Growing Balls

Topology, Morse theory and Tessellations

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Subdivide and Tile

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Outline

Growing balls

Take a point set $\{P_1, \cdots, P_N\} \subset \mathbb{R}^n$. We assume general position. Assign a weight w_i to each point P_i.

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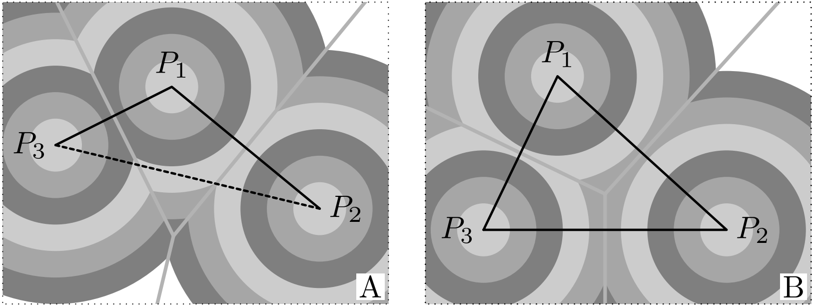
We show here a picture and next link to cabri

We are interested in the change of topology and geometry of the growing balls by increasing radius.



Local view

- topologically regular
- topological saddle point
- topological maximum
- two triangles
- vertex, topologically regular
-

Voronoi Power Diagram

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$$g_i(x) = \frac{1}{2}\|x - P_i\|^2 - \frac{1}{2}w_i \qquad g(x) = \min_{1 \le i \le N} g_i(x) \tag{1}$$

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Definition

For each of the P_i:

$$\mathrm{Pow}(P_i) = \mathrm{cl}\{x \in \mathbb{R}^n \mid g_i(x) = g(x) \text{ and } g_j(x) > g(x) \;\; j \neq i\}$$

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In case all $w_i = 0$ we have a tesselation with Voronoi cells.

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It is no restriction to assume that $w_i > 0$. We may add some number to all of the functions g_i: the cells will not change.

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Construct a dual tesselation.

- Vertices are the points $\{P_1, \cdots, P_N\}$,

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Note that *Pow*(α) and Del(α) lie in orthogonal hyperplanes of complementary dimension. The unique intersection point of these hyperplanes $c(\alpha)$ is the center of the circumscribed sphere of α in the spanning affine space

Critical points of the distance function

Let x be on the power diagram; then there is a set α (nearest neighbours) such that

$$g(x) = g_i(x) \Leftrightarrow P_i \in \alpha \quad ; \text{so} \quad x \in \text{Pow}(\alpha)$$

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From non-smooth critical point theory :

$$x \text{ critical iff } O \in \partial g(x)$$

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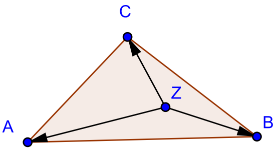
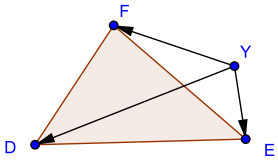
$x \in \mathrm{Del}(\alpha)$ critical

$x = \mathrm{Pow}(\alpha) \cap \mathrm{Del}(\alpha)$



$x \notin \mathrm{Del}(\alpha)$ non critical



The index of the critical point x is equal to $k = \dim < \mathrm{Del}(\alpha) >$.

Morse Formula

The following Morse formula is a non differentiable version of the 'mountaineering equation'.

Theorem

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$$\sum (-1)^i s_i = 1$$

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Proof.

g is a topological Morse function. In that case, as t grows, g passes through a number of non-degenerate critical values. When g passes a critical value of index i, an i-cell gets attached. In between we apply the (topological) regular interval theorem. For each intermediate function value t we have therefore :

$$\chi(\{g(x) \leq t\}) = \sum(-1)^i s_i(t)$$

The Morse poset

Critical points determine an active cell in the Delaunay tesselation. The set of active cells form the Morse poset: a combinatorial description of critical points.

PICTURES

- trangles
- 4-gon
- plane 4 points
- spatial ;
- tetrahedra .

