Using the Maple `multires` package

July 10, 2003

Abstract

We present implementations on the computer system `Maple` for doing elimination with multivariate polynomials, in particular for computing resultant matrices. We give a brief overview of classical resultants, sparse resultants, residual resultants and determinantal resultants, as well as applications and examples using the presented library which we detail in the appendix.

1 Introduction

The aim of this tutorial is to give a computational overview of resultant theory and its applications through a presentation of a library implemented in `Maple` and called `multires`, which is developed by the GALAAD team at INRIA. We only present here the functions dealing with resultant matrices. All along the paper we give some code to run some trivial examples but also to work with some more involved problems. We hope to illustrate the particular geometric properties of the resultant based methods. They differ from other classical methods, such as Gröbner basis techniques, in the sense that we first need to analyze the geometry of the solutions in order to apply the correct resultant construction for which the system of equations is *generic*. Once this analysis is performed, we are able to tune the resultant construction to the geometry of the solutions and thus to build efficient and controlled algorithms. The analysis step is a preprocessing or *off-line* step, which may take time. On the contrary, the resolution step just requires to instantiate the parameters, which usually is very fast.

2 Classical resultants

2.1 Definition and main properties

The Macaulay resultant, introduced by F.S. Macaulay in [Mac02], corresponds to the direct generalization of the well-known Sylvester resultant of two bivariate homogeneous polynomials. For each $i = 0, \ldots, n$ we are given a homogeneous polynomial of degree $d_i \geq 1$ in the variables $\mathbf{x} = (x_0, \ldots, x_n)$,

$$f_i(\mathbf{x}) = \sum_{|\alpha|=d_i} \mathbf{c}_{i,\alpha}\mathbf{x}^\alpha,$$

where α is a n-tuple of non negative integers $(\alpha_0, \ldots, \alpha_n)$, $\mathbf{x}^\alpha$ denotes the monomial $x_0^{\alpha_0} \ldots x_n^{\alpha_n}$ and $\mathbf{c}_{i,\alpha}$ denotes the coefficients which are in a field $\mathbb{K}$. Considering all the coefficients $\mathbf{c}_{i,\alpha}$ as indeterminates, there exists an irreducible homogeneous polynomial in the ring $\mathbb{K}[\mathbf{c}_{i,\alpha} : |\alpha| = d_i, i = 0, \ldots, n]$ which is homogeneous for all $i = 0, \ldots, n$ in the set of variables $\{\mathbf{c}_{i,\alpha}, |\alpha| = d_i\}$ of degree $\frac{d_0 d_1 \ldots d_n}{d_i}$. This polynomial is the so-called projective (or classical) resultant and we denote it by Res. It gives a necessary and sufficient condition on the $\mathbf{c}_{i,\alpha}$'s such that $f_0, \ldots, f_n$ have a common root in $\mathbb{P}^n$. Indeed it satisfies the property: for any given polynomials $f_0, \ldots, f_n$ with coefficients $\mathbf{c}_{i,\alpha}$ in $\mathbb{K}$

$$\mathrm{Res}(f_0, \ldots, f_n) = 0 \Leftrightarrow \exists x \in \mathbb{P}^n : f_0(x) = \cdots = f_n(x) = 0.$$

We now describe the matrices used by F.S. Macaulay to compute explicitly this resultant. We do it in the affine setting by substituting $x_0 = 1$, $x_1 = t_1, \ldots, x_n = t_n$. Let $\nu = \sum_{i=0}^{n} d_i - n$ and $\mathbf{t}^F$ be the set of all monomials in $\mathbf{t}$ of degree $\leq \nu$. It contains $\binom{\nu+n}{n}$ elements. Let $t_n^{d_n}\mathbf{t}^{E_n}$ be the set of all monomials of $\mathbf{t}^F$ which are divisible by $t_n^{d_n}$. For $i = n-1, \ldots, 1$, we define by induction $t_i^{d_i}\mathbf{t}^{E_i}$ to be the set of all monomials of $\mathbf{t}^F \setminus (t_n^{d_n}\mathbf{t}^{E_n} \cup \ldots \cup t_{i+1}^{d_{i+1}}\mathbf{t}^{E_{i+1}})$ which are divisible by $t_i^{d_i}$. The set $\mathbf{t}^F \setminus (t_n^{d_n}\mathbf{t}^{E_n} \cup \ldots \cup t_1^{d_1}\mathbf{t}^{E_1})$ is denoted by $\mathbf{t}^{E_0}$ and is equal to

$$\mathbf{t}^{E_0} = \{t_1^{\alpha_1} \cdots t_n^{\alpha_n} : 0 \leq \alpha_i \leq d_i - 1\}.$$

It has $d_1 \cdots d_n$ monomials. If $E \subset \mathbb{N}^n$, $\langle \mathbf{t}^E \rangle$ denotes the vector subspace generated by the set $\mathbf{t}^E$. The resultant matrix $\mathtt{S}$ is the matrix in monomial bases of the linear map:

$$\begin{aligned} \mathcal{S} : \langle \mathbf{t}^{E_0} \rangle \times \cdots \times \langle \mathbf{t}^{E_n} \rangle \quad &\rightarrow \quad \langle \mathbf{t}^F \rangle \\ (q_0, \ldots, q_n) \quad &\mapsto \quad \sum_{i=0}^{n} q_i f_i. \end{aligned}$$

The determinant of $\mathtt{S}$ is generically not 0 (for it does not vanish when we specialize f_i to $t_i^{d_i}$) and has the same degree $\prod_{i=1}^{n} d_i$ as the resultant with respect to the coefficients of f_0. Therefore

$$\det(\mathtt{S}) = \mathrm{Res}(f_0, \ldots, f_n)\, \Delta(f_1, \ldots, f_n),$$

where $\Delta(f_1, \ldots, f_n)$ is a subminor of $\mathtt{S}$ depending only on the coefficients of $f_1, \ldots, f_n$ [Mac02].

Example. Suppose that one wants to compute the necessary and sufficient condition so that three general plane conics intersect. We can use `multires` in the affine setting:

```
>F:=[a_0*x^2+a_1*x*y+a_2*x+a_3*y^2+a_4*y+a_5,
    b_0*x^2+b_1*x*y+b_2*x+b_3*y^2+b_4*y+b_5,
    c_0*x^2+c_1*x*y+c_2*x+c_3*y^2+c_4*y+c_5];
>mresultant(F,[x,y]);
```

or use `Macaulay2` in the projective setting

```
>R=QQ[a_0..a_5,b_0..b_5,c_0..c_5,x,y,z]
>F=matrix{{a_0*x^2+a_1*x*y+a_2*x*z+a_3*y^2+a_4*y*z+a_5*z^2,
           b_0*x^2+b_1*x*y+b_2*x*z+a_3*y^2+b_4*y*z+b_5*z^2,
           c_0*x^2+c_1*x*y+c_2*x*z+a_3*y^2+c_4*y*z+c_5*z^2}}
>MacRes(F,{x,y,z})
```

The Macaulay resultants have been widely studied and have a lot of properties; a quasi-complete list can be found in the works of Jouanolou [Jou91, Jou97]. We point out that some more compact matrices have been discovered by Jouanolou to compute these resultants [Jou97]. Without giving any details on their construction we mention that these matrices can be computed in `multires` with the command `jresultant`:

```
>F:=[a_0*x^2+a_1*x*y+a_2*x*z+a_3*y^2+a_4*y*z+a_5*z^2,
    b_0*x^2+b_1*x*y+b_2*x*z+a_3*y^2+b_4*y*z+b_5*z^2,
    c_0*x^2+c_1*x*y+c_2*x*z+a_3*y^2+c_4*y*z+c_5*z^2];
>jresultant(F,[x,y]);
```

2.2 Examples and applications

In this subsection we mention some applications of Macaulay resultants. Our aim is not to give a complete exposition on these topics but only to mention some situations where resultants may be useful.

2.2.1 Solving polynomial systems

It is well-known that resultants can be used to solve polynomial systems. The most used technique is the one called "U-resultant" which was introduced by B. Van der Waerden in its book *"Modern algebra"* (volume II) [VdW48]. Starting from n homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{K}[x_0, \ldots, x_n]$, the U-resultant technique consists in adding a linear form L, depending on parameters U, to this system. The general case corresponds to the choice $L(x) = u_0x_0 + u_1x_1 + \ldots + u_nx_n$. The U-resultant of $f_1, \ldots, f_m$ is then defined as the Macaulay resultant

$$\mathrm{Res}(L, f_1, \ldots, f_n) \in \mathbb{K}[u_0, u_1, \ldots, u_n].$$

Assume that the algebraic variety $V(f_1, \ldots, f_n)$ in $\mathbb{P}^n$ consists of a finite set of points, then

$$\mathrm{Res}(L, f_1, \ldots, f_n) = c \prod_{p \in V(f_1, \ldots, f_n)} L(p)^{\mu_p},$$

where μ_p denotes the multiplicity of the point $p \in P$ and c is a non zero constant in $\mathbb{K}$.

