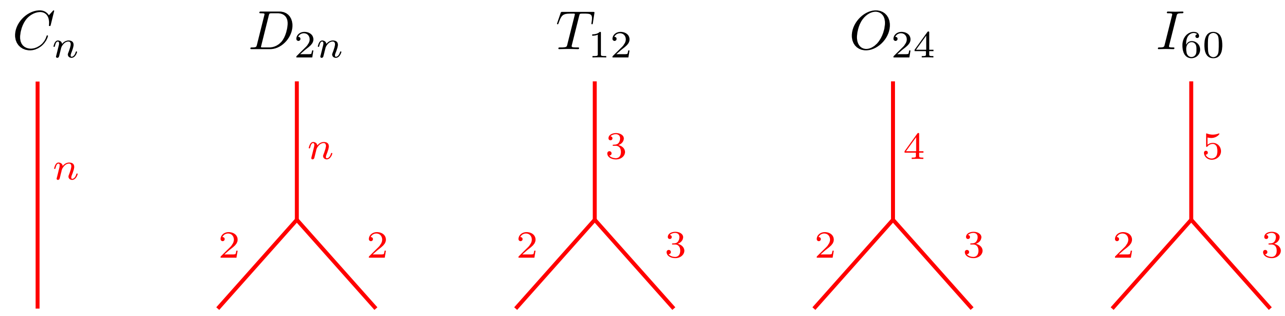
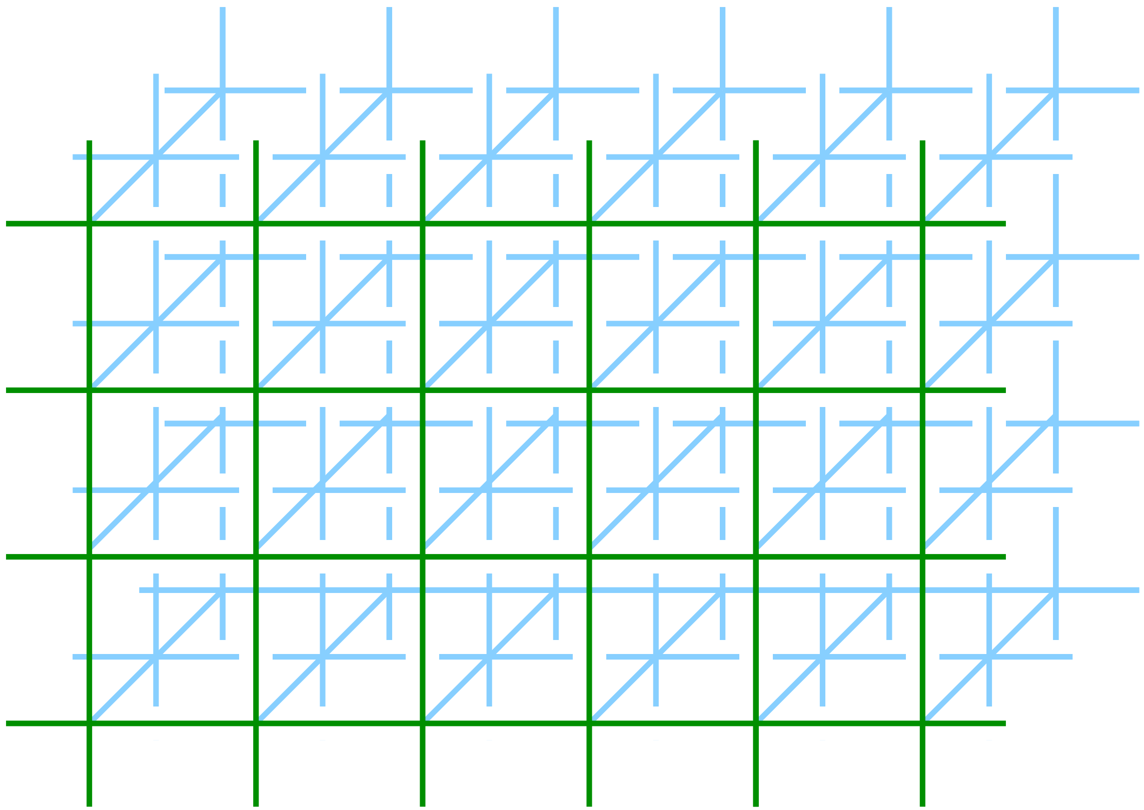


- Non orientable case: combine this with reflections along planes and antipodal map: $a(x, y, z) = (-x, -y, -z)$
- $\mathbf{R}^3/a =$ cone on $\mathbf{RP}^2$, is <u>not a manifold</u>.
- In dim 4 and larger, $\exists \mathcal{O}$ orientable and $\mathcal{O}$ possibly not a manifold.

More examples: tessellation by cubes

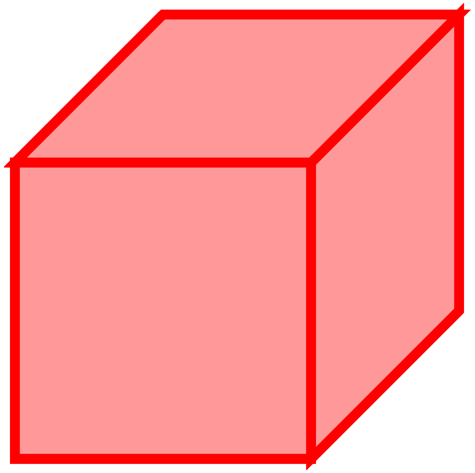


- $\mathbf{Z}^3$ translation group, $\mathbf{R}^3/\mathbf{Z}^3 = S^1 \times S^1 \times S^1$
- But we can also consider other groups

More examples: tessellation by cubes

- $\mathcal{O} = \mathbf{R}^3/\langle$reflections on the sides of the cube$\rangle$

$|\mathcal{O}|$ is the cube and $\Sigma_{\mathcal{O}}$ boundary of the cube



- x in a face $\Rightarrow \Gamma_x = \mathbf{Z}/2\mathbf{Z}$ reflexion
- x in an edge$\Rightarrow \Gamma_x = (\mathbf{Z}/2\mathbf{Z})^2$
- x in a vertex $\Rightarrow \Gamma_x = (\mathbf{Z}/2\mathbf{Z})^3$

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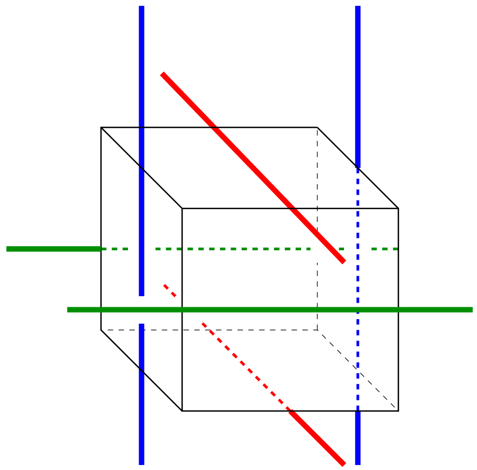
Consider the group generated by order 2 rotations around axis as in:



