Effective Polynomial system solving

Philippe Trébuchet, INRIA, Spaces/Lip6 Calfor

Équipe CALFOR/INRIA Projet SPACES
Université Pierre et Marie Curie
4 place Jussieu
75252 Paris Cedex 05

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Challenge : Dealing with semi-algebraic sets.

Solving Polynomial Systems

Complex solving for Real Geometry
Facts:

- Algorithms in real geometry (BPR, SaSc, Rouillier Roy Safey. . .) reduce the problems to a 0-dimensional system solving.

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- Need of resolution methods working over Non-archimedian Fields

- Intrinsic height of the variety may be VERY high $\Rightarrow$ resolution algorithms must not introduce artificial instabilities.

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- $\mathbb{K}$ a field.
- $\mathbb{K}[x_1, \ldots, x_n]$ the ring of n variate polynomials.
- $I = (f_1, \ldots, f_s)$ an ideal defining a variety $\mathcal{V}$.

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- finding an approximation of the coordinates of the points of $\mathcal{V}$.
- giving a univariate representation of $\mathcal{V}$.
- decomposing $\mathcal{V}$ into simple parts.

Different approaches

- Numerical methods
  - ⋆ homotopy (Somese, Vershelde,etc . . .).
  - ⋆ interval analysis (J.P. Merlet,).
  - ⋆ Weierstrass (Newton-like) (A. Bellido, O. Ruatta, J.C. Yakoubsohn,etc . . .).
- Algebraic methods
  - ⋆ Matrix methods(J. Canny, I. Emiris, B. Mourrain, . . .).
  - ⋆ Geometric methods (P. Aubry, M. Kalkbrener, D. Lazard, M. Moreno and M. Giusti, G. Lecerf, J. Heintz . . .).
  - ⋆ Rewriting methods (B. Buchberger, J.C. Faugère, . . .).

The quotient algebra

Key structure!

$$A = \mathbb{K}[x_1, \ldots, x_n]/I$$

Problems :

- Where to read the information about the points?
- How to compute with it? (representation)
- Is there a best representation? (numerical conditionning, memory size, **stability**. . .)

Where to read the information about the points?

From A to the ζ_i

Theorem : *[Stickelberger]If* $\mathcal{V} = \zeta_1, \ldots, \zeta_k$, $p \in \mathbb{K}[x_1, \ldots, x_n]$.
We will call the multiplication by p, $\mathcal{M}_p$ *the operator* $\begin{array}{c} A \to A \\ f \to fp \end{array}$. *The operator* $\mathcal{M}_p$ *has the following properties :*

- *The eigenvalues of* $\mathcal{M}_p$ *are the numbers* $p(\zeta_1), \ldots, p(\zeta_k)$ *counted with multiplicities.*
- *The common eigenvectors to all the* $\mathcal{M}_p^t$ *are the evaluations to the* ζ_i.

From here it is easy to compute the Chow form of I i.e. compute

$$Det(\mathcal{M}_{u_0+u_1x_1+\cdots+u_nx_n})$$

From A to the ζ_i (continued)

Theorem : *[Hermite]* $\mathbb{K} = \mathbb{R}$, $h \in \mathbb{R}[\mathbf{x}_1, \ldots, x_n]$ *and* Q_h *be the quadratic form*

$$Q_h : \begin{array}{l} A \rightarrow A \\ p \rightarrow Tr(hp^2) \end{array}.$$

Then we have the following two properties :

- *The number of complex root* ζ_i *such as* $h(\zeta_i) \neq 0$ *is the rank of the quadratic form* Q_h.

- *The number of real roots* ζ_i *such as* $h(\zeta_i) > 0$ *and the number of real roots such as* $h(\zeta_i) < 0$ *is the signature of* Q_h.

From A to the ζ_i (continued)

Theorem : *[Rouillier] Let* $u \in \mathbb{K}[x_1, \ldots, x_n]$ *we define :*

- $f_u(T) = \Pi_{i=1..k}(T - u(\zeta_i))^{\mu_i}$
- $g_0(T) = \Sigma_{i=1..k}\mu_i \Pi_{j \neq i}(T - u(\zeta_j))$
- $g_l(T) = \Sigma_{i=1..k}\mu_i \zeta_l.i \Pi_{j \neq i}(T - u(\zeta_j))$

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$$f_u(\zeta) = 0 \rightarrow \zeta_i = \begin{pmatrix} g_1(\zeta)/g_0(\zeta) \\ \vdots \\ g_n(\zeta)/g_0(\zeta) \end{pmatrix}$$

If u separates the ζ_i then the preceeding polynomials define a Rational Univariate Representation.