Thus we can recover the solutions of our system $f_1, \ldots, f_n$ by factorizing the U-resultant:

Example. We want to compute the common roots of $f_1 = yz + x^2 + y^2$ and $f_2 = -z^2 + 2x^2 + 2y^2$. Using `Macaulay2`:

```
>R=QQ[u0,u1,u2,x,y,z]
>F=matrix{{y*z+x^2+y^2,-z^2+2*x^2+2*y^2,u0*x+u1*y+u2*z}}
>M=(MacRes(F,{x,y,z}))_0
>factor(det(MaxCol(M)))
```

we obtain

$$-(u_0^2 + u_1^2)(u_0 - u_1 + 2u_2)(u_0 + u_1 - 2u_2)$$

and deduce immediately the four isolated roots of this (very simple) system.

However, we point out that the more efficient methods avoid the explicit computation of the U-resultant and work with one of its matrix representation, for instance the Macaulay matrices. Solving polynomial systems is then reduced to eigenvectors or eigenvalues computations [Mou96]. We refer to [CLO98], §5 chapter 3, for a nice exposition on this topic and also for other techniques as the one called *hidden variable*. We mention also that it is not necessary to work with complete intersection points, similar results hold for any number of polynomials defining points in $\mathbb{P}^n$ [Laz81].

2.2.2 Implicitizing rational curves and surfaces without base points

Any rational plane curve is represented as the closed image of a rational map

$$\rho : \mathbb{P}^1 \to \mathbb{P}^2 : (s:t) \mapsto (f_0(s,t) : f_1(s,t) : f_2(s,t)),$$

where f_0, f_1 and f_2 are homogeneous polynomials in $\mathbb{K}[s,t]$ of the same degree $d \geq 1$. Dividing each polynomial f_i by the gcd of f_0, f_1 and f_2 if necessary we may assume that the parameterization ρ has no base point (a base point is a common projective root of f_0, f_1 and f_2). Consequently, the implicit equation of ρ, which is an irreducible and homogeneous polynomial $P(x,y)$, can be obtained from the equality

$$\mathrm{Res}(f_1 - xf_0, f_2 - yf_0) = P(x,y)^{\deg(\rho)},$$

where Res denotes the Sylvester resultant (a particular Macaulay resultant) and $\deg(\rho)$ the degree of the map ρ onto its image, that is the number of points in a generic fiber.

Example. The implicit equation of the parameterized plane curve (t^2, t^3) is simply obtained with `multires` by

```
>det(mresultant([t^2-x,t^3-y],[t]));
```

Similarly, Macaulay resultants can be used for implicitizing rational surfaces without base points. Such surfaces are obtained as the image of a regular map

$$\rho : \mathbb{P}^2 \to \mathbb{P}^3 : (s : t : u) \mapsto (f_0(s,t,u) : f_1(s,t,u) : f_2(s,t,u) : f_3(s,t,u)),$$

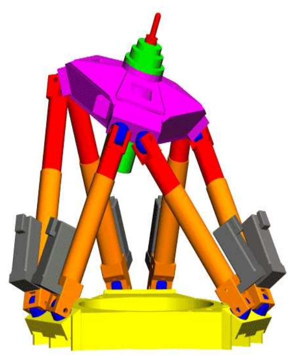
where f_0, f_1, f_2 and f_3 are homogeneous polynomials in $\mathbb{K}[s,t]$ of the same degree $d \geq 1$ without any common root (i.e. ρ has no base point). It follows that the implicit equation of ρ, which is an irreducible and homogeneous polynomial $P(x,y,z)$, can be computed by

$$\mathrm{Res}(f_1 - xf_0, f_2 - yf_0, f_3 - zf_0) = P(x,y,z)^{\deg(\rho)}.$$

We point out here the existence of *anisotropic resultants* [Jou96] which give a more efficient computational answer to this problem. We refer to [BEM03] for an overview on this point and other methods for implicitizing rational surfaces.

2.2.3 Stewart platforms

In this paragraph we show how Macaulay resultants can be used in the study of the so-called Stewart platforms which is practically a difficult problem. A Stewart platform looks like this:



We have six fixed points $(X_i)_{1 \leq i \leq 6}$ and six other points $(Y_i)_{1 \leq i \leq 6}$ attached to a solid platform which can move. Each point Y_i, for all $i = 1, \ldots, 6$, is joined to the point X_i. The problem of the Stewart platform is to compute all the possible positions of the platform if each leg (X_i, Y_i) has a given fixed length l_i, $i = 1, \ldots, 6$. It can be shown that this problem is equivalent to determine all the rotations R and the translations T in $\mathbb{P}^3$ satisfying the six following equations (we may suppose w.l.o.g. that $X_1 = Y_1 = 0$)

$$\begin{aligned}
&||T|| - l_1^2 = 0, \\
&2\langle R.Y_i, T\rangle - 2\langle T, X_i\rangle - 2\langle R.Y_i, X_i\rangle + \langle X_i, X_i\rangle + \langle Y_i, Y_i\rangle + l_1^2 - l_i^2 = 0, \quad i = 2, \ldots, 6.
\end{aligned}$$

It is known that these equations define 40 complex solutions [RV92, Laz92, Mou93]. In the following we use Macaulay resultants to reduce this problem to solving a polynomial system of 4 equations of resp. degree 6,6,4 and 4.

We set $T = [p_x : p_y : p_z]$ and denote by p the homogenizing variable in $\mathbb{P}^3$. All the rotations R in $\mathbb{P}^3$ can be described by

$$R = \frac{1}{\Delta}\begin{pmatrix} {c_1}^2 - {c_3}^2 - {c_2}^2 + 1 & 2\,c_1\,c_2 - 2\,c_3 & 2\,c_3\,c_1 + 2\,c_2 \\ 2\,c_1\,c_2 + 2\,c_3 & 1 - {c_1}^2 + {c_2}^2 - {c_3}^2 & 2\,c_2\,c_3 - 2\,c_1 \\ 2\,c_3\,c_1 - 2\,c_2 & 2\,c_2\,c_3 + 2\,c_1 & 1 - {c_1}^2 - {c_2}^2 + {c_3}^2 \end{pmatrix},$$

where $\Delta = c_0^2 + c_1^2 + c_2^2 + c_3^2$ and c_0 is the homogenizing variable. Introducing the vector $Q = R^tT = [q_x : q_y : q_z]$ which satisfies $(c_0\mathrm{Id}+C)Q = (c_0\mathrm{Id}-C)T$, we can rewrite our system with the 9 equations:

$$\begin{aligned} c_0q_x - c_3q_y + c_2q_z - c_0p_x - c_3p_y + c_2p_z &= 0, \\ c_3q_x + c_0q_y - c_1q_z + c_3p_x - c_0p_y - c_1p_z &= 0, \\ -c_2q_x + c_1q_y + c_0q_z - c_2p_x + c_1p_y - c_0p_z &= 0, \\ ||T|| - l_1^2 &= 0, \\ 2\langle Y_i, Q\rangle - 2\langle T, X_i\rangle - 2\langle R.Y_i, X_i\rangle + \langle X_i, X_i\rangle + \langle Y_i, Y_i\rangle + l_1^2 - l_i^2 &= 0, \quad i = 2, \ldots, 6. \end{aligned}$$

We would like to eliminate base points defined by the variables $p_x, p_y, p_z, q_x, q_y, q_z, p$ of this system of 9 equations. To do it we can compute the Macaulay resultants of 7 equations (in 7 homogeneous variables) from our 9 equations. Each such resultant has an extraneous factor of the form $\Delta^\alpha * p^\beta$ that we have to divide out. Here is the Maple source code using `multires` :

```
read(`multires`):
N2:=proc(A) normal(A[1]^2 + A[2]^2 + A[3]^2): end:
scprod:=proc(A,B) dotprod(evalm(A),evalm(B),'orthogonal') end:

#Some random points:
X[1]:=[0,0,0]:             X[2]:=[5,0,0]:
X[3]:=[12,-15,0]:          X[4]:=[18,-6,3]:
X[5]:=[20,1,-3]:           X[6]:=[10,8,5]:
Y[1]:=[0,0,0]:             Y[2]:=[4,0,0]:
Y[3]:=[8,-6,0]:            Y[4]:=[13,-3,-5]:
Y[5]:=[14,5,2]:            Y[6]:=[6,10,3]:
l:=[14,12,17,15,23,19]:

Id:=matrix([[c0,0,0],[0,c0,0],[0,0,c0]]):
C:=matrix([[0,-c3,c2],[c3,0,-c1],[-c2,c1,0]]):
R:=matrix([[c1^2-c3^2-c2^2+c0^2,2*(c1*c2-c3*c0),2*(c3*c1+c2*c0)],
   [2*(c1*c2+c3*c0),(c0^2-c1^2+c2^2-c3^2),2*(c2*c3-c1*c0)],
   [2*(c3*c1-c2*c0),2*(c2*c3+c1*c0),(c0^2-c1^2-c2^2+c3^2)]]):
T:=[px,py,pz]:
Q:=[qx,qy,qz]:
Rd:=c0^2+c1^2+c2^2+c3^2:

#The 9 equations:
S[1] := N2(T)-p^2*l[1]^2:
for i from 2 to 6 do
S[i] := expand( (N2(X[i])+N2(Y[i])-l[i]^2+l[1]^2)*p*Rd+
        2*Rd*scprod(Y[i],Q)-2*p*scprod(R&*Y[i],X[i])
        -2*Rd*scprod(X[i],T)):
od:
```

```
for i from 7 to 9 do
S[i]:=expand(evalm(evalm((Id+C)&*Q)-evalm((Id-C)&*T)))[i-6]:
od:

#The 4 needed Macaulay resultants:
M1:=jresultant([S[1],S[2],S[3],S[4],S[5],S[6],S[7]],
               [qx,qy,qz,px,py,pz,p]):
M2:=jresultant([S[1],S[2],S[3],S[4],S[5],S[6],S[8]],
               [qx,qy,qz,px,py,pz,p]):
M3:=jresultant([S[2],S[3],S[4],S[5],S[6],S[7],S[9]],
               [qx,qy,qz,px,py,pz,p]):
M4:=jresultant([S[2],S[3],S[4],S[5],S[7],S[8],S[9]],
               [qx,qy,qz,px,py,pz,p]):
P1:=expand( factor(det(M1))/((Rd^8*p^2))): #degree 6
P2:=expand( factor(det(M2))/((Rd^8*p^2)) ): #degree 6
P3:=expand( factor(det(M3))/((Rd^4*p)) ): #degree 4
P4:=expand( factor(det(M4))/((Rd^3*c0*p)) ): #degree 4
```