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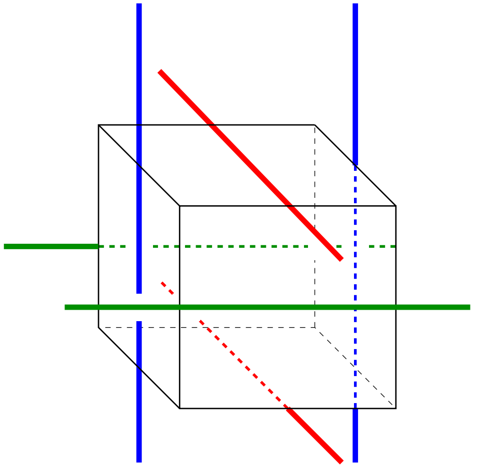
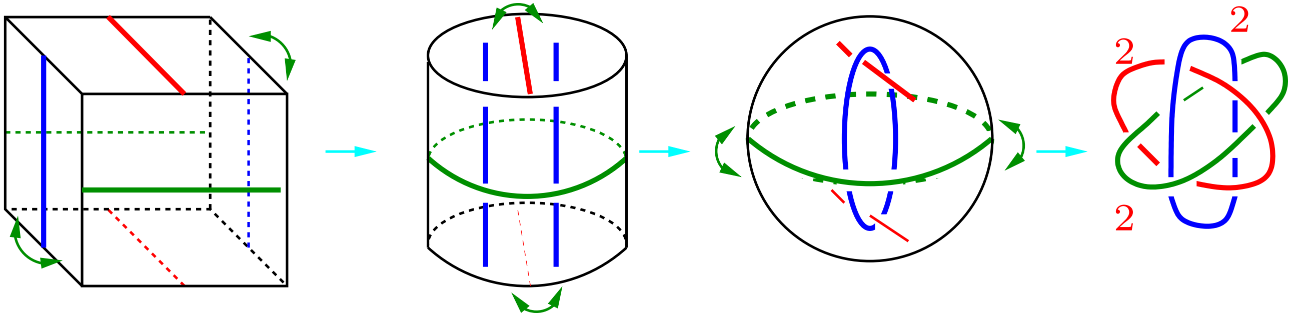
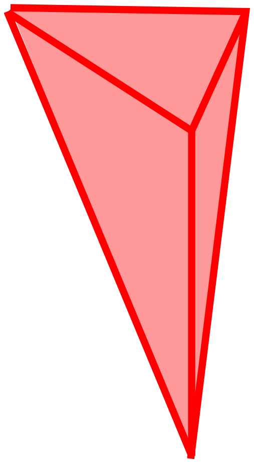




- $|\mathcal{O}| = S^3$ $\quad \Sigma_{\mathcal{O}} =$ Borromean rings. $\quad \Gamma_x \cong \mathbf{Z}/2\mathbf{Z}$ acting by rotations.

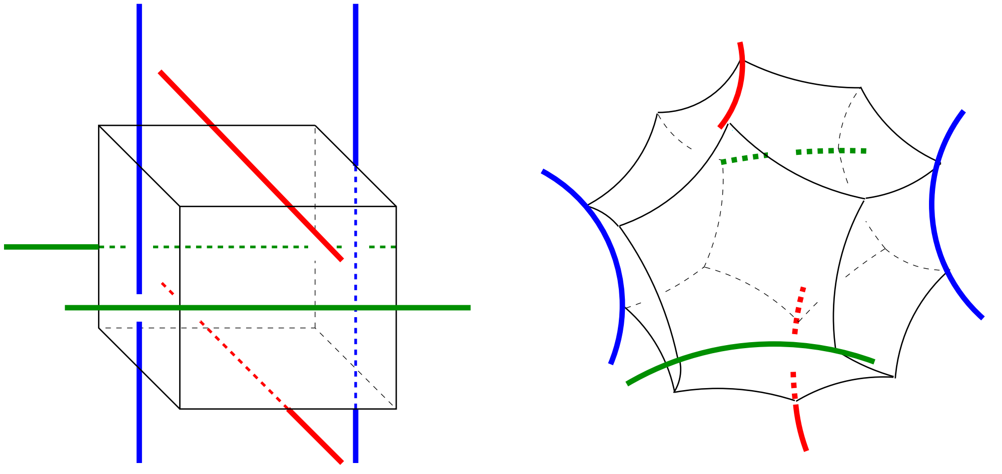
More examples: tessellation by cubes

$\mathbf{R}^3/\{$ Full isometry group of the tessellation $\}$



- x in a face $\Rightarrow \Gamma_x = \mathbf{Z}/2\mathbf{Z}$ reflexion
- x in an edge $\Rightarrow \Gamma_x =$ dihedral (extension by reflections of cyclic group of rotations)
- x in a vertex $\Rightarrow \Gamma_x =$ extension by reflections of dihedral, T_{12} or O_{24}

More examples: hyperbolic tessellation



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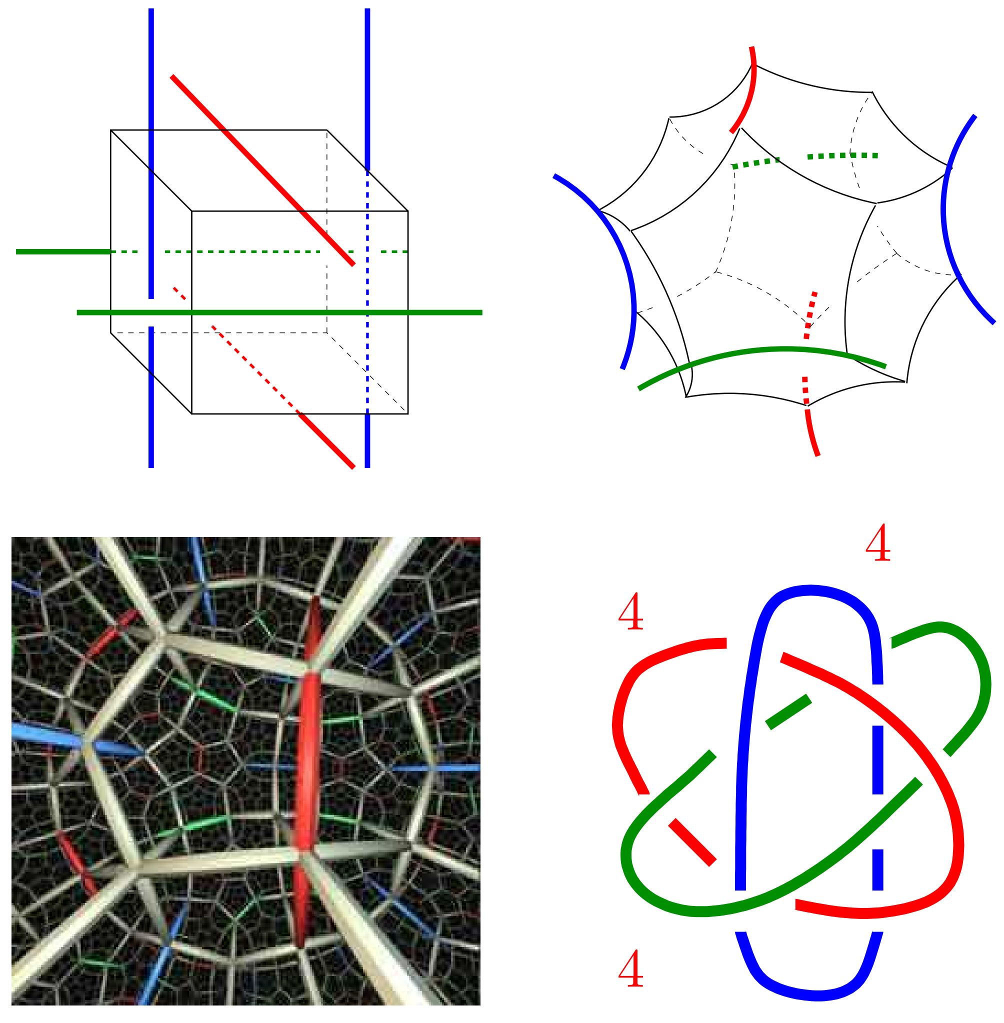
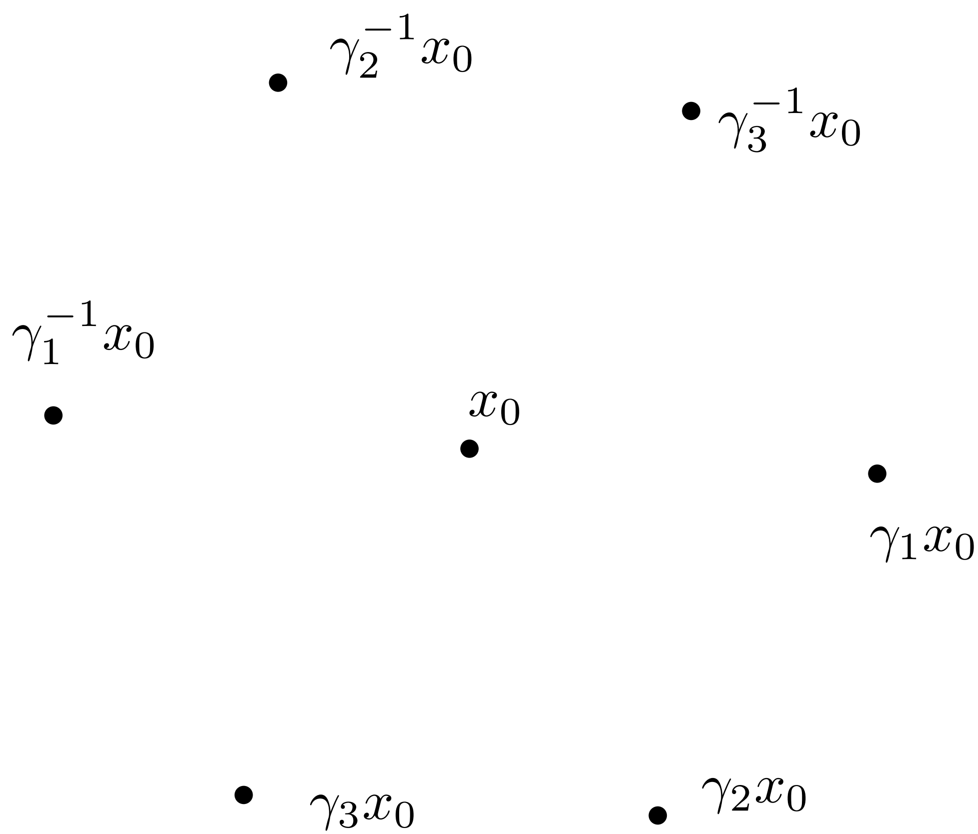