Digression on tropical geometry

We will connect the theory of power diagrams with the concept of tropical hypersurface in tropical geometry.

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it connects algebraic geometry problems with combinatorial questions on certain polytopes.

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Recall that algebraic geometry studies varieties: the zero set of polynomials with real or complex coefficients in affine space.

Min-plus algebra

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- tropical addition: $x \oplus y := \min(x, y)$, and
- tropical multiplication: $x \otimes y := x + y$.

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Polynomials in tropical geometry are defined in the usual way. The "dictionary" from algebraic geometry to tropical geometry works as follows: The ordinary polynomial

$$x^3 + y^3 + 3xy + 3$$

has a tropical version:

$$\min\{3x, 3y, x + y + 3, 3\}$$

.

Tropical hypersurfaces

Tropical polynomials are piecewise linear concave functions on $\mathbb{R}^n$ with integer coefficients. The vertex set is defined by the exponents; in the example $(3, 0)$, $(0, 3)$, $(1, 1)$, $(0, 0)$.

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Also tesselations of the polytope of the vertex set appear in a natural way in tropical geometry.
Because of the relation $\max(x,y) = -\min(-x,-y)$ one could also have used the operation $x \oplus y = \max(x,y)$ to define a tropical semi-ring. Tropical hypersurfaces are defined by piecewise linear convex functions.

Amoeba

Tropical hypersurfaces appear also as follows:
Let $V \subset (\mathbb{C}^\star)^n$ be an algebraic variety. Recall that $\mathbb{C}^\star = \mathbb{C} - 0$ is the group of complex numbers under multiplication. Let $\mathrm{Log}\colon (\mathbb{C}^\star)^n \to \mathbb{R}^n$ be the "logarithmic moment-map" defined by

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Gelfand-Kapranov-Zelevinski defined the amoeba of an algebraic variety V as the image $A = \mathrm{Log}(V) \in \mathbb{R}^n$.
Tropical hypersurfaces are "spines" of amoebas of algebraic varieties. Mikhalkin showed that the spine is a certain limit of the amoeba and carries all topological information of the amoeba.
amoeba1.jpg ; amoeba2.jpg .

Affine approach

$$g_i(x) = \frac{1}{2}\|x - P_i\|^2 - \frac{1}{2}w_i = \frac{1}{2}\|x\|^2 - < x, P_i > + \frac{1}{2}\|P_i\|^2 - \frac{1}{2}w_i$$

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We can define the power diagram solely using affine functions:

$$\text{Pow}(\alpha) = cl\{x \in \mathbb{R}^n \mid f_i(x) = f(x) \ \ P_i \in \alpha \text{ and } f_j(x) < f(x) \ \ P_j \notin \alpha\}$$

Forman's Discrete Morse Theory

Definition

A function $h\colon \mathcal{T} \to \mathbb{R}$ is called a Forman discrete Morse function if for all $\beta \in \mathcal{T}$

$$\#\{\alpha \in \mathcal{T} \mid 1 + \dim(\alpha) = \dim(\beta) \;\; \alpha \subset \beta \;\; h(\alpha) \geq h(\beta)\} \leq 1$$

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In case both numbers are zero for some $\beta \in \mathcal{T}$, β is called critical.

Non-critical occur in pairs ; criticals are single.

Power distance as a discrete Morse function

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There is a polyhedral collapse involving 4 simplices.

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Theorem

- *$\hat{h}$ extends h as a (generalized) discrete Morse function , with polyheral collapses,*
- *$\hat{h}$ can be perturbed to a usual discrete Morse function*

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We made a travel through :

- Voronoi and Power Tesselations
- Morse Theory
- Tropical Geometry
- Discrete Morse Theory