One can check that these four equations $P1, P2, P3, P4$ define an homogeneous ideal in the variables c_0, c_1, c_2, c_3 of codimension 3 and degree 40, i.e. our initial problem. In this way the resolution of the Stewart platform is reduced to the resolution of an over-determined polynomial system which can be achieved via the U-resultant technique, see for instance section 2.2. We point out how to use Bezoutian matrices and factorization algorithms to recover the solutions in one variable, say c_1, illustrating the so-called *hidden variable* method. Always using `multires` in `Maple`:

```
>  t:= melim(subs(c0=1,[P1,P3,P4]), [c2,c3]):
> degree(t); #returns 96
> ft:=  factor(t):
> map(degree,[op(ft)]);  #returns [0, 32, 40, 4, 20]
> sort(op(3,ft));
```

The result is a polynomial in the variable c_1 of degree 40 (that we can not reasonably print here).

3 Sparse resultants

In this section we recall briefly the construction of sparse (or toric) resultants and illustrate it with simple examples, using the function `spresultant`[1] in the `Maple` library `multires`.

Instead of considering homogeneous polynomials as in the previous section, we consider Laurent polynomials $f_i(\mathbf{t})$ (where $\mathbf{t} = (t_1, \ldots, t_n)$) with support into a fixed set $A_i \subset \mathbb{Z}^n$. They are of the form

$$f_i(\mathbf{t}) = \sum_{\alpha \in A_i} \mathbf{c}_{\alpha,i}\, \mathbf{t}^\alpha \quad , \quad i = 0 \ldots n.$$

Let X be the *toric variety* associated with the Minkowski sum of the supports $A = A_0 \oplus \cdots \oplus A_n$, it may be defined as the closure in $\mathbb{P}^N$ of the map

$$\begin{array}{rcl} \sigma : (\mathbb{C}^*)^n & \to & \mathbb{P}^N \\ \mathbf{t} & \mapsto & (\mathbf{t}^\alpha)_{\alpha \in A}. \end{array} \tag{1}$$

where $N = |A| - 1$. When the Minkowski sum A is n-dimensional, the sparse resultant, denoted Res, is well defined [CLO98, GKZ94, PS93], it is an irreducible (and multi-homogeneous) polynomial in

[1] this function, and all the other ones needed, have been developed by Ioannis Emiris.

all the coefficients $\mathbf{c}_{\alpha,i}$'s vanishing for a given specialization if and only if the corresponding Laurent polynomials f_i have a common root in X; in particular it vanishes as soon as these Laurent polynomials have a common root in the torus $(\mathbb{C}^*)^n$. When the f_i are completely dense, then we recover the projective resultant studied earlier.

When A generates $\mathbb{Z}^n$ as a $\mathbb{Z}$-module, we know the multi-degree of Res: its degree with respect to the coefficients of f_i is the *mixed volume* of $\{A_j\}_{j\neq i}$, that is the coefficient of $\prod_{j\neq i}\lambda_j$ in

$$\mathrm{Vol}\Big(\sum_{j\neq i}\lambda_i\,A_i\Big) = \mathrm{MV}\big(\{A_j\}_{j\neq i}\big)\prod_{j\neq i}\lambda_j + \cdots$$

where Vol denotes the usual Euclidean volume. When the supports correspond to simplices of edge length d_i, then their mixed volume equals $\prod_i d_i$.

Methods for constructing a Sylvester-type matrix are based on geometric properties of the supports A_i. See [CE93, CE00, CP93] for more details. They adopt the following scheme: For any polytope $A \subset \mathbb{Z}^n$ and for any non-zero vector $\delta \in \mathbb{R}^n$, let A^δ denote the set of integer points of A which are not on facets F of A such that the scalar product $n_F \cdot \delta > 0$, where n_F is the exterior normal vector of F. Consider now the following (well-defined) linear transformation

$$\begin{array}{rrcl} \tilde{\mathcal{S}}: & \langle \mathbf{t}^{E_0}\rangle \times \cdots \times \langle \mathbf{t}^{E_n}\rangle & \to & \langle \mathbf{t}^{F}\rangle \\ & (q_0,\ldots,q_n) & \mapsto & \displaystyle\sum_{i=0}^{n} q_i f_i, \end{array}$$

where $E_i = (\oplus_{i\neq j}A_i)^\delta, F = A^\delta$. Now we assume that the perturbation vector δ is sufficiently generic. Moreover, a sufficiently generic (affine) lifting is used to define a regular subdivision of A, called the mixed subdivision. Exploiting the properties of this subdivision, it is possible to extract from $\tilde{\mathcal{S}}$ a square matrix $\mathtt{S}(\mathbf{c})$, such that its determinant is not generically 0 and such that its degree in the coefficients of f_0 is exactly the mixed volume of $A_1,\ldots,A_n$ [CE93, CE00]. Therefore, its determinant is a non-trivial multiple of Res, the extraneous factor depending only on the coefficients of $f_1,\ldots,f_n$. Hence, the sparse resultant could be computed as the GCD of at most $n+1$ such determinants.

The algorithm in [D'A02] proposes a modification of the above algorithm in order to also define a submatrix whose determinant is the extraneous factor in $\det \mathtt{S}(\mathbf{c})$; this yields a Macaulay-type rational formula for the sparse resultant. The subdivision-based algorithm also yields information on the support of the sparse resultant [Stu94]. This information was used in [EK03] in order to predict the support of the implicit equation of a parametric (hyper)surface.

When the input coefficients are specialized, it is possible that the sparse resultant, or the determinant of its matrix, degenerates to zero. An infinitesimal perturbation is defined in [DE01], based on the mixed subdivision, so that the trailing coefficient of the perturbed determinant yields a nontrivial multiple of the sparse resultant. This randomized perturbation is implemented in `multires`.

Example. We illustrate this construction with a very simple example where the Macaulay resultant is degenerate. We refer to [BEM03] §4.6 for a nice application of the function `spresultant` in order to compute an implicit equation of a *Pillow patch*, problem occurring in Computer Aided Geometric Design.

We consider the system

$$\left\{\begin{array}{rcl} f_0 &=& a_0 + a_1x + a_2y + a_3xy \\ f_1 &=& b_0 + b_1x + b_2y + b_3xy \\ f_2 &=& c_0 + c_1x + c_2y + c_3xy. \end{array}\right.$$

It is easy to check that its Macaulay resultant is identically 0 (for there are two base points at infinity). However the sparse resultant of this system is not identically zero. We compute it below using `Maple`; the sparse resultant is a factor of the matrix determinant.

```
>spresultant([a_0+a_1*x+a_2*y+a_3*x*y,
             b_0+b_1*x+b_2*y+b_3*x*y,
             c_0+c_1*x+c_2*y+c_3*x*y],[x,y]);
```

Example. We illustrate the sparse resultant and its perturbation on a simple example of surface implicitization from [Bus01b], in the affine setting.

```
>f_0:=s^3+t^3: f_1=s^2: f_2=s^3: f_3=t^2:
>spresultant([f_1-x*f_0, f_2-y*f_0, f_3-z*f_0] ,[s,t]);
```

The surface has the base point $(0,0)$, of multiplicity 4, so the matrix determinant yields a nontrivial multiple of the sparse resultant, the latter being equal to the implicit equation $z^3y^2 - x^3y^2 + 2x^3y - x^3$.