- If Γ acts on $\mathbf{R}$ or $\mathbf{H}^n$

  Dirichlet domain: Voronoi cell of the orbit of a point x_0 with $\Gamma_{x_0} = 1$.



The points with nontrivial stabilizer are on the boundary

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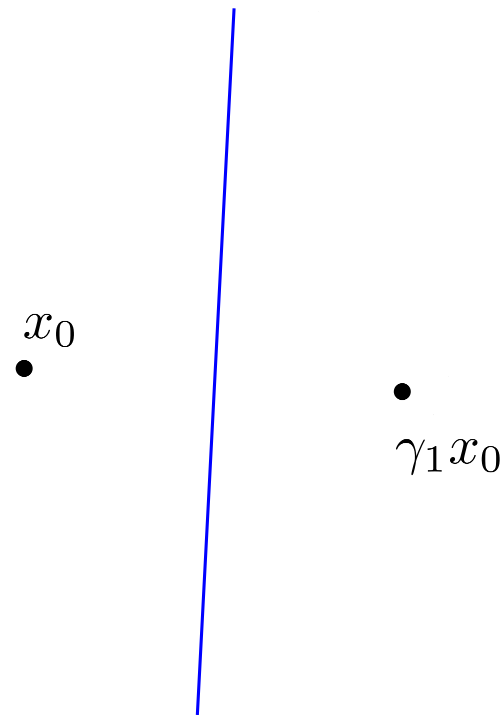

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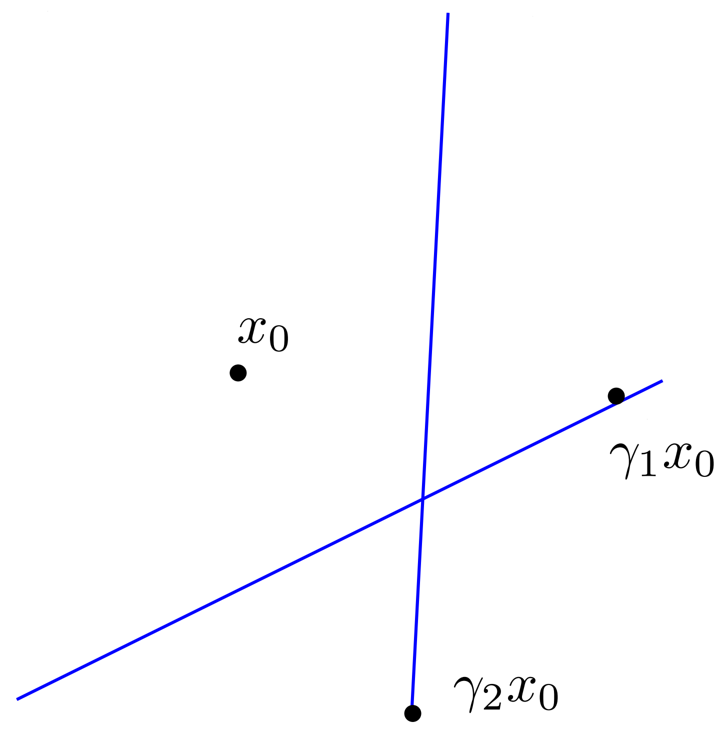

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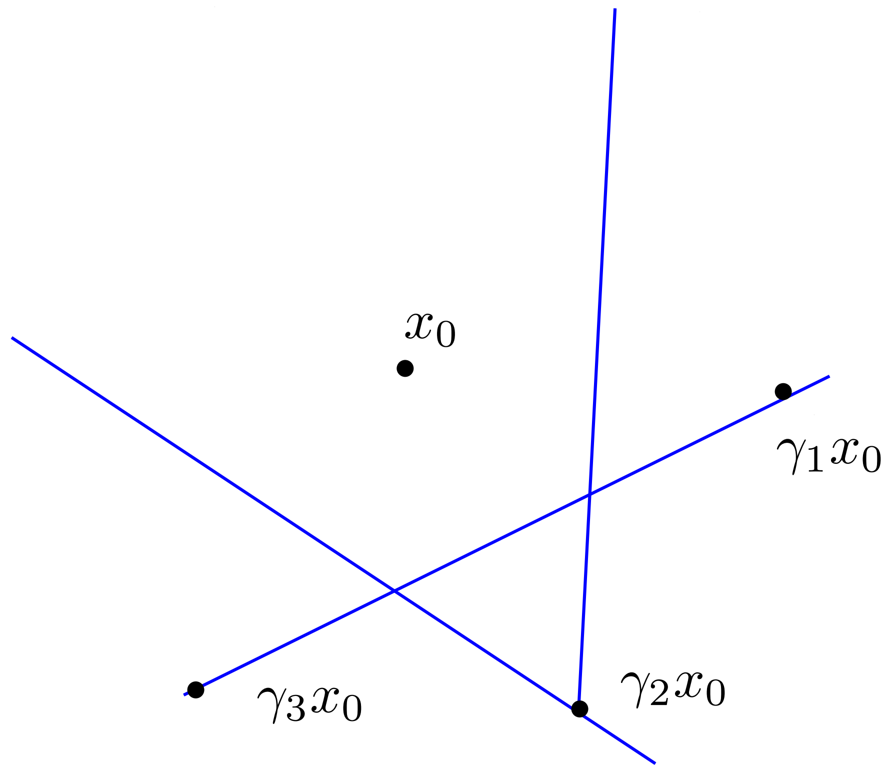