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If u separates the ζ_i then the preceeding polynomials define a Rational Univariate Representation.

Proposition : *A separating element can be chosen in the family :*

$$\mathcal{U} = \{x_1 + jx_2 + \cdots + j^{n-1}x_n, j = 0 \dots nk(k-1)/2\}$$

we also have :

$$u \text{ is separating the } \zeta_i \Leftrightarrow deg(minpol((M_u))) == dim(A)$$

Representation of A

Working in A

Finding Canonical representation of the elements of $\mathbb{K}[x_1, \ldots, x_n]$

(Normal Forms)

- $\mathcal{V}$ contains only points $\Rightarrow$ A is a finite dimensionnal $\mathbb{K}$-vector space.
- As a $\mathbb{K}$-vector space $\mathbb{K}[x_1, \ldots, x_n]$ is spanned by the monomials.
- Finding a basis of A can be reduced to finding a **monomial** basis of A (noted B).

Suppose that we know B, a monomial basis of A. Then we have the equality :

$$\mathbb{K}[x_1, \ldots, x_n] = \langle B \rangle \oplus \langle I \rangle$$

($\langle I \rangle$ denotes the $\mathbb{K}$-vector space generated by I)

Here Comes Macaulay!

Computing the canonical expression of a polynomial $p \in \mathbb{K}[x_1, \ldots, x_n]$ in B is simple:

Algorithm 1. INPUT: $p \in \mathbb{K}[x_1, \ldots, x_n]$
$I = (f_1, \ldots, f_s)$
B *a monomial basis of* A
OUTPUT *the representation of* p *in* A.

- *k=0, reduced=*`false`
- *while !reduced*
  - *construct the matrix* $Mat_k(p)$
  - *echelonize* $Mat_k(p)$ *without permuting lines*
  - *if the last line has no nonzero coefficients outside the columns corresponding to* B *then*
    reduced=`true`
- *return the last line of the echelonized matrix.*

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$$\begin{array}{c} \quad B^c \quad B \\ \left(\begin{array}{c} \mathtt{M}f_1 \\ \vdots \\ \mathtt{M}f_s \\ p \end{array} \right) \end{array}$$

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Macaulay Construction

If the f_i are $generic$ of degree d_i then we have :

- the set $E_0 = \{x_1^{\alpha_1} \dots x_n^{\alpha_n}, \alpha_i < d_i\}$ is a basis of A.

- the sets $E_i = \{x_1^{\alpha_1} \dots x_n^{\alpha_n}, \Sigma_{j=1..n}\alpha_j \leq \Sigma_{j=1..n}(d_j - 1) + 1,\ \forall n \geq j \geq i,\ \alpha_j < d_j\}$

Algorithm 2. Input: $f_0 \in \mathbb{K}[x_1, \dots, x_n]$ *whom we want the multiplication operator*
$f_1, \dots, f_s$ *generic polynomials*
Output: *The multiplication matrix of* f_0 *in* A

- *construct the matrix :*

$$\begin{array}{c} \begin{array}{cccc} E_0 f_0 & E_1 f_1 & \dots & E_n f_n \end{array} \\ \left(\begin{array}{c|c} \mathtt{A} & \mathtt{C} \\ \hline \mathtt{B} & \mathtt{D} \end{array}\right) \end{array}$$

- *return the Schur complement*$= A - CD^{-1}B$

Matrix methods

Study of the sylvester endomorphism :

$$\begin{array}{ccc} < E_0 > \times < E_1 > \times \cdots \times < E_s > & \longrightarrow & \mathbb{K}[x_1, \ldots, x_n] \\ (p_0, p_1, \ldots, p_s) & \longrightarrow & \Sigma_{i=0..s} p_i f_i \end{array}$$

- E_0 must be a basis of A.
- E_i must be wide enought for the Schur complement of the bloc `A` gives the multiplication operator.

Ex: Sparse resultant (J. Canny, I. Emiris. . .), etc...