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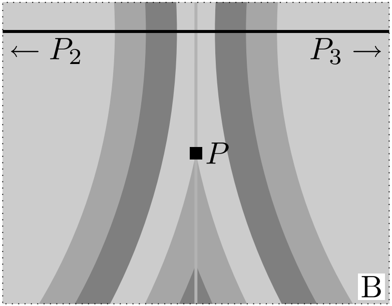
Thank you

figure2



topologically regular situation

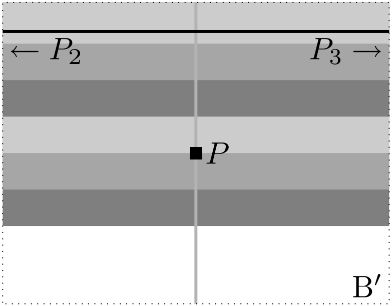
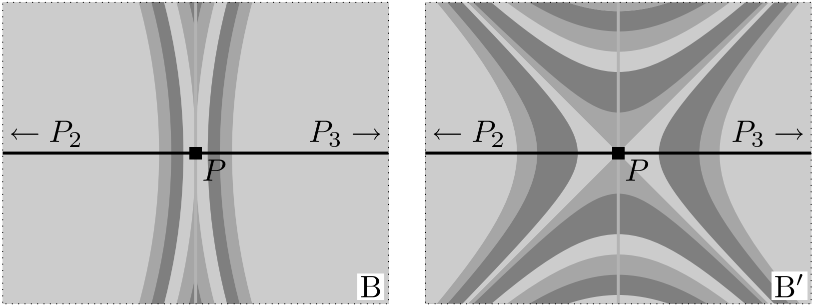
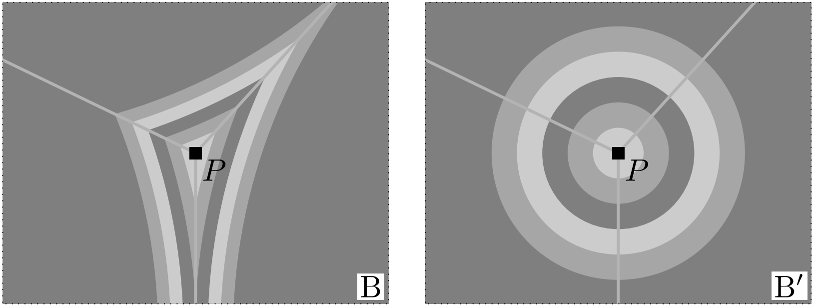


figure3



topological saddle point

figure6



topological maximum

figure5

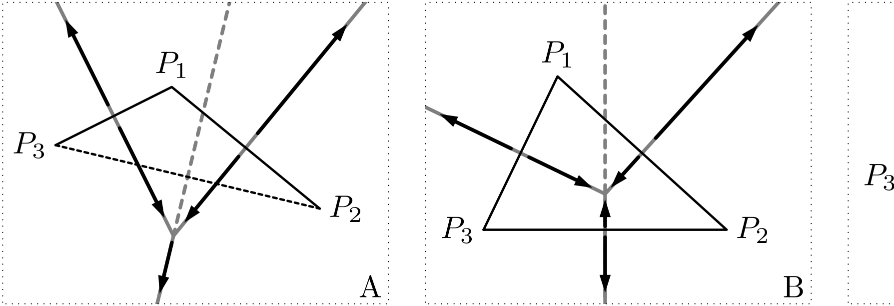
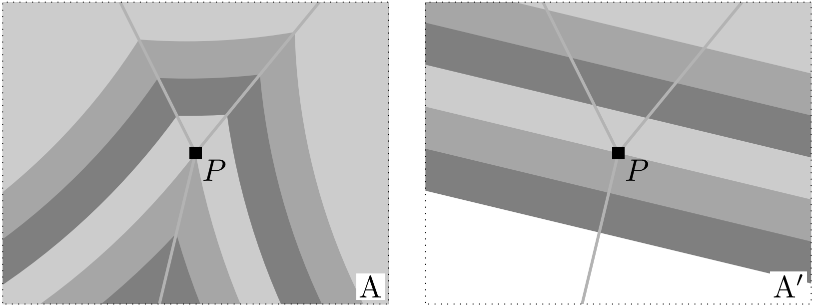
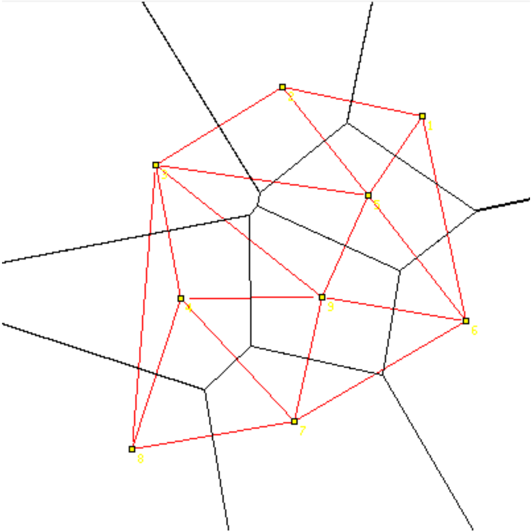