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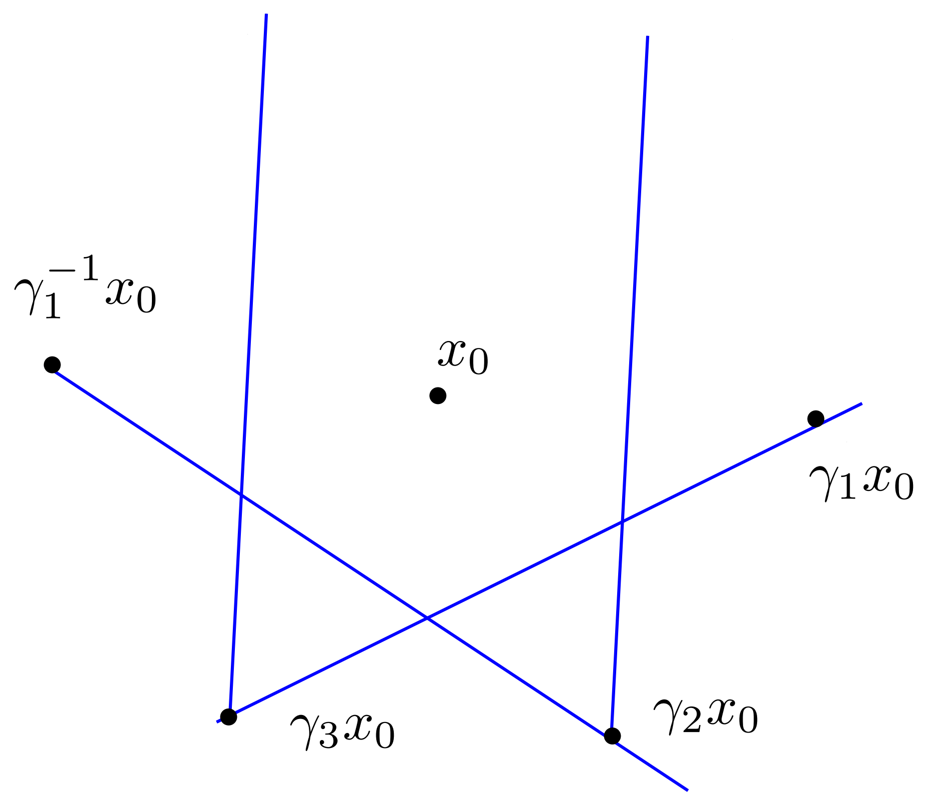

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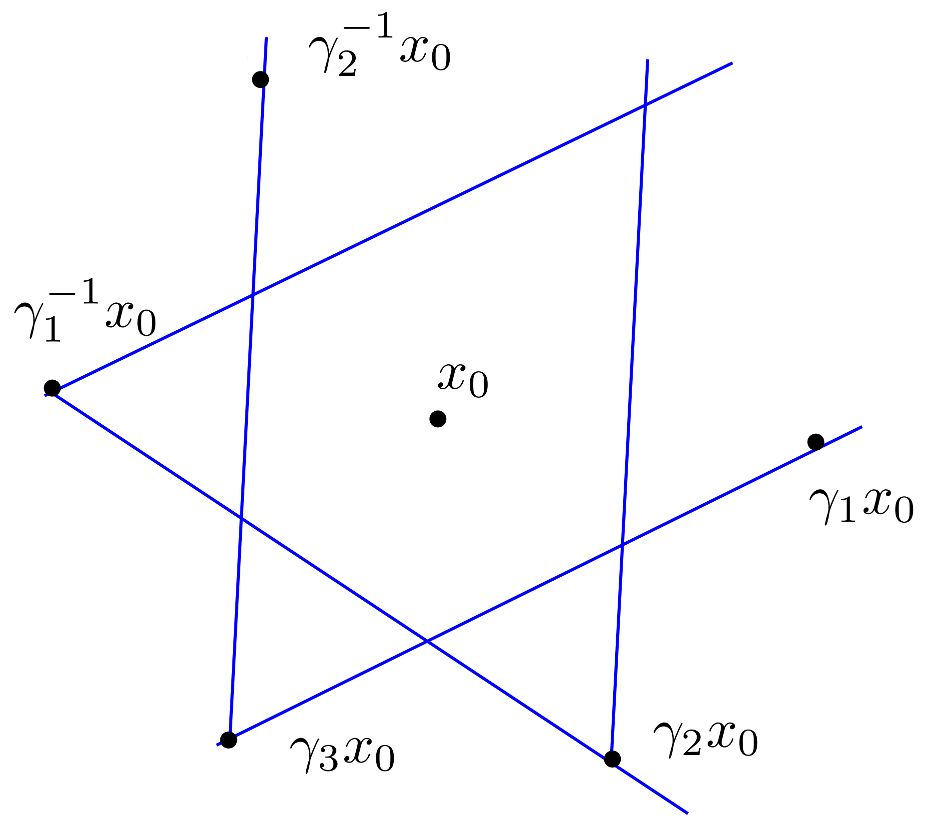

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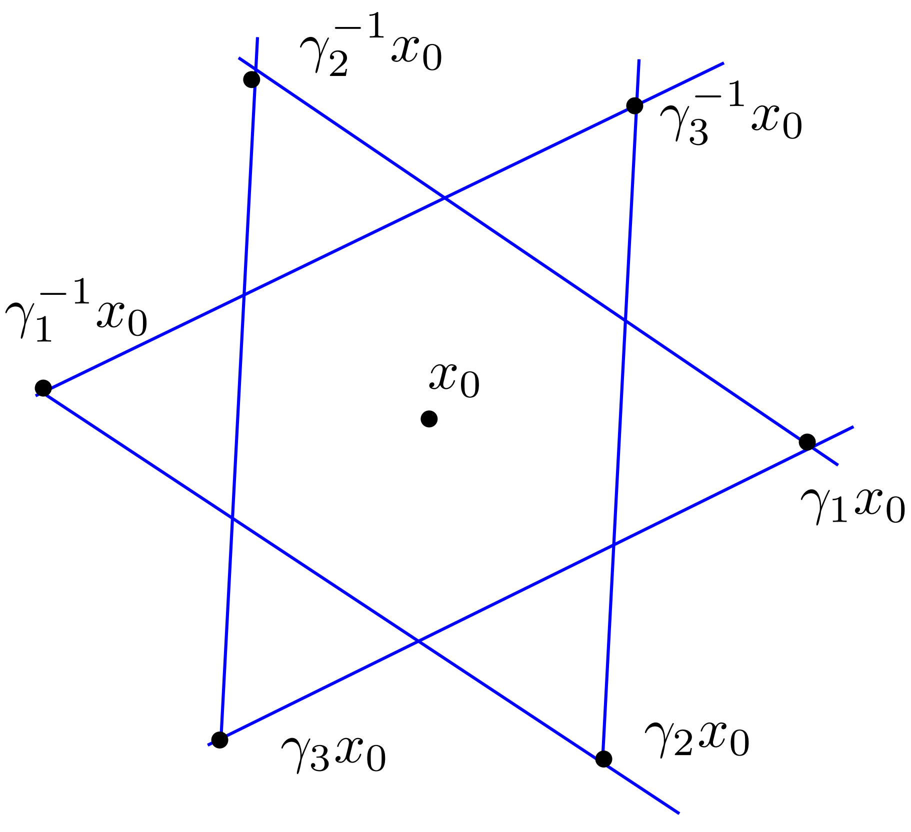

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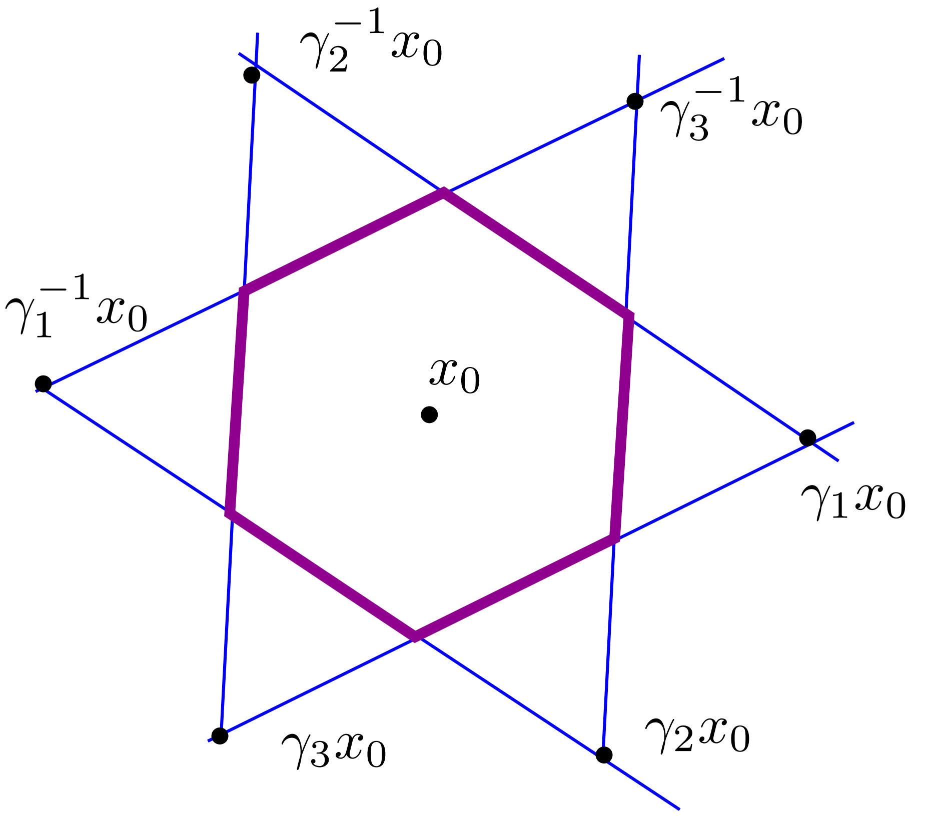