Gröbner bases

Algorithm 3. INPUT : $f_1, \ldots, f_s$ *generating* I*.(any dimensional)*
An admissible monomial order γ
OUTPUT : *A representation of* A

- $G = \{f_1, \ldots, f_s\}$
- **repeat**
  - ⋆ $G' = G$
  - ⋆ *for all pair* $(p, q), \ p, q \in G'$ *do*
    - ∗ $S = \overline{S(p,q)}^{G'}$
    - ∗ *if* $S \neq 0$ *then* $G = G \cup S$
- **until** $G' == G$
- *return* G

Gröbner bases (Faugère,Lazard,Lombardi)

At a second thought. . .

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Then we can do the following changes :

- see the polynomials of I as linear dependence relations

- substitute S-polynomial reductions with echelonisations of matrices

Reduce S-pol

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$$
\begin{array}{c}
\text{monomials} \\
\left(\begin{array}{c}
\frac{lcm(\gamma(p),\gamma(q))}{\gamma(p)}p \\
\frac{lcm(\gamma(p),\gamma(q))}{\gamma(q)}q \\
\vdots \\
\hline \\
\text{Reductor polynomials}
\end{array}\right)
\end{array}
$$

Gröbner bases (Faugère,Lazard,Lombardi)

At a second thought. . .

- Can substitute polynomial algebra by linear algebra

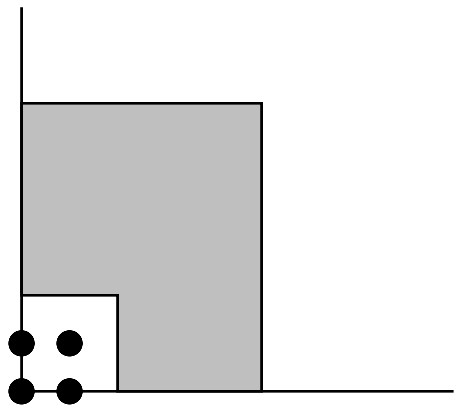
Then we can do the following changes :

- see the polynomials of I as linear dependence relations
- substitute S-polynomial reductions with echelonisations of matrices
- proceed incrementaly

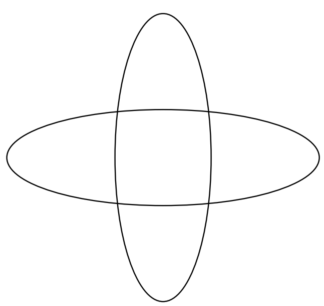
$$\begin{array}{c} \text{monomials} \\ \left(\begin{array}{c} \frac{lcm(\gamma(p),\gamma(q))}{\gamma(p)} p \\ \frac{lcm(\gamma(p),\gamma(q))}{\gamma(q)} q \\ \vdots \\ \hline \\ \text{Reductor polynomials} \end{array} \right) \end{array}$$

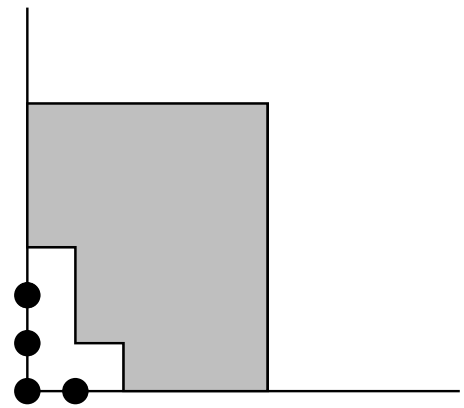
Unhappy with Gröbner though

ex: $p_1 = ax_1^2 + bx_2^2 + n_1(x_1, x_2)$
$p_2 = cx_1^2 + dx_2^2 + n_2(x_1, x_2)$

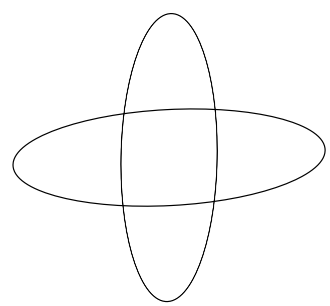


$$n_1 = n_2 = 0$$





$$n_1 = \varepsilon_1 x_1 x_2$$
$$n_2 = \varepsilon_2 x_1 x_2$$



Computing better representations

Notations

- A Normal form is :• A monomial basis B of A
  • An algorithm to project $\mathbb{K}[x_1, \ldots, x_n]$ onto B

- A choice function refining the degree, γ, is a function that takes a polynomial p and returns one monomial $\gamma(p)$ of the support of p such that $deg(\gamma(p)) = deg(p)$.

