Understanding hyperbolic surfaces

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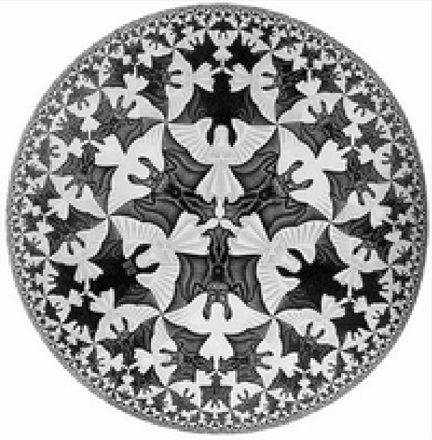
Tiling the hyperbolic plane



From tilings to hyperbolic surfaces

A tiling determines an identification by isometries between different areas of $\mathbb{H}^2$ corresponding to the same tile. So a group action on $\mathbb{H}^2$. The quotient is a *hyperbolic surface*, possibly with *orbifold points* (points where total angle $2\pi/k$.

Some remarks on hyperbolic surfaces.

- There are many ! (Much "more" than Euclidean surfaces).
- They stand at a "center" of maths : relations to complex analysis, algebraic geometry, representation theory, number theory, etc.
- Key point : understand the geometry of the space of hyperbolic metrics on a surface (Teichmüller space).

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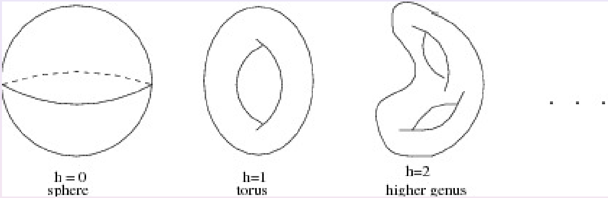
Hyperbolic surfaces

Closed (orientable) surfaces are classified by their genus.



- The sphere has a spherical metric ($K = 1$),
- The torus has (several) Euclidean metrics (the space of those metrics is... the hyperbolic plane),
- All other surfaces have (many) hyperbolic metrics.

Often useful to allow for point singularities : *cone singularities*, where the angle is $\theta < 2\pi$.
In particular, *orbifold points* where angle $2\pi/k$ (cf Porti's talk).

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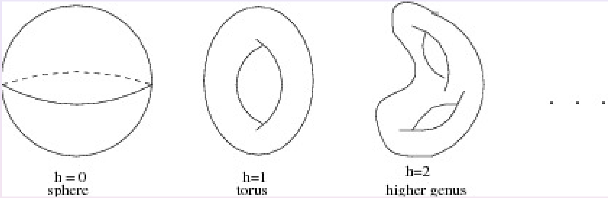

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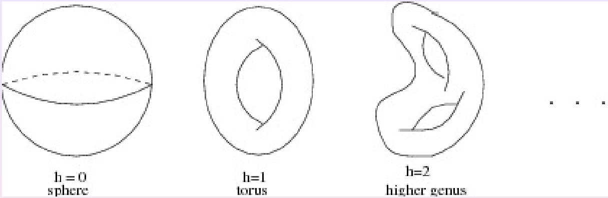

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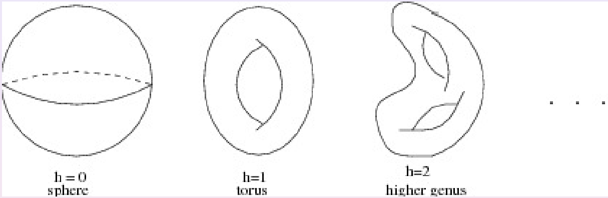

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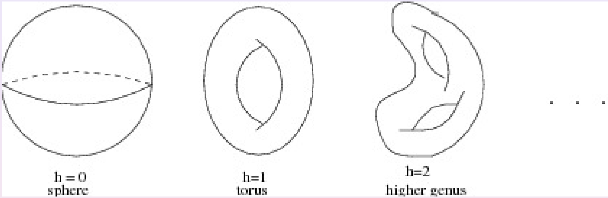

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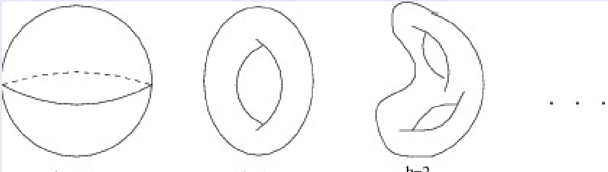

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Riemann surfaces

A *Riemann surface* is a surface with a complex structure. Modeled on $\mathbb{C}$, considered up to isotopies (=deformations).
A complex structure is the same (for *orientable* surfaces) as a conformal structure : enough to measure angles (not lengths).