Under the change of variable $t \rightarrow t-1$ the new system has zero sparse resultant. The determinant of the perturbed resultant matrix has a trailing coefficient, with respect to the infinitesimal variable, which is precisely the implicit equation. To apply the perturbation, the user must set the following global variable:

```
>PERT_DEGEN_COEFS := 1;
>spresultant([f_1-x*f_0, f_2-y*f_0, f_3-z*f_0] ,[s,t]);
```

4 Residual resultants

4.1 Definition and main properties

The residual resultant is a recent extension of the classical resultant theory [BEM00, BEM01, Bus01b, Bus01a]. Consider a polynomial system depending on parameters. In many situations coming from practical problems, polynomial systems depending on parameters have common zeros which do not depend on these parameters, and which we are not interested in. We are going to present here how to compute a resultant in such a situation, which is called a residual resultant, under suitable assumptions.

Let $g_1, \ldots, g_m$ be m homogeneous polynomials of degree $k_1 \geq \ldots \geq k_m \geq 1$ in $S = \mathbb{K}[x_0, \ldots, x_n]$. Being given $n+1$ integers $d_0 \geq \ldots \geq d_n \geq k_1$ such that $d_m \geq k_m + 1$, there exists a resultant (called a residual resultant) associated to systems of the form:

$$\mathbf{f_c} := \left\{ \begin{array}{lcl} f_0(\mathbf{x}) & = & \sum_{i=1}^m h_{i,0}(\mathbf{x})\, g_i(\mathbf{x}) \\ & \vdots & \\ f_n(\mathbf{x}) & = & \sum_{i=1}^m h_{i,n}(\mathbf{x})\, g_i(\mathbf{x}) \end{array} \right. \tag{2}$$

where $h_{i,j}(\mathbf{x}) = \sum_{|\alpha|=d_j-k_i} c_\alpha^{i,j}\, \mathbf{x}^\alpha$ is a homogeneous polynomial of degree $d_j - k_i$. It is an irreducible homogeneous polynomial in the ring of coefficients $\mathbb{K}[c_\alpha^{i,j}]$. Being given some specialized polynomials $f_0, \ldots, f_n$, we have the property

$$\exists\, x \notin V(g_1, \ldots, g_m) : f_0(x) = \cdots = f_n(x) = 0 \Rightarrow \mathrm{Res}(f_0, \ldots, f_n) = 0.$$

Notice that the polynomials $g_1, \ldots, g_m$ describe exactly the variety of base points we are not interested in. Notice also that this last property can be stated as an equivalence on what are called blow-up varieties, but we are not going to describe them here, we refer to [BEM01, Bus01a] for more details.

We now show how it is possible to compute these residual resultants.

4.1.1 General residual resultants

Whatever the base points are, that is to say whatever the polynomials $g_1, \ldots, g_m$ are, it is always possible to compute a non zero multiple of the residual resultant using Bezoutian matrices (see [BEM00, Bus01a]).

The Bezoutian $\Theta_{f_0,\ldots,f_n}$ of $f_0,\ldots,f_n \in S$ is the element of $S \otimes_{\mathbb{K}} S$ defined by

$$\Theta_{f_0,\ldots,f_n}(\mathbf{t},\mathbf{z}) := \begin{vmatrix} f_0(\mathbf{t}) & \theta_1(f_0)(\mathbf{t},\mathbf{z}) & \cdots & \theta_n(f_0)(\mathbf{t},\mathbf{z}) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(\mathbf{t}) & \theta_1(f_n)(\mathbf{t},\mathbf{z}) & \cdots & \theta_n(f_n)(\mathbf{t},\mathbf{z}) \end{vmatrix},$$

where

$$\theta_i(f_j)(\mathbf{t},\mathbf{z}) := \frac{f_j(z_1,\ldots,z_{i-1},t_i,\ldots,t_n) - f_j(z_1,\ldots,z_i,t_{i+1},\ldots,t_n)}{t_i - z_i}.$$

Let $\Theta_{f_0,\ldots,f_n}(\mathbf{t},\mathbf{z}) = \sum \theta_{\alpha\beta}\, \mathbf{t}^\alpha \mathbf{z}^\beta, \theta_{\alpha,\beta} \in \mathbb{K}$. The Bezoutian matrix of $f_0,\ldots,f_n$ is defined as the matrix $B_{f_0,\ldots,f_n} = (\theta_{\alpha\beta})_{\alpha,\beta}$. And we have:

Theorem 4.1 *Any maximal minor of the Bezoutian matrix $B_{f_0,\ldots,f_n}$ is divisible by the resultant* $\mathrm{Res}(f_0,\ldots,f_n)$.

Notice that we do not need to know the polynomials $g_1,\ldots,g_m$ to perform the computation of the Bezoutian matrix. In fact the only thing we have to check is that the polynomials $f_0,\ldots,f_m$ separate points and tangent vectors on an open subset of $\mathbb{P}^n$ (see [BEM00] for more details on this point).

Example. Consider the three following polynomials ([BEM00], example 1.5):

$$\begin{cases} f_0 = c_{0,0} + c_{0,1}t_1 + c_{0,2}t_2 + c_{0,3}({t_1}^2 + {t_2}^2) \\ f_1 = c_{1,0} + c_{1,1}t_1 + c_{1,2}t_2 + c_{1,3}({t_1}^2 + {t_2}^2) + c_{1,4}({t_1}^2 + {t_2}^2)^2 \\ f_2 = c_{2,0} + c_{2,1}t_1 + c_{2,2}t_2 + c_{2,3}({t_1}^2 + {t_2}^2) + c_{2,4}({t_1}^2 + {t_2}^2)^2. \end{cases}$$

Using the command `mbezout` of `multires` we can compute the Bezoutian matrix, which is of size 12×12 and of rank 10. The determinant of a maximal minor yields a huge polynomial in $(c_{i,j})$ containing 207805 monomials. It can be factorized as $q_1 q_2 (q_3)^2 \rho$, with

$$\begin{array}{rcl} q_1 & = & -c_{0,2}c_{1,3}c_{2,4} + c_{0,2}c_{1,4}c_{2,3} + c_{1,2}c_{0,3}c_{2,4} - c_{2,2}c_{0,3}c_{1,4} \\ q_2 & = & c_{0,1}c_{1,3}c_{2,4} - c_{0,1}c_{1,4}c_{2,3} - c_{1,1}c_{0,3}c_{2,4} + c_{2,1}c_{0,3}c_{1,4} \\ q_3 & = & {c_{0,3}}^2{c_{1,1}}^2{c_{2,4}}^2 - 2{c_{0,3}}^2c_{1,1}c_{2,1}c_{2,4}c_{1,4} + {c_{0,3}}^2{c_{2,4}}^2{c_{1,2}}^2 + \cdots \\ \rho & = & {c_{2,0}}^4{c_{1,4}}^4{c_{0,2}}^4 + {c_{2,0}}^4{c_{1,4}}^4{c_{0,1}}^4 + {c_{1,0}}^4{c_{2,4}}^4{c_{0,2}}^4 + {c_{1,0}}^4{c_{2,4}}^4{c_{0,1}}^4 + \cdots \end{array}$$

The polynomials q_3 and ρ contain respectively 20 and 2495 monomials. As for generic equations f_0, f_1, f_2, the number of points in the varieties $\mathcal{Z}(f_0,f_1)$, $\mathcal{Z}(f_0,f_2)$, $\mathcal{Z}(f_1,f_2)$ is 4 (see for instance [Mou96]), $\mathrm{Res}(f_0,f_1,f_2)$ is homogeneous of degree 4 in the coefficients of each f_i. Thus, it corresponds to the last factor ρ.

4.1.2 Residual resultants of a complete intersection

We suppose here that the ideal $G = (g_1,\ldots,g_m)$ is a *complete intersection*, that is defines a variety of codimension m in $\mathbb{P}^n$. In this particular case we know how to compute exactly the residual resultant and also its degree. Indeed, its degree in the coefficients $(c_\alpha^{i,j})$ of each f_j is given by

$$N_j = \frac{P_{m_j}}{P_1}(k_1,\ldots,k_m)$$

where, $m_j(T) = \sigma_n(\mathbf{d}) + \sum_{l=m}^n \sigma_{n-l}(\mathbf{d})\, T^l$, with the notations $\mathbf{d} = (d_0,\ldots,d_{j-1},d_{j+1},\ldots,d_n)$, $\sigma_0(\mathbf{d}) = (-1)^n$, $\sigma_1(\mathbf{d}) = (-1)^{n-1}\sum_{l\neq j} d_l$, $\sigma_2(\mathbf{d}) = (-1)^{n-2}\sum_{j_1\neq j, j_2\neq j, j_1<j_2} d_{j_1}d_{j_2}$, $\ldots$, $\sigma_n(\mathbf{d}) = \prod_{l\neq j} d_l$, and

$$P_{m_j}(y_1,\ldots,y_m) = \det\begin{pmatrix} m_j(y_1) & \cdots & m_j(y_m) \\ y_1 & \cdots & y_m \\ \vdots & & \vdots \\ y_1^{m-1} & \cdots & y_m^{m-1} \end{pmatrix}.$$

Example As an example we suppose that $n=3$ and $m=2$. Then we can obtain the critical degree and the multi-degree of the residual resultant with `Macaulay2`:

```
>R=ZZ[d_0..d_4,k_1,k_2]:
>CiResDeg({d_0,d_1,d_2,d_3},{k_1,k_2})
```

With `Maple` we can obtain partial multi-degree: for instance the degree of the residual resultant in the coefficient of the polynomial f_0 is given by

```
>bkmdegree([d1,d2,d3],[k1,k2]);
```

which returns $(-d_3-d_2-d_1+k_2+k_1)k_1k_2+d_1d_2d_3$.