- Let S be a subset of $\mathbb{K}[x_1, \ldots, x_n]$ S^+ is the set :

$$S^+ = x_1 S \cup \cdots \cup x_n S$$

- $P \subset \mathbb{K}[x_1, \ldots, x_n], M \subset \{x^\alpha,\ \alpha \in \mathbb{Z}^n\}$,$(P|M)$, is the matrix whose columns are index by M and lines by P.

- Let p_1, p_2 $\in$ $\mathbb{K}[x_1, \ldots, x_n]$, the C-polynomial of p_1 and p_2 is $C(p_1, p_2) = \frac{lcm(\gamma(p_1),\gamma(p_2)),\gamma(p_1)}{p}_1 - \frac{lcm(\gamma(p_1),\gamma(p_2)),\gamma(p_2)}{p}_2$, and $deg_C(C(p_1, p_2)) = deg(lcm(\gamma(p_1), \gamma(p_2)))$.

Macaulay revisited

Provide a way to write any polynomial p under the form
$p = b + i,\ b \in \langle B \rangle,\ i \in \langle I \rangle$

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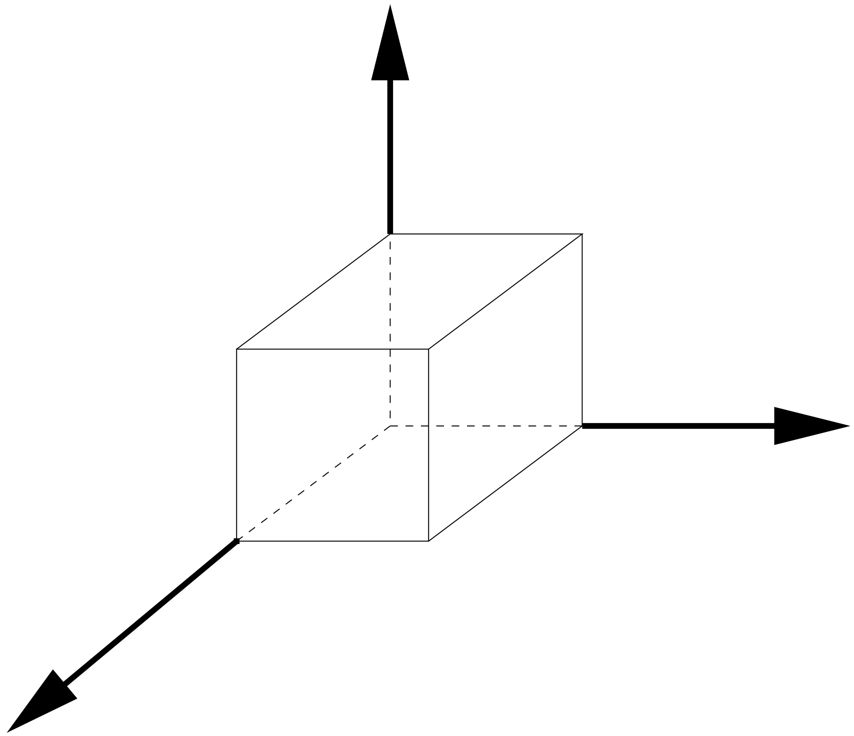
Algorithm 4. [Mourrain, T.] INPUT : $F = f_1, \ldots, f_s$ `generic` *polynomials*
OUTPUT : *The representation of A given by Macaulay.*

- $k = min(deg(f_i),\ i = 1..s)$.
- $P_k = F[k]$, $M_k = \{x_i^k,\ f_i \in P_k\}$
- **while** $k \leq \Sigma_{i=1..s}(d_i - 1) + 1$ **do**
  - ⋆ $P_{k+1} = P_k^+ \cap B^+ \cup proj(P[k+1])$, $M_{k+1} = M_k^+ \cap B^+ \cup \{x_i^{k+1}, f_i \in P[k+1]\}$
  - ⋆ *Solve the linear system* $(P_{k+1}|M_{k+1})X = P_{k+1}$.
  - ⋆ $k = k + 1$
- *return* $P_i, i = min(deg(f_i),\ i = 1..s)..\Sigma_{i=1..s}(d_i - 1) + 1$