Poincaré Uniformization Thm. Any conformal class of metrics on a surface of genus ≥ 2 contains a unique hyperbolic metric.

Extension to surfaces with cone singularities : choose where the cone points are, and their "angle".

Riemann Mapping Thm. Given $\Omega, \Omega' \subset \mathbb{C}$ simply connected domains, $\exists!$ conformal diffeo $\Omega \to \Omega'$, with given images for $x, y, z \in \partial\Omega$.

Discrete version based on circle packings (Koebe, Thurston), converges well to Riemann map (Rodin-Sullivan,Zhe,...).

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Motivations and applications

Conformal maps can be used for texture mapping. Picture : Kharevych, Springborn, Schröder. Conformality seems to be visually relevant. Discrete version $\Longrightarrow$ circle patterns $\Longrightarrow$ hyperbolic geometry.

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Teichmüller space

Def. The space of all hyperbolic metrics on a given surface (of genus $g \geq 2$), up to isotopy (deformation). $\mathcal{T}_g$.

Thm. $\mathcal{T}_g$ is topologically a ball, of dimension $6g - 6$.

Def. With n marked points : $\mathcal{T}_{g,n}$, topologically a ball of dimension $6g - 6 + 2n$. One can fix the cone angles, $\theta_i (< \pi)$.

Rich geometric structure on $\mathcal{T}_g$: Kähler (Weil-Petersson) metric + Finsler metric. Action on $\mathcal{T}_g$ of the *mapping-class group*.
Recent development : discrete/combinatorial models of $\mathcal{T}_g$, e.g. the "curve complex" (proof of Ending Lamination Conjecture, Minski).

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Fenchel-Nielsen coordinates

Consider a surface S of genus ≥ 2, with a *pant decomposition* : a family of n disjoint simple closed curves with complement pairs of pants.

Let g be a hyperbolic metric on S. The lengths of the curves define n numbers $l_1, \cdots, l_n$.

Moreover, the *gluing angles* define $\theta_1, \cdots, \theta_n$. The l_i, θ_i describe the metric up to diffeo.

Thm : the $\theta_i \in \mathbb{R}$ are well-defined on $\mathcal{T}_S$.

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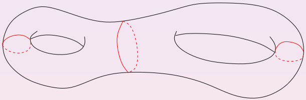

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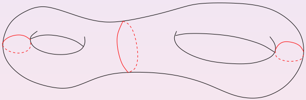

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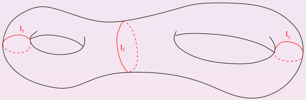

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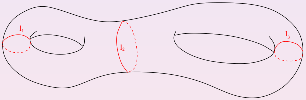

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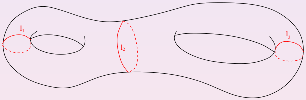

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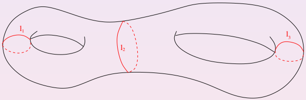

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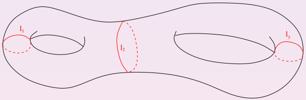

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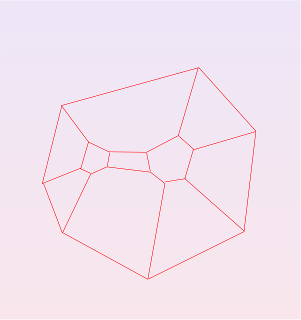
Angles of Delaunay tesselations

Let $x_1, \cdots, x_n \in S^2/\mathbb{R}^2/H^2$, the corresponding Delaunay tesselation defines an "ideal" circle pattern.

Consider the angles between the circles, and the incidence graph. Two conditions :

- the sum on the boundary of a face is 2π,
- the sum on any other simple closed curve is $> 2\pi$.