We denote by H the matrix $(h_{i,j})_{1\le i\le m, 0\le j\le n}$ and by $\Delta_{i_1\ldots i_m}$ the $m\times m$ minors of H corresponding to the columns $i_1,\ldots,i_m$. We also define the homogeneous ideal $F=(f_0,\ldots,f_n)\subset S$.

Theorem 4.2 *For any $\nu\ge\sum_{i=0}^n d_i-n-(n-m+2)k_m$, the morphism*

$$\partial_\nu:\left(\bigoplus_{0\le i_1<\ldots<i_m\le n} S_{\nu-d_{i_1}-\cdots-d_{i_m}+\sum_{i=1}^m k_i} e_{i_1}\wedge\ldots\wedge e_{i_m}\right)\bigoplus\left(\bigoplus_{i=0}^{i=n} S_{\nu-d_i}e_i^{'}\right)\longrightarrow S_\nu$$

$$e_{i_1}\wedge\ldots\wedge e_{i_m}\longrightarrow\Delta_{i_1\ldots i_m}$$
$$e_i^{'}\longrightarrow f_i$$

is surjective if and only if $V(F:G)=\emptyset$ (or $F^{sat}=G^{sat}$). In this case, all nonzero maximal minors of size $\dim_{\mathbb{K}}(S_\nu)$ of the matrix ∂_ν is a multiple of the residual resultant, and the gcd of all these maximal minors is exactly the residual resultant.

Example. We consider the following example

$$\left\{\begin{array}{l} f_0=a_0z+a_1x+a_2y+a_3(x^2+y^2)\\ f_1=b_0z+b_1x+b_2y+b_3(x^2+y^2)\\ f_2=c_0z+c_1x+c_2y+c_3(x^2+y^2),\end{array}\right.$$

of three circles in the plane. We would like to know when they intersect outside the two trivial points given by $V(z,x^2+y^2)$. We use `Macaulay2` to compute the associated residual resultant matrix:

```
>R=QQ[a_0,a_1,a_2,a_3,a_4,b_0,b_1,b_2,b_3,b_4,c_0,c_1,c_2,c_3,c_4,x,y,z];
>G=matrix{{z,x^2+y^2}};
>H=matrix{{a_0*z+a_1*x+a_2*y,b_0*z+b_1*x+b_2*y,c_0*z+c_1*x+c_2*y},
          {a_3,b_3,c_3}};
>F=G*H;
>L=CiRes(G,H,{x,y,z});
>MaxCol oo_0
```

which returns:

$$\begin{pmatrix} a_3 & b_3 & c_3 & -a_3b_1+a_1b_3 & 0 & -a_3c_1+a_1c_3\\ 0 & 0 & 0 & -a_3b_2+a_2b_3 & -a_3b_1+a_1b_3 & -a_3c_2+a_2c_3\\ a_1 & b_1 & c_1 & -a_3b_0+a_0b_3 & 0 & -a_3c_0+a_0c_3\\ a_3 & b_3 & c_3 & 0 & -a_3b_2+a_2b_3 & 0\\ a_2 & b_2 & c_2 & 0 & -a_3b_0+a_0b_3 & 0\\ a_0 & b_0 & c_0 & 0 & 0 & 0\end{pmatrix}$$

whose determinant is the desired condition multiplied by $a_3(-a_2b_3+a_3b_2)$.

4.1.3 Residual resultants of a local complete intersection ACM of codimension 2

We have just seen that if the ideal $G = (g_1, \dots, g_m)$ is a complete intersection we know how to compute the corresponding residual resultant. There is another case where we have similar results, the case where G is a *local* complete intersection of codimension 2 arithmetically Cohen-Macaulay (abbreviated ACM) ideal [Bus01a]. For simplicity we restrict ourselves to the case of three homogeneous variables [Bus01b], i.e. $n = 2$, since in this case G has only to be an ideal of $\mathbb{P}^2$ defining isolated points. We refer to [Bus01a] chapter 3 for the general situation.

First we compute the syzygies of G, i.e. the matrix ψ which is such that:

$$0 \to \bigoplus_{i=1}^{m-1} S[-l_i] \xrightarrow{\psi} \bigoplus_{i=1}^{m} S[-k_i] \xrightarrow{\gamma=(g_1,\dots,g_m)} G \to 0, \qquad (3)$$

with $\sum_{i=1}^{m-1} l_i = \sum_{i=1}^{m} k_i$. At this point we can compute the degree of the residual resultant: it is homogeneous in the coefficient of each f_i, $i = 0, 1, 2$, of degree

$$\frac{d_0 d_1 d_2}{d_i} - \frac{\sum_{j=1}^{m-1} l_j^2 - \sum_{j=1}^{m} k_j^2}{2}.$$

Now we construct the $m \times (m+2)$ glued matrix

$$\bigoplus_{i=1}^{m-1} S[-l_i] \bigoplus_{i=0}^{2} S[-d_i] \xrightarrow{\psi \oplus \phi} \bigoplus_{i=1}^{m} S[-k_i],$$

where ϕ is the matrix $(h_{i,j})_{1 \le i \le m, 0 \le j \le 2}$. And we have:

Theorem 4.3 *We denote by $\Delta_{i_1,\dots,i_m}$ the determinant of the submatrix of the map $\phi \oplus \psi$ corresponding to columns $i_1, \dots, i_m$, and by $\alpha_{i_1,\dots,i_m}$ its degree. Then, for any $\nu \ge \sum_{i=0}^{n} d_i - n(k_m + 1)$, the morphism*

$$\begin{array}{rccc} \partial_\nu : & \bigoplus_{0 \le i_1 < \dots < i_m \le n} S_{\nu - \alpha_{i_1,\dots,i_m}} e_{i_1} \wedge \dots \wedge e_{i_m} & \longrightarrow & S_\nu \\ & e_{i_1} \wedge \dots \wedge e_{i_m} & \mapsto & \Delta_{i_1 \dots i_m} \end{array}$$

is surjective if and only if $V(F : G) = \emptyset$ (or $F^{sat} = G^{sat}$). In this case, all non-zero maximal minors of size $\dim_{\mathbb{K}}(S_\nu)$ of the matrix ∂_ν is a multiple of the residual resultant, and the gcd of all these maximal minors is exactly the residual resultant.

Example. As a simple example we consider the residual resultant of three cubics in $\mathbb{P}^2$ passing through the same three points. Here is the `Macaulay2` code:

```
>R=ZZ/32003[a_0..a_8,b_0..b_8,c_0..c_8,x_0,x_1,x_2];
>G=matrix{{x_0*x_1,x_0*x_2,x_1*x_2}};
>l0=for i from 0 to 2 list a_(0+3*i)*x_0+a_(1+3*i)*x_1+a_(2+3*i)*x_2;
>l1=for i from 0 to 2 list b_(0+3*i)*x_0+b_(1+3*i)*x_1+b_(2+3*i)*x_2;
>l2=for i from 0 to 2 list c_(0+3*i)*x_0+c_(1+3*i)*x_1+c_(2+3*i)*x_2;
>H=matrix{l0,l1,l2};
>Cm2Res(G,H,{x_0,x_1,x_2})
```

We obtain a 10×10 matrix which is too big to be printed here.

Example. What is the condition so that four cubics in $\mathbb{P}^3$ containing the twisted cubic have a common point outside this twisted cubic? We consider the following polynomials, $i = 0, 1, 2, 3$,

$$f_i = h_{1,i}(x)(x_1^2 - x_0x_2) + h_{2,i}(x)(x_1x_2 - x_0x_3) + h_{3,i}(x)(x_2^2 - x_1x_3),$$

where $h_{i,j}(x) = c_{i,j}^0 x_0 + c_{i,j}^1 x_1 + c_{i,j}^2 x_2 + c_{i,j}^3 x_3$ are linear forms. We just have to compute the residual resultant of this system, taking for the ideal G the ideal of the twisted cubic, that is to say $G = (-x_1^2 + x_0x_2, -x_1x_2 + x_0x_3, -x_2^2 + x_1x_3)$. Its syzygies are given by the matrices

$$\psi = \begin{pmatrix} -x_2 & x_3 \\ x_1 & -x_2 \\ -x_0 & x_1 \end{pmatrix}, \quad \gamma = (x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3).$$

Now we can use `Cm2Res` in `Macaulay2` or `cm2resultant` in `multires` to compute the residual resultant matrix.

4.2 Applications and examples

The residual resultants can be applied in situations where the classical resultants are usually used, with the advantage of being not degenerated if there exists base points. We briefly present, with examples, two such applications in what follows.