Macaulay revisited an example



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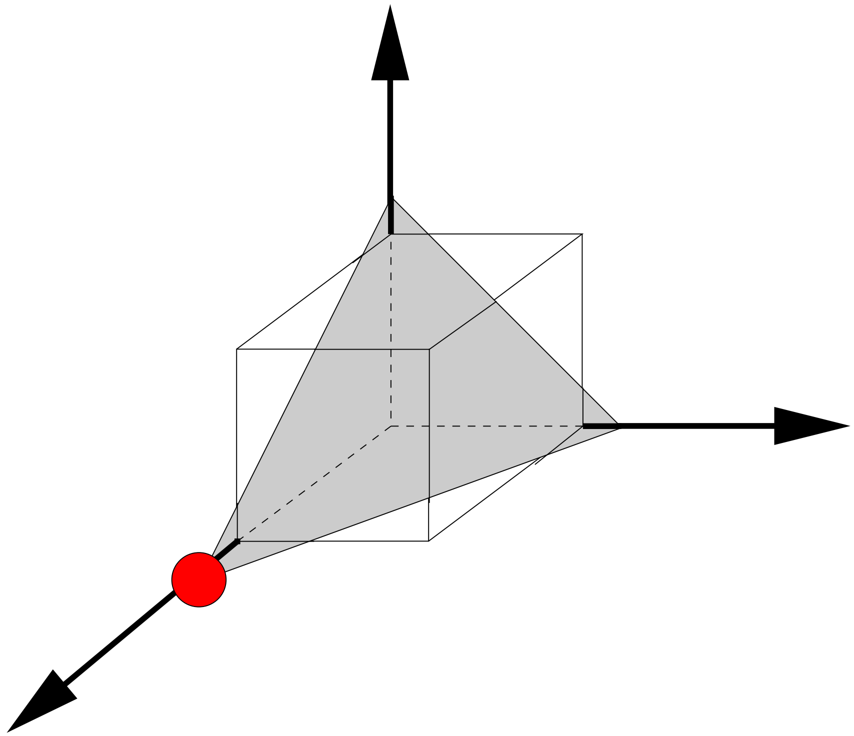

$$f_i = x_i^{d_i} + \dots$$

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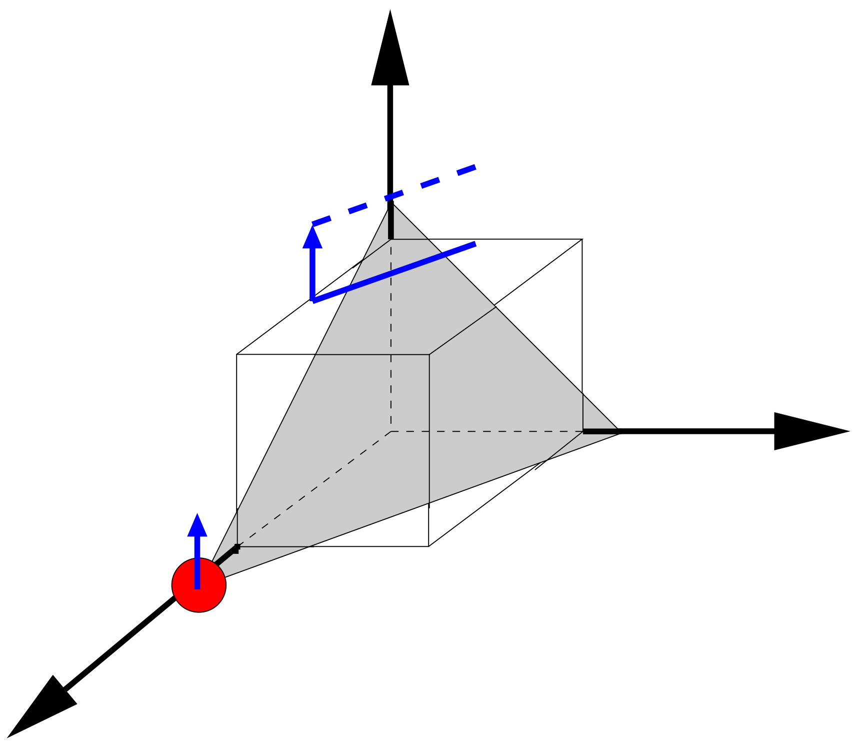

$$f_i = x_i^{d_i} + \dots$$

$$x_1 f_i = x_1 x_i^{d_i} + \alpha_2 x_1^{d_1} x_2 + \cdots + \alpha_n x_1^{d_1} x_n + \dots$$