The points with nontrivial stabilizer are on the boundary

Structures on orbifolds

- Can define geometric structures on orbifolds by taking equivariant definitions on charts $\widetilde{U}_i, \Gamma_i$.
- A Riemanian metric on the charts $(\widetilde{U}_i, \Gamma_i)$ requires:
  - $\widetilde{U}^i$ has a Riemannian metric
  - coordinate changes are isometries
  - Γ_i acts isometrically on $\widetilde{U}_i$
- e.g. Analytic, flat, hyperbolic, etc...

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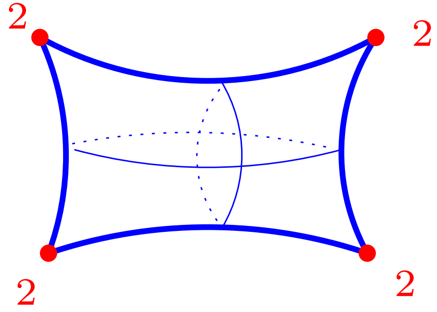

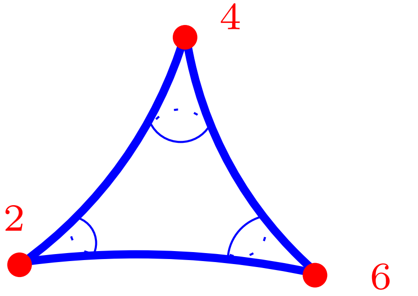

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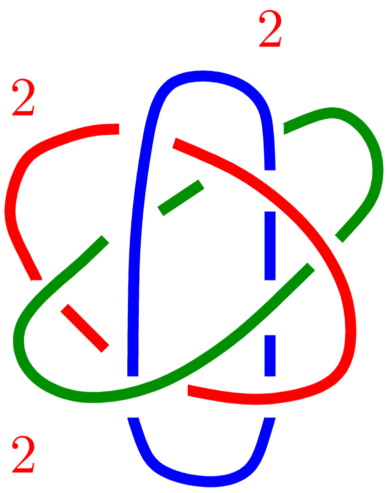

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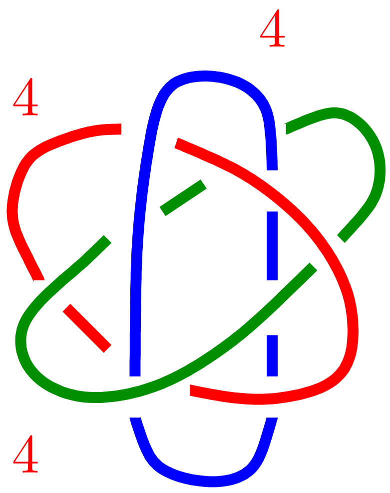

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Coverings

$p : \mathcal{O}_0 \to \mathcal{O}_1$ is an orbifold covering if

Every $x \in \mathcal{O}_1$ is in some $U \subset \mathcal{O}_1$ s.t. if $V =$ component of $p^{-1}(U)$:

then $\widetilde{V} \to V \xrightarrow{p} U$ is a chart for U

$$\begin{tikzcd} V \cong \widetilde{V}/\Gamma_0 \arrow[d, "p"] & \widetilde{V} \subset \mathbf{R}^n \arrow[l] \arrow[d, equal, "\Gamma_0-\text{equiv}"] \\ U \cong \widetilde{U}/\Gamma_1 & \widetilde{U} \subset \mathbf{R}^n \arrow[l] \end{tikzcd} \qquad \Gamma_0 < \Gamma_1$$

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- If Γ acts properly discontinuously on M manifold then $M \to M/\Gamma$ is an orbifold covering.

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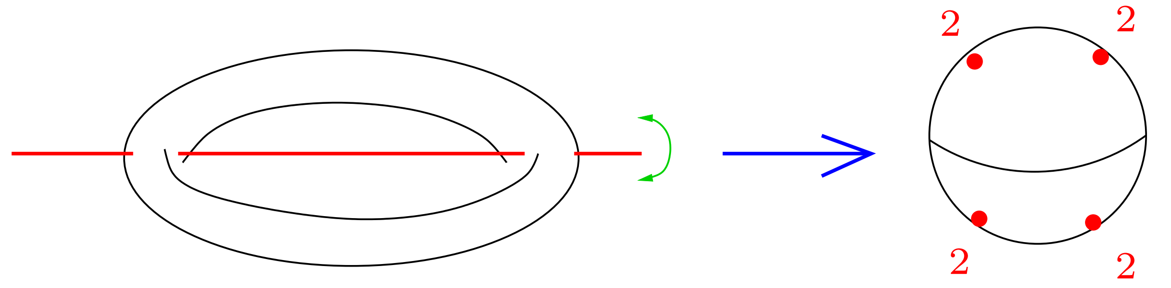

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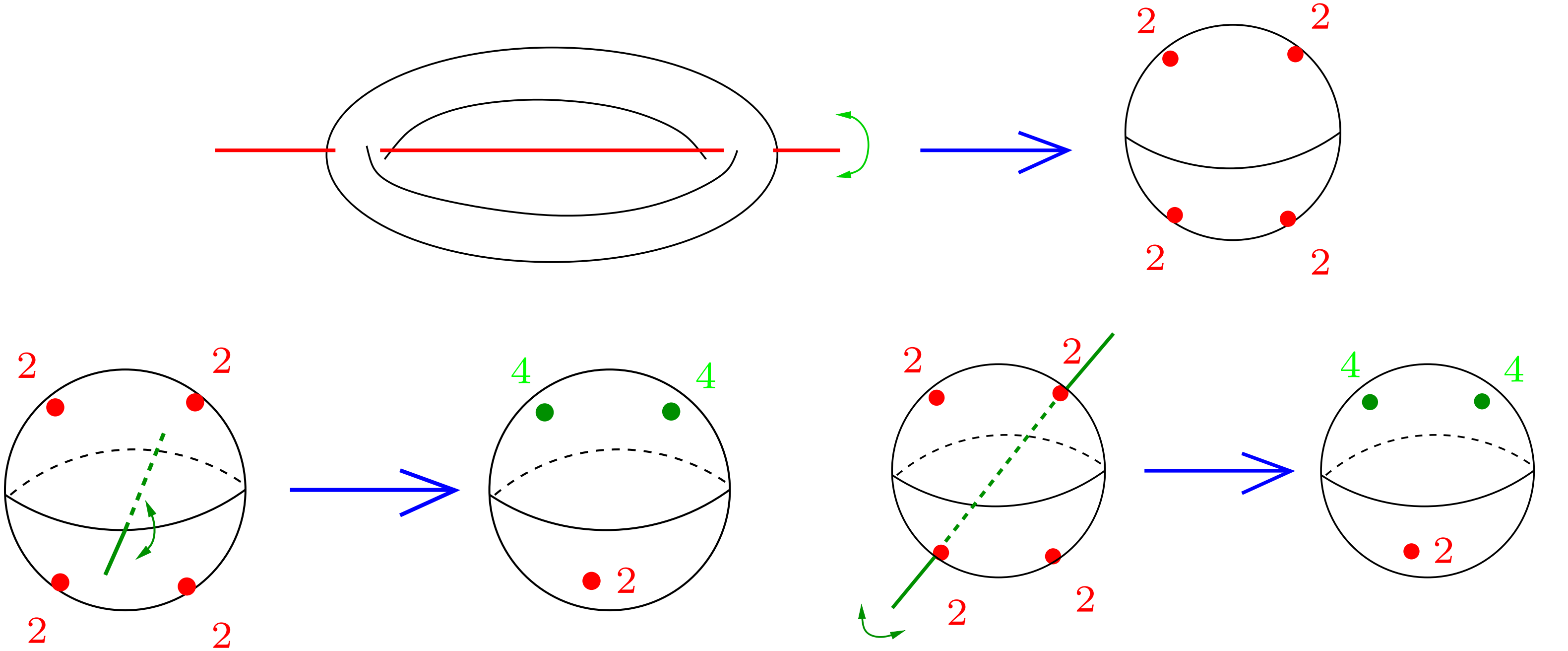