4.2.1 Solving polynomial systems with base points

The U-resultant can be generalized to the case of residual resultants, giving the U-residual resultant [Bus01a]. Let $f_1, \ldots, f_n$ be n homogeneous polynomials of degree $d_1, \ldots, d_n$ in the ideal G such that $d_1 \geq \ldots \geq d_n \geq k_1$, $d_n \geq k_m + 1$. Denoting by $L(x)$ the generic linear form $u_0x_0 + u_1x_1 + \ldots + u_nx_n$, we have

Proposition 4.4 *Let f be a homogeneous polynomial in the ideal $G \subset S$ of degree $d \geq k_m$. If $(f_1, \ldots, f_n) : (g_1, \ldots, g_m)$ is a geometric m-residual intersection and that $(f, f_1, \ldots, f_m) : (g_1, \ldots, g_n)$ is a geometric (m+1) -residual intersection, then, in $\mathbb{K}[u_0, \ldots, u_m]$, we have :*

$$\mathrm{Res}(Lf, f_1, \ldots, f_m) = \prod_{\xi \in V(f_1, \ldots, f_m) \setminus V(g_1, \ldots, g_n)} L(\xi)^{\mu_\xi}$$

where μ_ξ is the multiplicity of the root ξ in the ideal $(f_1, \ldots, f_m)$.

This basically means that we can recover the "residual points", that is the points not in the ideal G, by computing a residual resultant of our system with a linear generic form. We refer to [Bus01a] chapter 4 for a matrix presentation of this result which do not involve a polynomial f.

Example. We take again the simple system $f_1 = yz + x^2 + y^2$ and $f_2 = -z^2 + 2x^2 + 2y^2$. We know that the points given by $z = 0$ and $x^2 + y^2 = 0$ are solutions but we are not interested in. So we are going to compute a U-residual resultant to *only* compute the other common solutions. We use `Macaulay2`:

```
>R=QQ[u_0,u_1,u_2,x,y,z]
>G=matrix{{z,x^2+y^2}}
>F=matrix{{y*z + x^2+y^2,-z^2 + 2*x^2+2*y^2,(u_0*x+u_1*y+u_2*z)*z}}
>M=(CiRes(G,F // G,{x,y,z}))_0
>factor(det(MaxCol(M)))
```

The last command returns the polynomial $2(u_0 - u_1 + 2u_2)(u_0 + u_1 - 2u_2)$ from we easily deduce the solutions we desired. Notice the use of `F // G` to compute the matrix denoted H previously.

Example: cylinders through five points. As an example we mention the problem of finding all the cylinders passing through five points sufficiently generic (see [DMPT01] and [Bus01a] §4.2.4). Given five points p_1, p_2, p_3, p_4, p_5 in the space we would like to compute all the cylinders passing through them. For this we compute only the possible directions of such a cylinder (which is sufficient to solve the problem): a direction is a unitary vector $\overrightarrow{t} = (l, m, n)$ which can be itself identify to a point $t = (l : m : n)$ in $\mathbb{P}^2$. Figure 4.2.1 illustrates the situation.

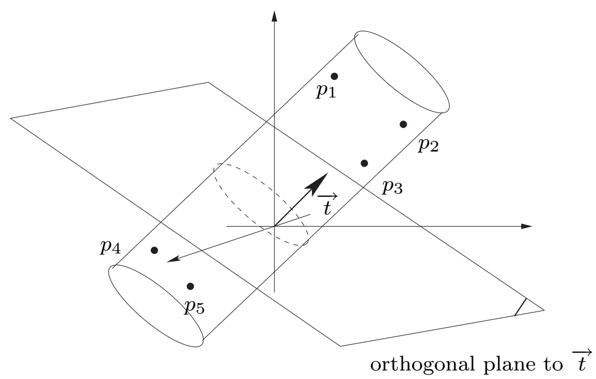
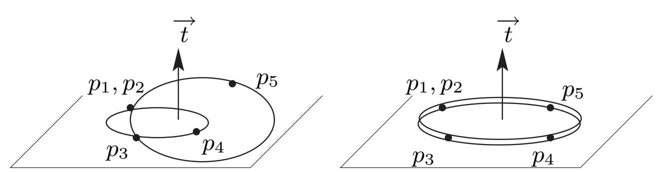


Figure 1: A cylinder through five points

Projecting the five points on the orthogonal plane to $\overrightarrow{t}$ we can obtain two conditions of cocyclicity: one saying that the points p_1, p_2, p_3, p_4 are on a common circle, and the other one saying that p_1, p_2, p_3, p_5 are on a common circle. We may obtain in this way two homogeneous polynomials $C_{1234}(l, m, n)$ and $C_{1235}(l, m, n)$ in the variables l, m, n of degree 3 [DMPT01]. These equations define 9 points in $\mathbb{P}^2$ that we can compute with classical resultant, and in particular U-resultant. However, among these 9 points 3 do not correspond to cylinders we are looking for, they correspond to the points (in fact to the directions since we have identified points and directions) $\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_3}$. In [Bus01a] §4.2.4 it is shown how we can use residual resultant to obtain only the 6 interesting points. In this way, taking into account some geometric properties of our problem we avoid the problem of identifying the 6 good solutions among 9 possible: the U-residual resultant technique yields only the 6 desired points. Finally we mention that the technique based on residual resultant is really well behaved. Indeed, even if we have to "forget" about the solutions $\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_2p_3}$, this does not mean that there is no cylinder in this direction. In fact if there is a cylinder in such a direction the corresponding point is a multiple solution of the system C_{1234}, C_{1235}. In such case the residual resultant keep the solution! The following picture, showing projection on the orthogonal plane to $\overrightarrow{t}$, illustrates this situation.



forget the solution p_1p_2 keep the solution p_1p_2

4.2.2 Implicitizing rational surfaces in the presence of base points

In 2.2.2 we have seen that classical resultants can be used for implicitizing rational surfaces without base points. In [Bus01b] it is proved that residual resultants can be used for implicitizing rational surfaces in the presence of base points. Suppose given a rational map

$$\rho : \mathbb{P}^2 \to \mathbb{P}^3 : (s:t:u) \mapsto (f_0(s,t,u) : f_1(s,t,u) : f_2(s,t,u) : f_3(s,t,u)),$$

where f_0, f_1, f_2 and f_3 are homogeneous polynomials of the same degree d. Then if the ideal $G = (g_1, \ldots, g_m)$ of base points of f_0, f_1, f_2, f_3 consists in a finite number of points and is generated in degree at most d and is not empty in degree $d-1$ then we have

$$\mathrm{Res}(f_1 - xf_0, f_2 - yf_0, f_3 - zf_0) = P(x,y,z)^{\deg(\rho)},$$

where Res denotes the residual resultant mentioned in 4.1.3 and $P(x,y,z)$ an implicit equation of ρ. Here is an example with `Macaulay2`:

Example.

```
>R=QQ[X,Y,Z,x_0,x_1,x_2];
>F=matrix{{x_0*x_1^2,x_1^3,x_0*x_2^2,x_1^3+x_2^3}};
>G=matrix{{x_1^2,x_2^2}};
>H=F//G;
>M=matrix{{1,0,0},{0,1,0},{0,0,1},{-X,-Y,-Z}};
>H=H*M;
>mr=(Cm2Res(G,H,{x_0,x_1,x_2}))#0 --the matrix of the residual resultant
>trim minors(10,mr) --gives the implicit equation
```

which returns

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -X & 1 & 0 & -Y+1 & 0 & 0 \\
0 & 0 & -Y & 0 & 0 & 0 & -Z & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -X & 0 & 0 & -Y+1 & 0 \\
0 & 0 & X & -Y & 0 & 0 & 0 & -Z & -X & -Z & 0 & -Y+1 \\
0 & 0 & 0 & 0 & -Y & -1 & 0 & 0 & -Z & 0 & 0 & 0 \\
X & Y-1 & 0 & 0 & 0 & Z & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & -Z & 0 \\
0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & -Z \\
X & Y & 0 & 0 & 0 & Z & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We deduce the implicit equation, for instance here with the command

```
>trim minors(10,mr)
```

which returns: $(X^3Y^2 - Y^2Z^3 - 2X^3Y + X^3)$.

We refer to [BEM03] for an overview and further developments around the implicitization problem and the use of resultants in CAGD.

5 Determinantal resultants

5.1 Definition and main properties

Determinantal resultants have been introduced in [Bus01a] and further studied in [Bus03] and [BG03]. They correspond to a generalization of the classical resultants. We here restrict ourselves to the case of homogeneous polynomials and refer to the cited papers for more general situations.