figure4

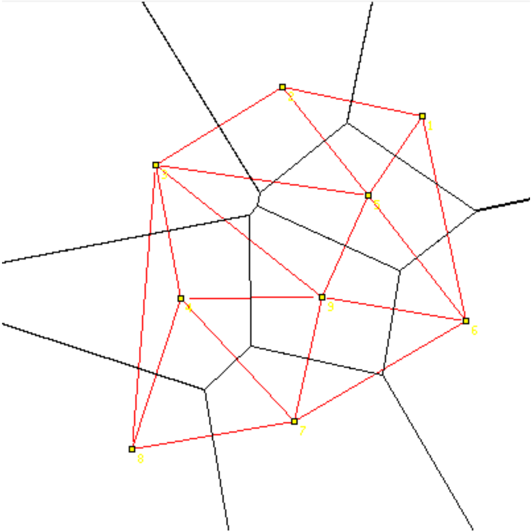


vertex, which is topologically regular

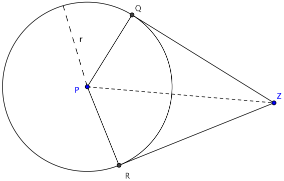
Voronoi-Delaunay



Voronoi-Delaunay



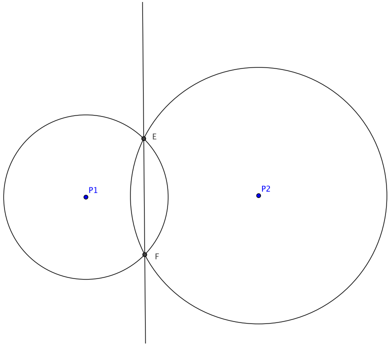
Power distance



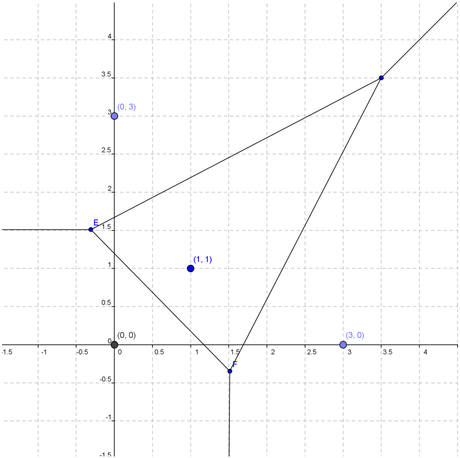
Pyhagoras:

$$|ZQ|^2 = |ZP|^2 - |PQ|^2 = \|z - P_i\|^2 - r^2$$

Power distance is square of tangent length (devided by 2).
Power line of two circles:


E
P1
P2
F

pictures



Amoeba

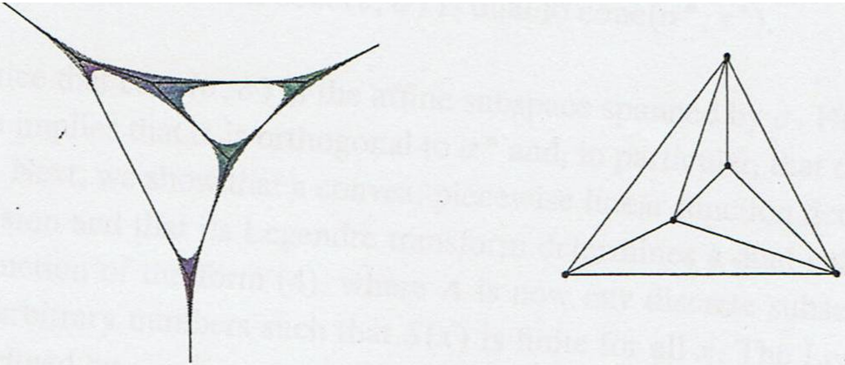


Figure 1. Amoeba of the polynomial $1 + z_1^5 + 80z_1^2z_2 + 40z_1^3z_2^2 + z_1^3z_2^4$ (shaded) together with its spine (solid) and the dual triangulation of the Newton polytope

Amoeba

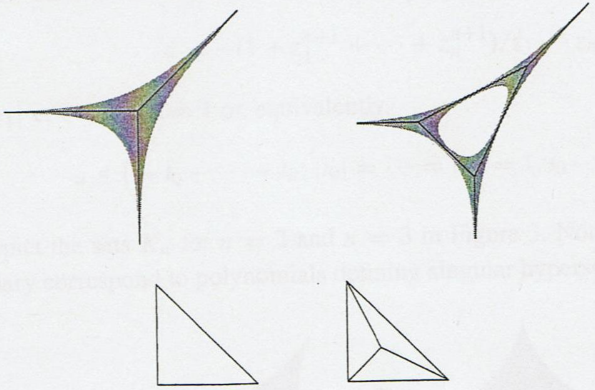
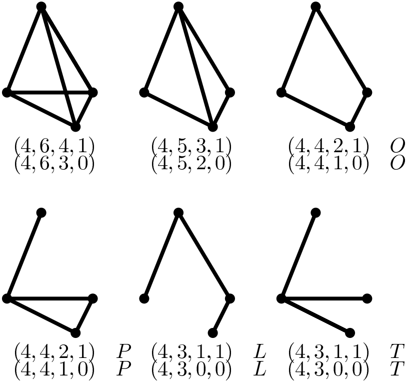


Figure 2. Amoebas, spines, and triangulated Newton polytopes of the polynomial $1 + z_1^3 + z_2^3 + az_1z_2$ for $a = 0$ and $a = -6$

Nine Morse Tetrahedra

a priori possibilites,
listed with number of critical points of index $0, 1, 2, 3$.

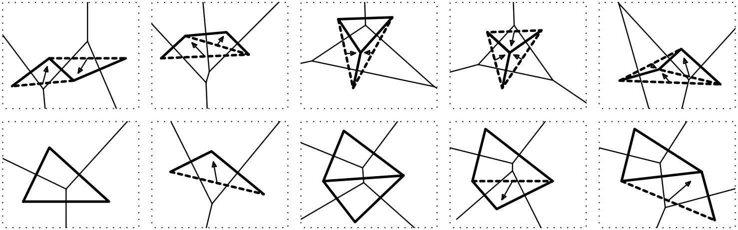


Theorem

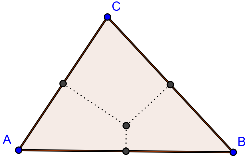
There are nine generic tetrahedra:
$(4,6,4,1)$, $(4,6,3,0)$, $(4,5,3,1)$, $(4,5,2,0)$, $(4,4,2,1)$ O, $(4,4,1,0)$ O, $(4,4,1,0)$ P, $(4,3,0,0)$ L *and* $(4,3,0,0)$ T,