The Delaunay tesselation is then unique..



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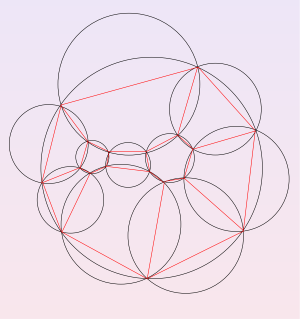

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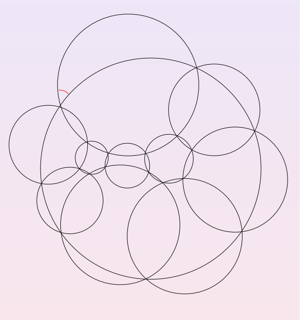

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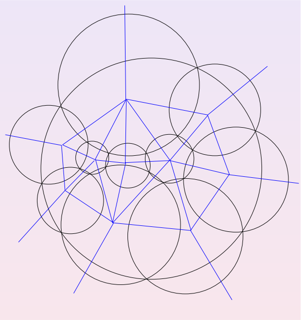

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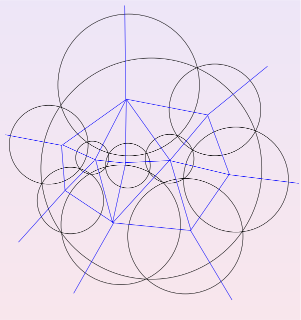

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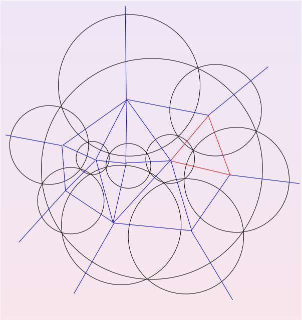

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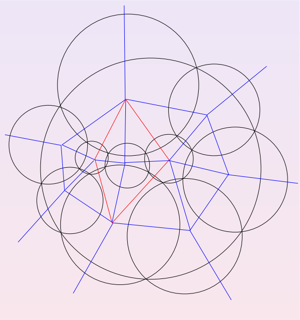

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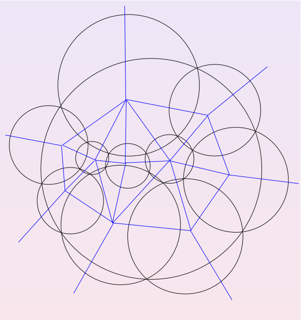

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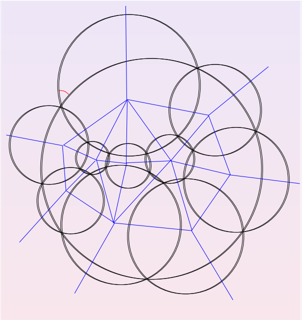

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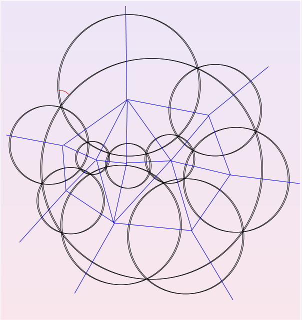

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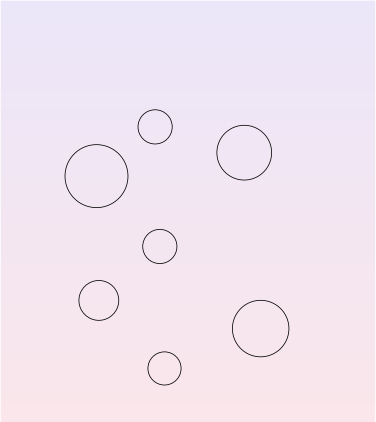
Delaunay tesselations are defined for a family of disjoint disks. ∃! tesselation with vertices at centers of disks, faces corresponding to circles orthogonal to the disks/adjacent vertices, with interior angle $< \pi/2$ with other disks.

They form an "hyperideal" circle pattern.