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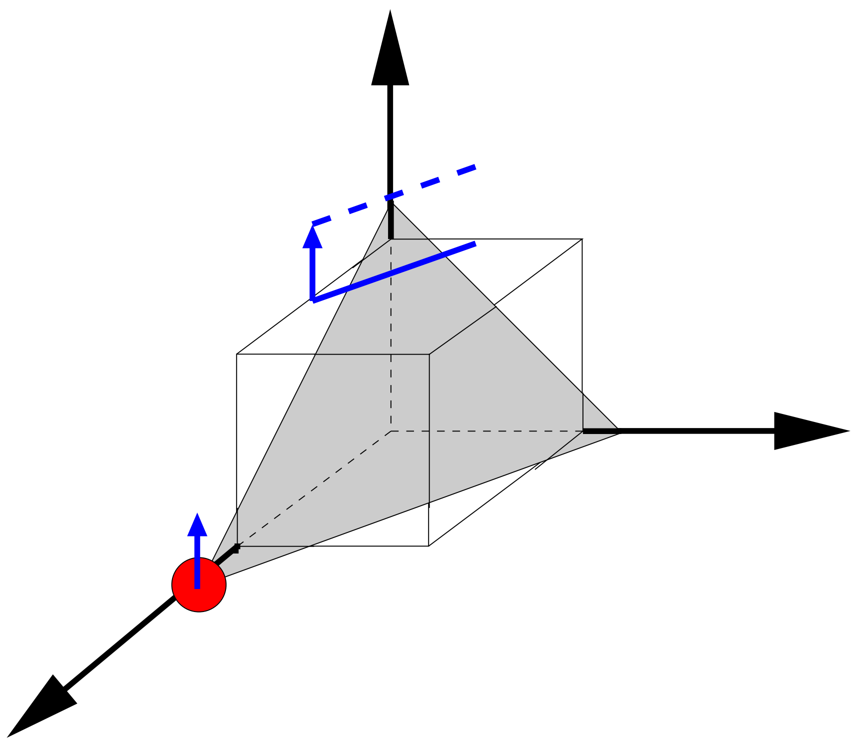