2 dimensional Morse Posets

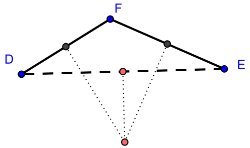
with 3 or 4 points



Two types of triangles

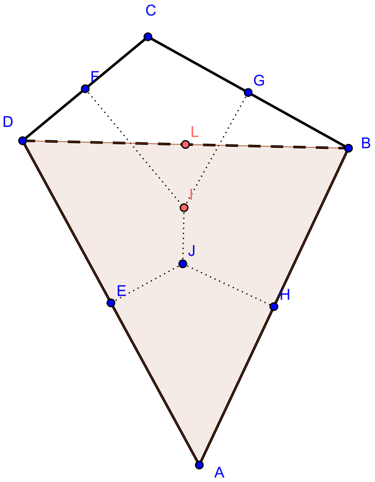


ACUTE triangle

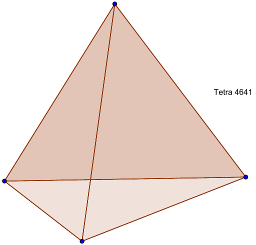


OBTUSE triangle

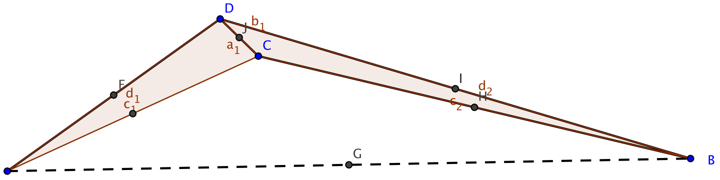
4-gon



tetra 4641

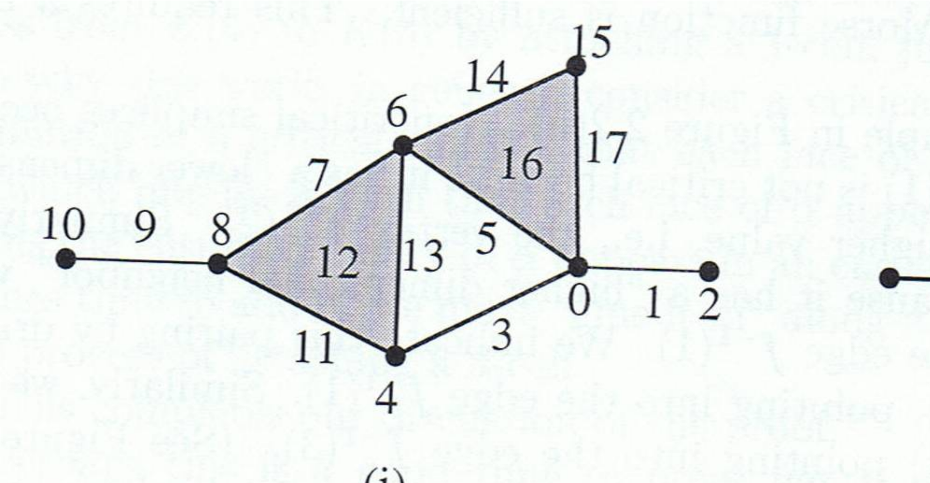


tetra 4520

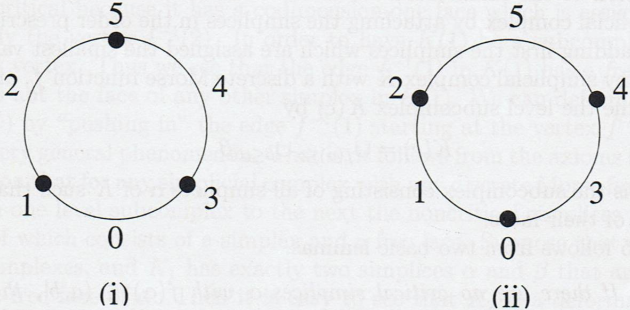


Polyhedral collaps

Basics of discrete Morse Theory

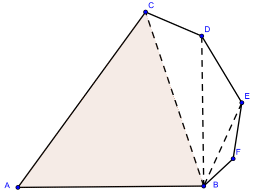


Example

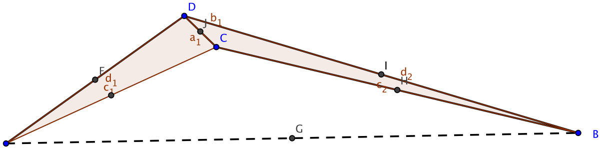


(i). This is not *a discrete Morse function. (ii). This is a discrete Morse function.*

example



collapse



Polyhedral collaps