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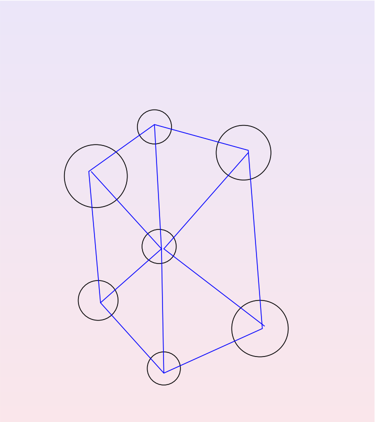

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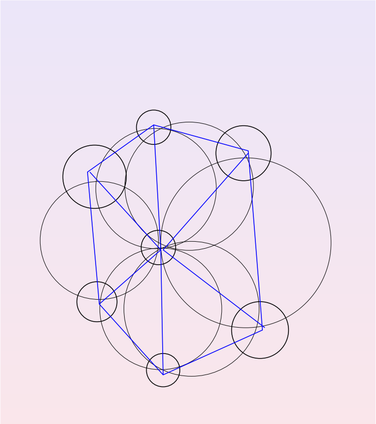

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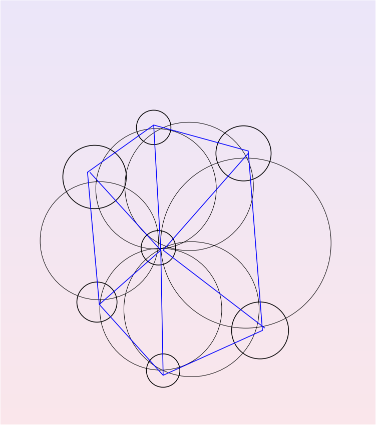

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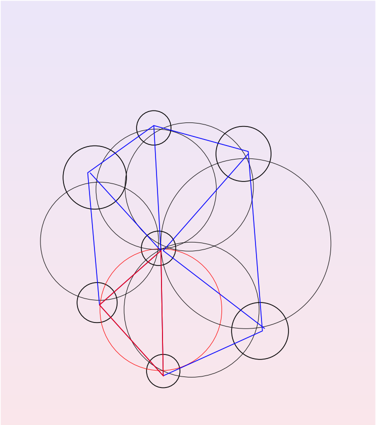

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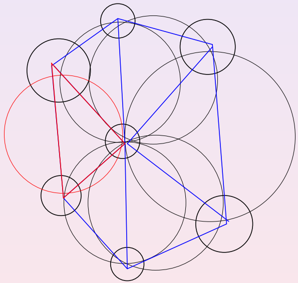

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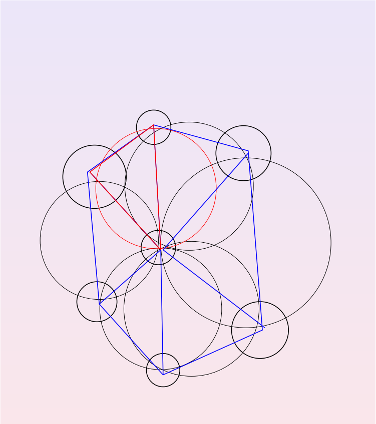

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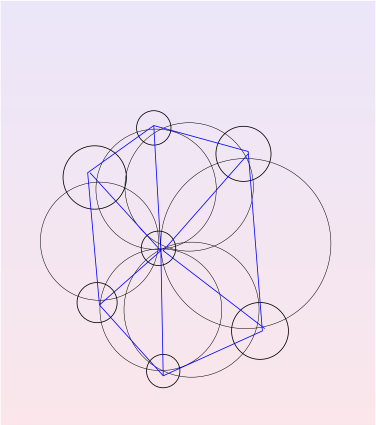

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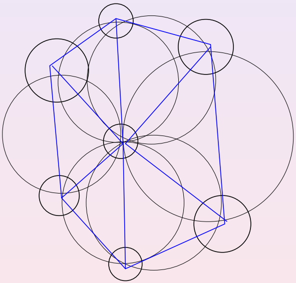

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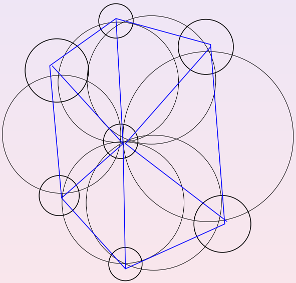

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