Good and bad

Definition

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Γ acts properly discontinuously on a manifold M

- $\mathcal{O}$ is good iff $\mathcal{O}$ has a covering that is a manifold.

Question: When is an orbifold good?

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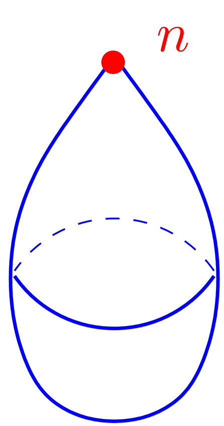
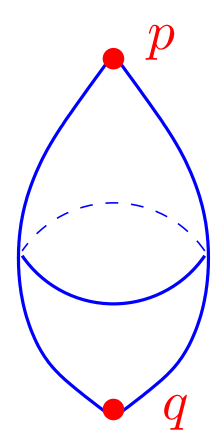
Question: When is an orbifold good?





are BAD

Teardrop $T(n)$ Spindle $S(p,q)$ ($p \neq q$)

Definition

$\mathcal{O}$ is good if $\mathcal{O} = M/\Gamma$

Γ acts properly discontinuously on a manifold M

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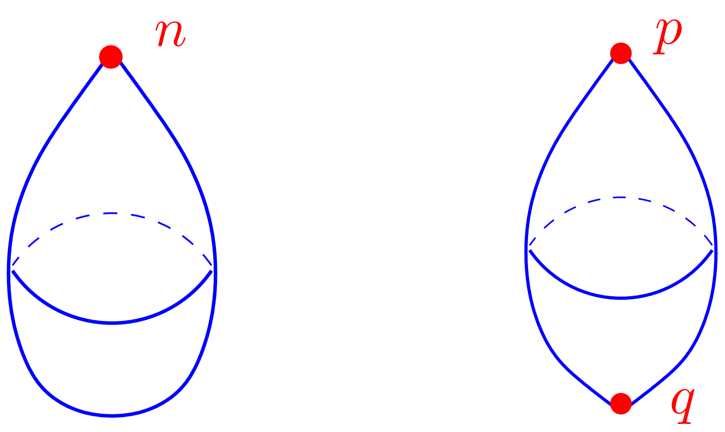


are BAD

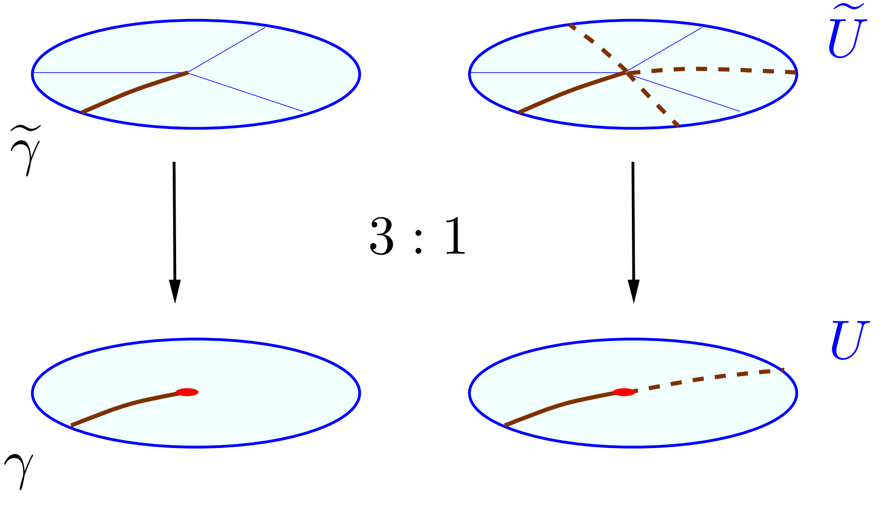
Teardrop $T(n)$ Spindle $S(p, q)$ ($p \neq q$)

- Those and their nonorientable quotients are the only bad 2-orbifolds

Def: A loop based at $x \in \mathcal{O} \setminus \Sigma_{\mathcal{O}}$:

$\gamma : [0,1] \to \mathcal{O}$ such that $\gamma(0) = \gamma(1) = x$

with a choice of lifts at branchings:



- Define homotopies as continuous 1-parameter families of paths.
- $\pi_1(\mathcal{O}, x) = \{$ loops based at x up to homotopy relative to $x\}$

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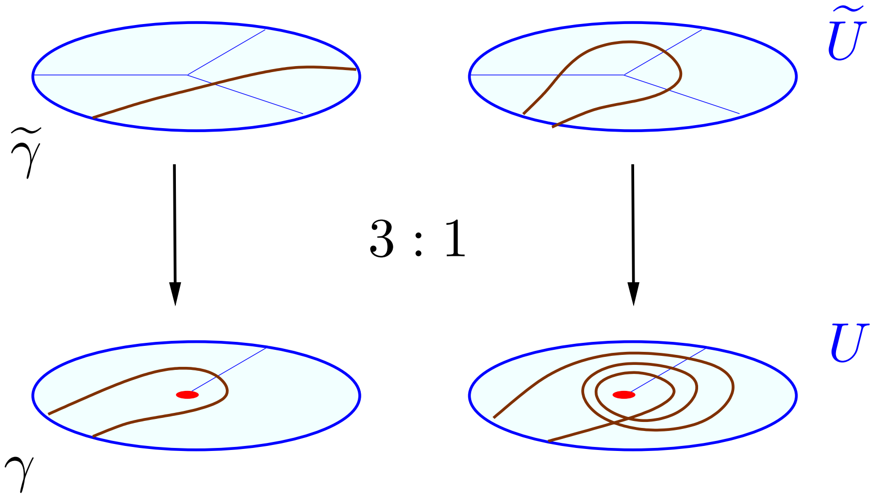
$\pi_1(D^2/\text{rotation order } n) \cong \mathbf{Z}/n\mathbf{Z}$



$\gamma^3 = 1$

Fundamental group

- $\pi_1(\mathcal{O}, x) = \{$ loops based at x up to homotopy relative to $x\}$

Seifert-Van Kampen theorem (Haëfliger):

If $\mathcal{O} = U \cup V$, $U \cap V$ conected, then:

$$\boxed{\pi_1(\mathcal{O}) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)}$$

$\pi_1\mathcal{O}$ = free product of $\pi_1(U)$ and $\pi_1(V)$ quotiented by $\pi_1(U \cap V)$