Let m, n and r be three integers such that $m \geq n > r \geq 0$. Given two sequences of integers $\{d_1, \ldots, d_m\}$ and $\{k_1, \ldots, k_n\}$ (not necessary positive) satisfying $d_i > k_j$ for all i, j, we consider matrices of size $n \times m$ of homogeneous polynomials in variables $\mathbf{x} = (x_1, \ldots, x_{(m-r)(n-r)})$

$$H = \begin{pmatrix} h_{1,1}(\mathbf{x}) & h_{1,2}(\mathbf{x}) & \ldots & h_{1,m}(\mathbf{x}) \\ h_{2,1}(\mathbf{x}) & h_{2,2}(\mathbf{x}) & \ldots & h_{2,m}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ h_{n,1}(\mathbf{x}) & h_{n,2}(\mathbf{x}) & \ldots & h_{n,m}(\mathbf{x}) \end{pmatrix},$$

where $h_{i,j}(\mathbf{x}) = \sum_{|\alpha|=d_j-k_i} c_\alpha^{i,j}\mathbf{x}^\alpha$ is of degree $d_j - k_i$ and have coefficients $c_\alpha^{i,j}$ with value in a field $\mathbb{K}$. The determinantal resultant of H, denoted hereafter $\mathrm{Res}(H)$ is a polynomial in the coefficients $c_\alpha^{i,j}$'s such that for any specialization of all these coefficients in $\mathbb{K}$ we have

$$\mathrm{Res}(H) = 0 \Leftrightarrow \exists\, x \in \mathbb{P}^{(m-r)(n-r)} : \mathrm{rank}(H(x)) \leq r.$$

In other words determinantal resultants give a necessary and sufficient condition so that a polynomial matrix depending on parameters is not of generic rank (w.r.t. its coefficients). We know how to compute them, as well as their multi-degree. They are multi-homogeneous in the coefficients of each column i (that is in the coefficients of the polynomials $h_{1,i}, h_{2,i}, \ldots, h_{n,i}$), $i = 1, \ldots, m$; their partial degree is the coefficient of α_i of the multivariate polynomial (in variables $\alpha_1, \ldots, \alpha_m$)

$$(-1)^{(m-r)(n-r)}\Delta_{m-r,n-r}\left(\frac{\prod_{i=1}^m (1-(d_i+\alpha_i)t)}{\prod_{i=1}^n (1-k_i t)}\right),$$

where for all formal series $s(t) = \sum_{k=-\infty}^{+\infty} c_k(s)t^k$, we set

$$\Delta_{p,q}(s) = \det\begin{pmatrix} c_p(s) & \ldots & c_{p+q-1}(s) \\ \vdots & & \vdots \\ c_{p-q+1}(s) & \ldots & c_p(s) \end{pmatrix}.$$

Example. Using `Macaulay2` you can compute the multi-degree of the determinantal resultant corresponding to $m = 3$, $n = 2$, $r = 1$:

```
>DetResDeg({d1,d2,d3},{k1,k2},1,ZZ[d1,d2,d3,k1,k2])
```

returns $\{\mathrm{d1}+\mathrm{d2}+\mathrm{d3}-\mathrm{k1}-2\mathrm{k2}-1, \{\mathrm{d2}+\mathrm{d3}-\mathrm{k1}-\mathrm{k2}, \mathrm{d1}+\mathrm{d3}-\mathrm{k1}-\mathrm{k2}, \mathrm{d1}+\mathrm{d2}-\mathrm{k1}-\mathrm{k2}\}\}$ giving the critical degree (see below) and the multi-degree of the determinantal resultant.

We now describe how to compute explicitly determinantal resultants. Consider the map

$$\bigoplus_{i_1<\ldots<i_{r+1},\, j_1<\ldots<j_{r+1}} R_{[d-\sum_{t=1}^{r+1} d_{i_t} + \sum_{t=1}^{r+1} k_{i_t}]} e_{i_1,\ldots,i_{r+1},j_1,\ldots,j_{r+1}} \xrightarrow{\sigma_d} R_{[d]}$$

which associates to each $e_{i_1,\ldots,i_{r+1},j_1,\ldots,j_{r+1}}$ the polynomial $\Delta_{i_1,\ldots,i_{r+1},j_1,\ldots,j_{r+1}}$ denoting the determinant of the minor

$$\begin{pmatrix} h_{j_1,i_1}(\mathbf{x}) & h_{j_1,i_2}(\mathbf{x}) & \ldots & h_{j_1,i_{r+1}}(\mathbf{x}) \\ h_{j_2,i_1}(\mathbf{x}) & h_{j_2,i_2}(\mathbf{x}) & \ldots & h_{j_2,i_{r+1}}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ h_{j_{r+1},i_1}(\mathbf{x}) & h_{j_{r+1},i_2}(\mathbf{x}) & \ldots & h_{j_{r+1},i_{r+1}}(\mathbf{x}) \end{pmatrix},$$

R denoting the polynomial ring $\mathbb{K}[x_1, \ldots, x_{(m-r)(n-r)}]$ and $R_{[t]}$ being the vector space of homogeneous polynomials of fixed degree t. We define the critical degree to be the integer

$$\nu_{\mathbf{d},\mathbf{k}} = (n-r)(\sum_{i=1}^m d_i - \sum_{i=1}^n k_i) - (m-n)(k_{r+1}+\ldots+k_n) - (m-r)(n-r) + 1.$$

Proposition 5.1 *Choose an integer $d \geq \nu_{\mathbf{d},\mathbf{k}}$. All nonzero maximal minor (of size $\sharp R_{[d]}$) of the map σ_d is a multiple of the determinantal resultant* $\mathrm{Res}(H)$. *Moreover the greatest common divisor of all the determinants of these maximal minors is exactly* $\mathrm{Res}(H)$.

This proposition gives us an algorithm to compute explicitly the determinantal resultant, completely similar to the one giving the expression of the Macaulay resultant. Notice that it is also possible to give the equivalent (in a less explicit form) of the so-called Macaulay matrices (of the Macaulay resultant) for the principal (i.e. $r = n-1$) determinantal resultant [Bus01a, Bus03].

In [Bus03] §5.3 determinantal resultant with $m = n+1$ and $r = n-1$ are used to compute Chow forms of rational normal scrolls. In the following subsection we present another use of such determinantal resultant for dealing with the problem of detecting the intersection of two space curves [BG03].

5.2 Application: intersecting family of curves in the projective space

We consider the problem of intersecting bicubic Bézier surfaces with the aid of determinantal resultants. We make our computations with `Maple`.

A bicubic Bézier surface is usually represented in homogeneous coordinates as:

$$C(s,t) = (X(s,t), Y(s,t), Z(s,t), T(s,t)) = \sum_{i=0}^{3}\sum_{j=0}^{3} V_{i,j} B_i^3(s) B_j^3(t),$$

where $V_{i,j} = (X_{i,j}, Y_{i,j}, Z_{i,j}, T_{i,j})$ are the homogeneous control points and $B_i^3(s)$ corresponds to the Bernstein polynomial

$$B_i^3(s) = \binom{3}{i} s^i (1-s)^{3-i}.$$

We can generate such a surface, with generic control points, as follows:

```
>Bt:=matrix([[(1-t)^3,t*(1-t)^2,t^2*(1-t),t^3]]):
>Bs:=transpose(matrix([[(1-s)^3,s*(1-s)^2,s^2*(1-s),s^3]]));
>P1:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bs);
>P2:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bs);
>P3:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bs);
>P4:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bs);
>C:=evalm((1-t)^3*P1+t*(1-t)^2*P2+t^2*(1-t)*P3+t^3*P4);
```

Such a surface can be seen as a family of space curves with parameter s, i.e. for each given value s_0 of s, $C(s_0, t)$ parameterizes a cubic Bézier space curve in $\mathbb{P}^3$. Such curve is generically (that is except for a finite number of values of s) a rational normal curve, that is to say projectively equivalent to the twisted cubic $(1, t, t^2, t^3)$. It appears that such rational normal curves are implicitly determinantal varieties, and hence we can obtain a semi-implicit determinantal representation of our surface C. Computing the projective transformation sending the twisted cubic on C by

```
>A:=transpose(matrixof([C[1,1],C[2,1],C[3,1],C[4,1]],[[1,t,t^2,t^3]])):
```

we obtain the desired representation as follows:

```
>V:=evalm(inverse(A)&*matrix([[X],[Y],[Z],[T]]));
>XX:=simplify(V[1,1]*det(A));
>YY:=simplify(V[2,1]*det(A));
>ZZ:=simplify(V[3,1]*det(A));
>TT:=simplify(V[4,1]*det(A));
>MT:=matrix(2,3,[[XX,YY,ZZ],[YY,ZZ,TT]]);
>Q1:=det(submatrix(MT,[1,2],[1,2]));
>Q2:=det(submatrix(MT,[1,2],[1,3]));
>Q3:=det(submatrix(MT,[1,2],[2,3]));
```

The matrix `MT` is of size 2×3. Its entries are linear forms in X,Y,Z,T with coefficients polynomials in s of degree at most 9. Its 2×2 minors `Q1,Q2,Q3` give a semi-implicit representation of C. More precisely, for all s such that $\det(A)\neq 0$ polynomials `Q1,Q2,Q3` describe a rational normal curve in $\mathbb{P}^3$ which is contained in C.