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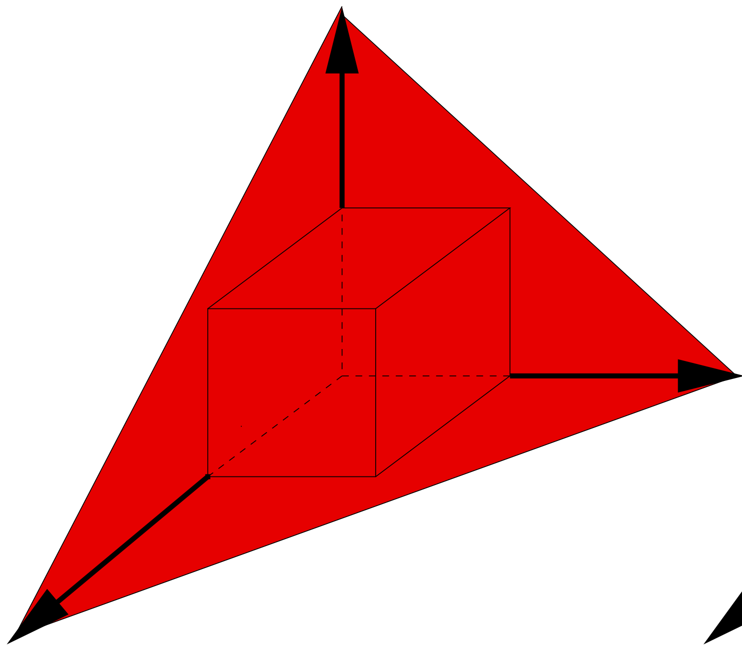
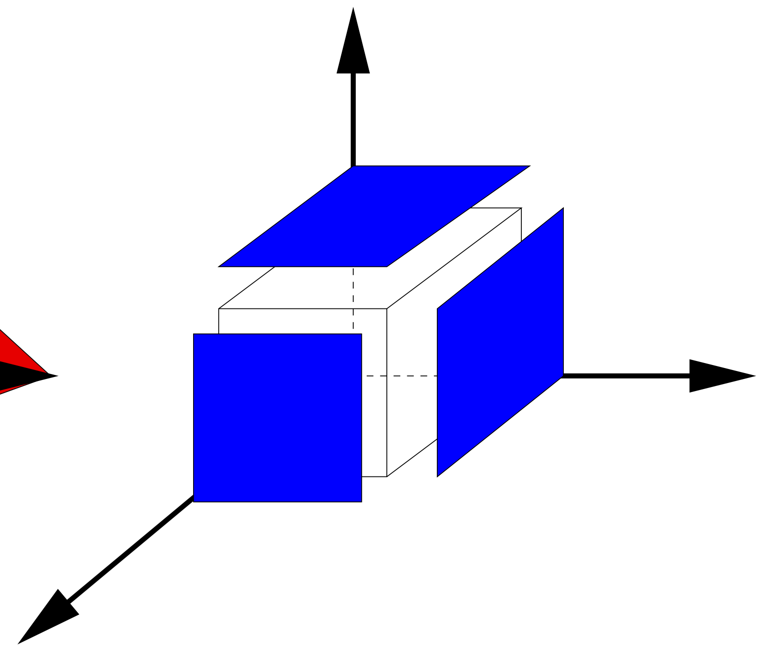
$$\left(\begin{array}{cccccccc} & & \dots\dots\dots\dots\dots\dots & & & & & \\ 1 & \dots & \alpha_2 & \alpha_3 & \dots & \alpha_n & \dots \\ & & \dots\dots\dots\dots\dots\dots & & & & & \end{array} \right)$$

comparing to the original





Comparing to the original

n	5	6	7	8	9	10	11
size	5	6	7	8	9	10	11
of the	20	30	42	56	72	90	110
matrices	30	60	105	168	252	360	495
	20	60	140	280	504	840	1320
	5	30	105	280	630	1260	2310
		6	42	168	504	1260	2772
			7	56	252	840	2310
				8	72	360	1320
					9	90	495
						10	110
							11
Sum	80	192	448	1024	2304	5120	11264
Macaulay	462	1 716	6 435	24 310	92 378	352 716	1 352 078
Nb points	32	64	128	256	512	1024	2048

Table 1: Size of the systems to invert

What to do if B is not known?

Algorithm 5. [Mourrain] INPUT : $F = f_1, \ldots, f_n$
L *a* $\mathbb{K}$*-vector space connex to* 1
OUTPUT : *The multiplicative structure of* A

1) $K_0 = \langle f_1, \ldots, f_n \rangle$, $n = 0$

2) **repeat**

- $\star$ $K_{n+1} = K_n^+ \cap L$
- $\star$ $n = n + 1$

3) **until** $K_n == K_{n-1}$

4) Compute B *a supplementary of* K_n *in* L

5) If $B^+ \not\subset L$, $L = L^+$ *and go back to 1)*

What to do if not generic

a new criterion :

- $\mathbb{K}[x_1, \ldots, x_n] =< B > \oplus < I >$.

$\Longleftrightarrow$

- The multiplication operators by the variables commute.

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Properties

$\oplus$ Very good numerical stability.

$\oplus$ Possibility to take into account the geometry of the problem.

$\ominus$ **VERY** expensive computation.

Computing better for computing less

What went wrong :

- Postpone the effective computation of B to the last step.

Computing better for computing less

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What we should do to mimic Gröbner bases :

- try since the first step to guess what will be B
- check our guess is correct
- do not compute polynomials that go $too\ far$ from B

Computing better for computing less

Algorithm 6. [T.] NORMAL FORMS

INPUT : $F = f_1, \ldots, f_s$ *defining* I *(*0*-dimensionnal)*

γ *a choice* **function** *refining the degree.*

Initilisation : *Choose the* f_i *of minimal degree*

$b = (\gamma(f_i))$, $k = deg(f_i)$, $P_k = \{f_i\}$, $M_k = \{\gamma(f_i)\}$

Core Loop : `While` $(newmon||k \leq Maxdeg(F))$ *do*

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- *Compute* $P_{k+1} = P_k^+ \cap b^+$ *and take into account the* f_i *of degree* $k+1$