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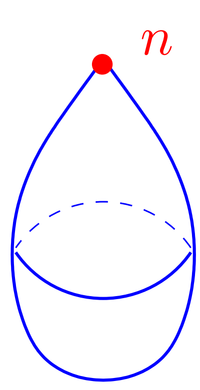
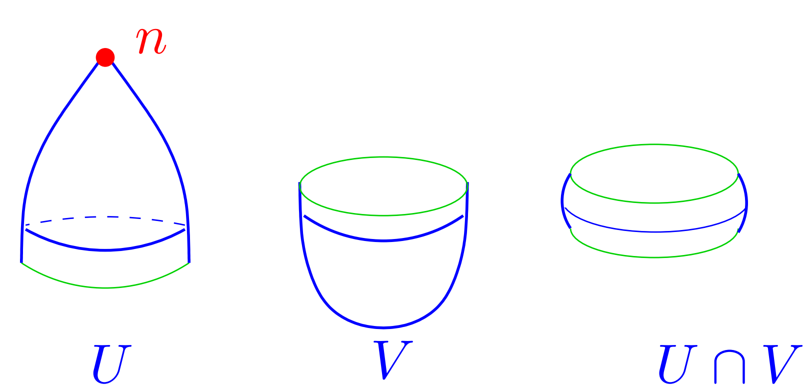
- $\pi_1(T(n)) = \{1\}$





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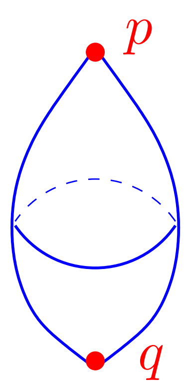

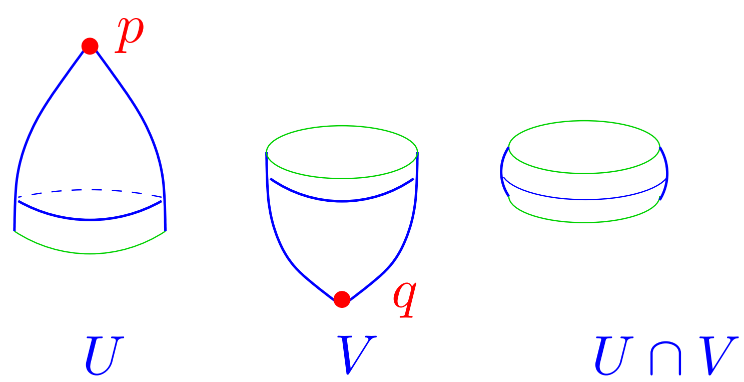

Universal covering

- Universal covering: $\widetilde{\mathcal{O}} \to \mathcal{O}$ such that every other covering $\mathcal{O}' \to \mathcal{O}$:

$$\begin{tikzcd} \widetilde{\mathcal{O}} \arrow[d] \arrow[dr] & \\ \mathcal{O}' \arrow[r] & \mathcal{O} \end{tikzcd}$$

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- Existence: $\widetilde{\mathcal{O}} = \{$rel. homotopy classes of paths starting at $x\}$
- $\pi_1(\mathcal{O}) \cong$ deck transformation group of $\widetilde{\mathcal{O}} \to \mathcal{O}$.
- $\pi_1(T(n)) = \{1\}$ and $\pi_1(S(p,q)) = \mathbf{Z}/\gcd(p,q)\mathbf{Z}$,

hence $\boxed{T(n) \text{ and } S(p,q),\ p \neq q, \text{ are bad.}}$

Theorem

1. If an orbifold has a metric of constant curvature, then it is good.
2. If an orbifold has a metric of nonpositive curvature, then it is good.

Proof 1: use developing maps $\widetilde{U} \to \mathbf{H}^n,\ S^n,\ \mathbf{R}^n$

Proof 2: use developing maps, convexity, uniqueness of geodesics.

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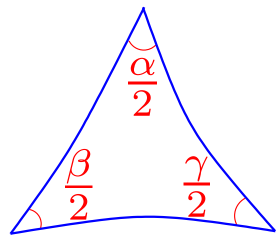
Proof 2: use developing maps, convexity, uniqueness of geodesics.

Corollary:

$\boxed{\begin{array}{c}\text{All orientable 2-orbifolds other than } T(n) \text{ and } S(p,q),\ p \neq q \\ \text{have a constant curvature metric, hence are good.}\end{array}}$

Can put an orbifold metric of constant curvature by using polygons.

2 dim example: turnovers

The triangle 

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$$\begin{cases} \text{hyperbolic if } \alpha+\beta+\gamma < 2\pi \\ \text{Euclidean if } \alpha+\beta+\gamma = 2\pi \\ \text{spherical if } \alpha+\beta+\gamma > 2\pi \end{cases}$$

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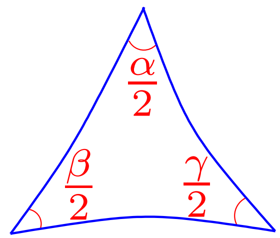
The triangle 

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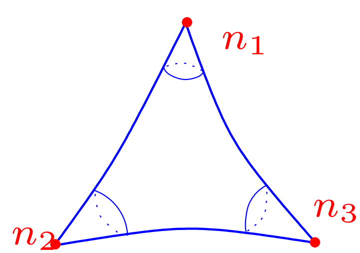
$|\mathcal{O}| = S^2$, $\Sigma_{\mathcal{O}} =$three points, cyclic isotropy groups of orders n_1, n_2, n_3.

The orbifold 

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$$\begin{cases} \text{hyperbolic if } \frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}<1 \\ \text{Euclidean if } \frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=1 \\ \text{spherical if } \frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}>1 \end{cases}$$

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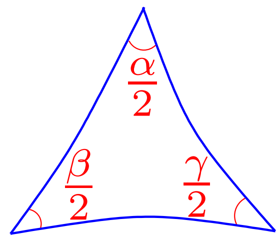
The triangle 

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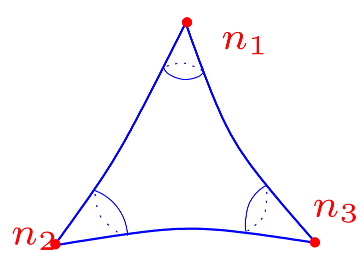
The orbifold 

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The metric on S^2 is singular, but not in $\mathcal{O}$

Euler characteristic

$$\chi(\mathcal{O}) = \sum_e (-1)^{\dim e} \frac{1}{|\Gamma_e|}$$

The sum runs over the cells of a cellulation of $\mathcal{O}$
that preserves the stratification of the branching locus.

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Properties:

- If $\mathcal{O} \to \mathcal{O}'$ is a covering of degree $n \Rightarrow \chi(\mathcal{O}) = n\chi(\mathcal{O}')$
- Gauss-Bonnet formula. If $\dim \mathcal{O} = 2$, then:

$$\int_{\mathcal{O}} K = 2\pi\chi(\mathcal{O})$$

where $K =$ curvature.

Normalized Ricci flow:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric} + \frac{2}{n}\overline{r}\,g$$

- g Riemmannian metric,
- $\bar{r} = \int_{\mathcal{O}} scal / \int_{\mathcal{O}} 1$ average scalar curvature
- Ric Ricci curvature.

- In dim 2, $\mathrm{Ric} = K\,g$.
  So write $g_t = e^u\,g_0$ for some function $u = u(x,t)$.
  The conformal class is preserved and

$$\frac{\partial}{\partial t} u = e^{-u} \nabla_{g_0} u$$

Ricci flow on two orbifolds

Normalized Ricci flow:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric} + \frac{2}{n}\bar{r}\,g$$

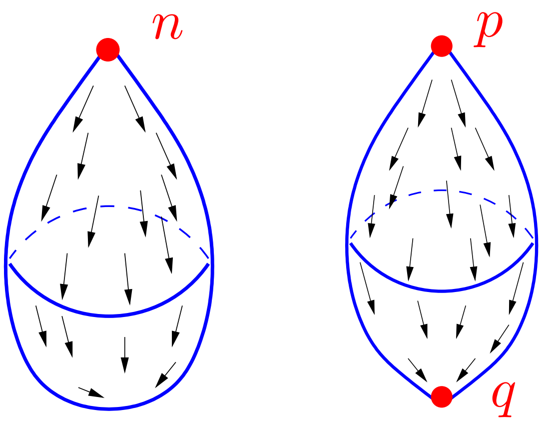
- Hamilton, Chow, Wu, Chen-Lu-Tian:

  Either it converges to a metric of constant curvature,

  or to a gradient soliton on $T(n)$ or $S(p,q)$

- Gradient soliton: $g_t = a_t \phi_t^* g_0$,

  with $\frac{\partial}{\partial t}\phi_t = \mathrm{grad}(F)$



Dimension 3

A 3-orbifold is good iff it does not contain bad 2-suborbifolds.