We know consider another bicubic Bézier surface $CC(u,v)$ (notice that we introduce a new variable z in the definition of $CC(u,v)$; this variable is here only to homogenize the variable u in order to use the function `detres` hereafter):

```
>Bv:=transpose(matrix([[(1-v)^3,v*(1-v)^2,v^2*(1-v),v^3]])):
>Q1:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bv);
>Q2:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bv);
>Q3:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bv);
>Q4:=evalm(randmatrix(4,4,entries=rand(-3..3))&*Bv);
>CC:=evalm((z-u)^3*Q1+u*(z-u)^2*Q2+u^2*(z-u)*Q3+u^3*Q4);
```

Substituting the parametric representation of CC in the semi-implicit representation of C, i.e. the matrix MT, the determinantal resultant yield a condition on s and v so that both surfaces intersect.

```
>H:=evalm(subs(X=CC[1,1],Y=CC[2,1],Z=CC[3,1],T=CC[4,1],evalm(MT)));
>Res:=detres(H,[u,z],8):
```

The resultant matrix `Res` we obtain is a 9×9 matrix,

$$\begin{pmatrix} \star & 0 & 0 & \star & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 & 0 & \star & 0 & 0 & \star \\ \star & \star & 0 & \star & \star & 0 & \star & \star & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & \star & \star & 0 & \star & \star & 0 & \star & \star \end{pmatrix},$$

($\star$ stands for some polynomials in variables s and v) whereas classical use of Dixon resultant for such a problem (which do not use the geometric property of being determinantal) yields a 18×18 matrix.

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Appendix – Functions for constructing resultant matrices with the `Maple` package `multires`

Hereafter we present some functions extracted from the `Maple` library called `multires` which is developed in the GALAAD team at INRIA. The whole library can be downloaded at `http://www-sop.inria.fr/galaad/logiciels/multires.html`. It is a big file containing the source code and the description of each function (from which the following is quoted).

- `mresultant` - Macaulay resultant matrix of a list of polynomials
  - Calling sequences:
    - mresultant(lp)
    - mresultant(lp, var)
  - Parameters:
    - lp - list of polynomials
    - var - list of variables
  - Description:
    - Compute the Macaulay resultant matrix of lp.
    - The second argument (optional) is a list of variables. The number of polynomials in lp should one more than the number of variables in var.
    - The default value of var corresponds to the case where var is the set of indeterminates in lp.
    - The determinant of this matrix is a multiple of the Macaulay resultant of the polynomials lp w.r.t. var.
    - The size of the matrix is the number of monomials of degree less than $(d_1 - 1) + ... + (d_n + 1 - 1) + 1$ where d_i is the degree of lp[i] and n is the number of polynomials in lp.
- `jresultant` - Jouanolou's matrix for computing the Macaulay resultant.
  - Calling sequences:
    - jresultant(lp,var)

    * jresultant(lp,var,μ)

  - Parameters:
    * lp - a list of polynomials
    * var - a list of variables
    * μ - an integer

  - Description:
    * Compute the matrix introduces by J.P. Jouanolou in [Jou97] of lp.
    * The second argument is a list of variables. The number of polynomials in lp should be one more than in var. The third argument is an integer with value between 0 and ν (which is the sum of the degree of all the polynomials minus one).
    * The determinant of this matrix is a multiple of the resultant of the polynomials lp w.r.t. var.
    * The size of the matrix is the sum of the number of monomials of degree μ and the number of monomials "d-repu" of degree $d - \mu$.

- `spresultant` - Sparse (toric) resultant matrix of polynomials

  - Calling sequence:
    * spresultant(pols, vars);
    * spresultant(pols, vars, delta);
    * spresultant(pols, vars, delta, lifting);

  - Parameters:
    * pols - a list of polynomials
    * vars - a list of variables
    * delta - a vector or a list of real numbers
    * lifting - a 2-dimensional array of lifting

  - Description:
    * Compute the sparse (toric) resultant matrix of the given polynomials by eliminating variables vars. The number of rows in the coefficients of the first polynomial is optimal (equal to the respective degree of the sparse resultant).
    * Letting n be the number of variables, n+1 must be the number of polynomials.
    * If delta is given, it is used as the geometric perturbation applied to the Minkowski sum of the Newton polytopes. It must contain n entries, sufficiently small in order not to perturb points in the sum outside adjacent cells in the sum's subdivision. If delta is not given, it is chosen randomly.
    * If lifting is given, it specifies the affine lifting functions of the Newton polytopes, thus defining the mixed subdivision of their Minkowski sum which will specify the matrix. It must be a square matrix of dimension n+1, each row corresponding to a Newton polytope. If lifting is not given, it is chosen randomly.
    * The function returns a square matrix, whose determinant is a multiple of the sparse resultant for generic coefficients. For degenerate coefficients, a perturbation can be defined by setting global variable PERT_DEGEN_COEFS, as described by D'Andrea and Emiris (Computing Sparse Projection Operators, In "Symbolic Computation: Solving Equations in Algebra, Geometry, and Engineering, AMS, 2001). The 2nd returned item is a list of the monomials indexing the columns.
    * The matrix construction implements the greedy version, by Canny and Pedersen (Tech. Report 1394, C.S. Dept, Cornell University, 1993), of the algorithm by Canny and Emiris (Proc. AAECC-1993, LNCS 263, pp. 89. Final version J. ACM, 2000).

- `bkmresultant` - Compute a resultant matrix for the residual resultant of a complete intersection
  - Calling sequences: bkmresultant(lp,M,var,reg)
  - Parameters:
    * lp - a list of homogeneous polynomials
    * M - a matrix
    * var - a list of variables
    * reg - an integer
  - Description:
    * Compute the matrix of the first application in the resolution of (I:J) given in the article of Bruns, Kustin and Miller (BKM) [BKM90].
    * The first argument is a list of homogeneous polynomials $I = (f_1, .., f_m)$. Given a homogeneous complete intersection $J = (g_1, .., g_n)$, such that I is included in J and $(I : J)$ is a residual intersection, we want to compute the residual resultant of I w.r.t. J. The matrix M is the matrix such that I=J.M. The integer reg must be superior or equal to the regularity of the ideal $(I : J)$.
    * The result of `bkmresultant` is a surjective nxm matrix such that the determinant of a nxn minor is a multiple of the resultant of I on the closure of $V(I) \setminus V(J)$. This minor can be obtain with the function `hmaxminor`.
- `bkmdegree` - Compute the degree of the residual resultant `bkmresultant`
  - Calling sequence: bkmdegree(ld,lk)
  - Parameters:
    * ld - a list of integers
    * lk - a list of integers
  - Description: Calling with $ld := [d_1, \ldots, d_m]$ and $lk := [k_1, \ldots, k_n]$, `bkmdegree` gives the degree of `bkmresultant` in the coefficients of the polynomial f_0.
- `cm2resultant` - Compute a resultant matrix for the residual resultant of a ACM local complete intersection of codimension 2.
  - Calling sequence: cm2resultant(H,R,var,reg)
  - Parameters:
    * H - a matrix
    * R - a matrix
    * var - a list of variables
    * reg - an integer
  - Description:
    * Compute the first map of the complex which computes the residual resultant of a local complete intersection Arithmetically Cohen-Macaulay of codimension two [Bus01a].
    * Given a homogeneous ideal l.c.i. ACM codimension 2 $J = (g_1, \ldots, g_n)$, such that $I = (f_1, .., f_m)$ is included in J and $(I : J)$ is a residual intersection, the matrix H is such that $I = J.H$. The matrix R is the matrix of the first syzygies of J. var denotes the variables and reg the critical degree.
    * The result of `cm2resultant` is a surjective matrix such that the determinant of a maximal minor is a multiple of the residual resultant of I w.r.t. J. This minor can be obtain with the function `hmaxminor`.

- `detres` - Compute the principal determinantal resultant of a matrix
  - Calling sequence: detres(M,var,reg)
  - Parameters:
    * M - a matrix of homogeneous polynomials with parameters
    * var - a list of variables
    * reg - an integer
  - Description:
    * Compute the determinantal resultant of an $n \times m$-matrix ($n < m$) of homogeneous polynomials in $n - m + 1$ homogeneous variables, i.e. a condition on the parameters of these polynomials to have rank$(M) < n$.
    * The third argument is the degree in which we compute the resultant matrix.
- `mbezout` - Compute the Bezoutian matrix of a list of polynomials
  - Calling sequence:
    * mbezout(lp)
    * mbezout(lp, var)
    * mbezout(lp, var, l1)
    * mbezout(lp, var, l1, l2)
  - Parameters:
    * lp - a list of polynomials
    * var - a list of variables (optional)
    * l1 - an unassigned variable where are stored the monomials indexing the rows of the matrix (optional)
    * l2 - an unassigned variable where are stored the monomials indexing the columns of the matrix (optional)
  - Description:
    * Compute the Bezoutian matrix of lp with respect to the variables var
    * The default value for var is the set of indeterminates of lp.
- `matrixof` - Coefficient matrix of a list of polynomials
  - Calling sequence:
    * matrixof(lp)
    * matrixof(lp, var)
    * matrixof(lp, lm)
    * matrixof(lp, 'l')
  - Parameters:
    * lp - a list of polynomials
    * var - a list of a list of variables
    * lm - a list of monomials
    * l - the name of a variable
  - Description:
    * Compute the coefficient matrix of lp.
    * If the second argument is a list of list var, it is the coefficient matrix with respect to all the monomials in the variables var.

∗ If the second argument is a simple list of monomials, it is the coefficient matrix with respect to this list of monomials. The other coefficients are ignored.
∗ If the second argument is the name of variable, the monomials indexing the columns are stored in this variable.
∗ The default value corresponds to the case var.