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Thurston's orbifold theorem

If $\mathcal{O}$ has no bad 2-suborbifolds, then

$\mathcal{O}$ decomposes canonically into locally homogeneous pieces

- $\mathcal{O}$ Locally homogeneous, if $\mathcal{O} = M/\Gamma$,
  $M =$ homogeneous manifold eg. $\mathbf{R}^3$, $\mathbf{H}^3$, S^3, $\mathbf{H}^2 \times \mathbf{R}$, $PSL_2(\mathbf{R})$...

- Canonical decomposition:
  1. Orbifold connected sum: $(\mathcal{O}_1 \setminus B^3/\Gamma) \cup_{S^2/\Gamma} (\mathcal{O}_2 \setminus B^3/\Gamma)$
  2. Cut along T^2/Γ π_1-injective in $\mathcal{O}$ (orbifold JSJ-theory)

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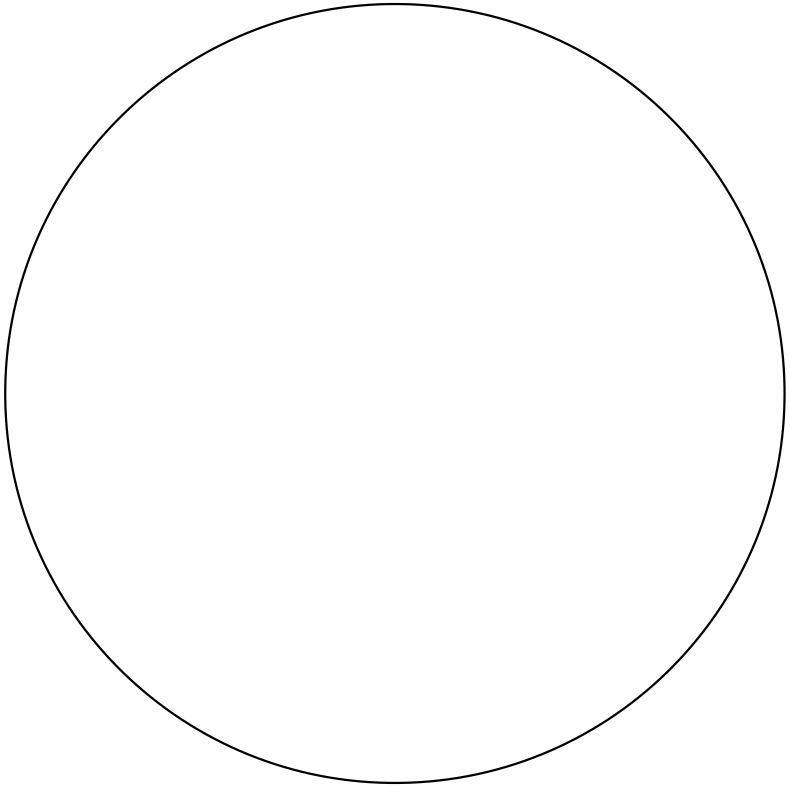
- $\widetilde{\mathcal{O}} \cong_{\text{diff}} \mathbf{R}^3$, $\mathbf{S}^3$ or infinite connected sums.

Dimension 3

$\boxed{\text{A 3-orbifold is good iff it does not contain bad 2-suborbifolds.}}$

- $\mathcal{O} = \mathcal{O}_1 \# \mathcal{O}_2$, $\mathcal{O}_1$, $\mathcal{O}_2$ hyperbolic, then $\widetilde{\mathcal{O}}$ infinite connected sum:



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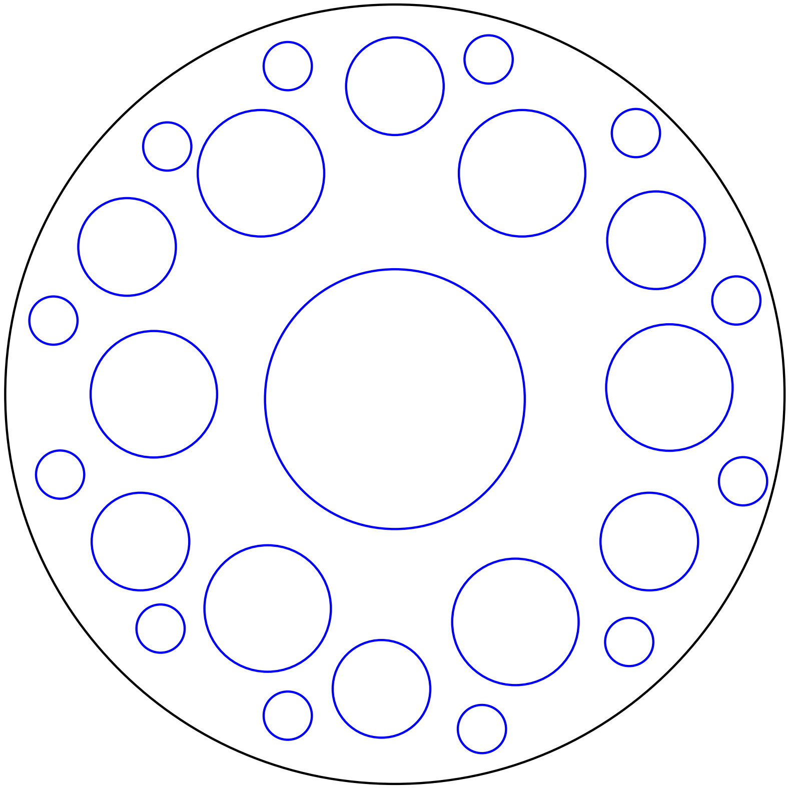

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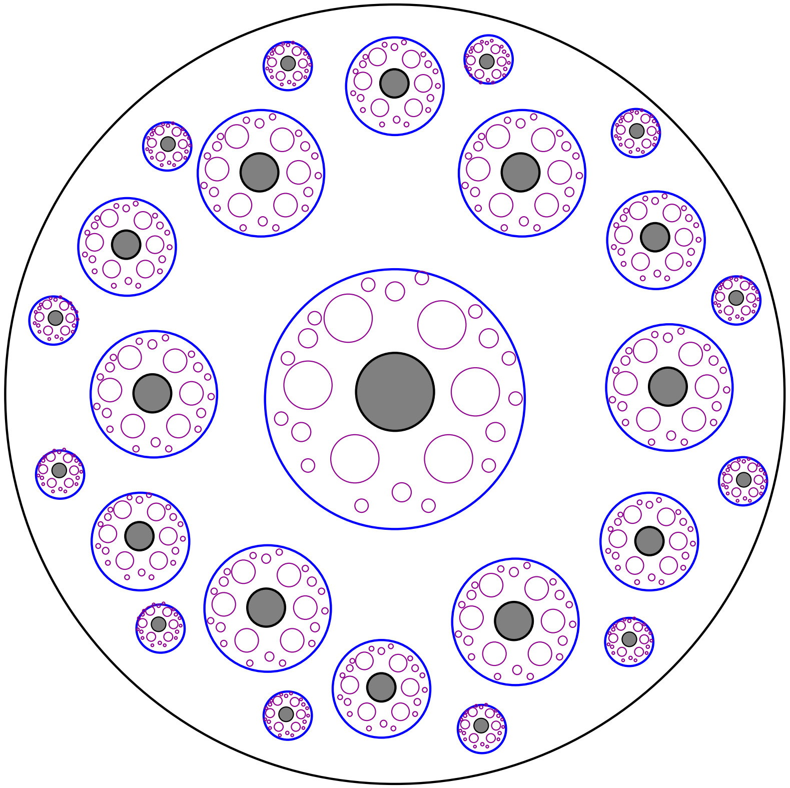

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