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- $M_{k+1} = \{M_k^+ \cap b^+\}$

Computing better for computing less

Algorithm 6. [T.] NORMAL FORMS

INPUT : $F = f_1, \ldots, f_s$ *defining* I *(*0*-dimensionnal)*

γ *a choice* **function** *refining the degree.*

Initilisation : *Choose the* f_i *of minimal degree*

$b = (\gamma(f_i))$, $k = deg(f_i)$, $P_k = \{f_i\}$, $M_k = \{\gamma(f_i)\}$

Core Loop : `While` $(newmon || k \leq Maxdeg(F))$ *do*

- *Compute the* C*-pol of degree* $k+1$
- *Compute* $P_{k+1} = P_k^+ \cap b^+$ *and take into account the* f_i *of degree* $k+1$
- $M_{k+1} = \{M_k^+ \cap b^+\}$
- $PseudoSolve\,(P_{k+1}|M_{k+1})X = P_{k+1}$

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Algorithm 6. [T.] $\textsc{Normal Forms}$

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- $M_{k+1} = \{M_k^+ \cap b^+\}$
- $PseudoSolve$ $(P_{k+1}|M_{k+1})X = P_{k+1}$
- *Reduce the* C*-*pol *with respect to* P_j
- *Whether the* C*-*pol *reduce to* 0 *or not and whether* $(P_{k+1}|M_{k+1})$ *is of maximal rank update* b, P_{k+1}, k, M_{k+1}

End `While`

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End `While`

OUTPUT : $\{P_j, j = 0..k\}$ *that allow to construct a system of normal form* $\forall k \in \mathbb{N}$

$$r = \#M_{k+1} - Rank((M_{k+1}|P_{k+1}))$$
$$c = \#\{\text{non reduced to } 0\}$$

$\mathbf{r = 0}$ && $\mathbf{c = 0}$	$\mathbf{r = 0}$ && $\mathbf{c \neq 0}$	$\mathbf{r \neq 0}$ && $\mathbf{c \neq 0}$	$\mathbf{r \neq 0}$ && $\mathbf{c = 0}$
Go to the next step	Treat the non reduced and go back at their degree	Add to B and treat the nonzero	Add to b Go to the next step

An example

$$F = \begin{cases} x^2 + xy \\ y^2 + xy \\ xy^3 \end{cases} \qquad \gamma = \text{ask user}$$

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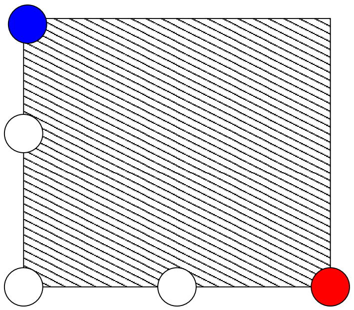

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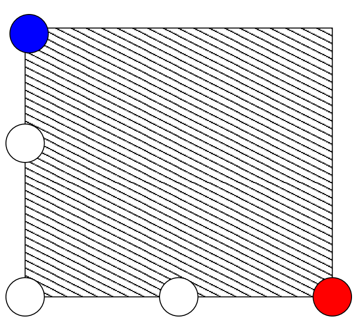

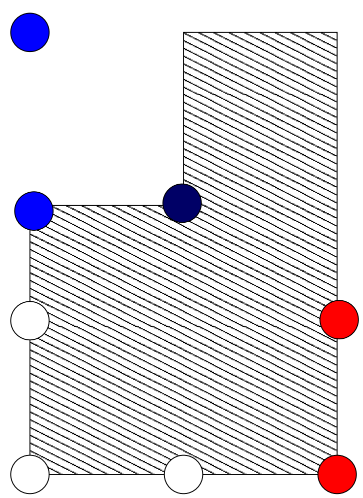

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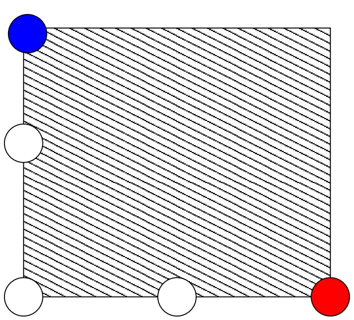

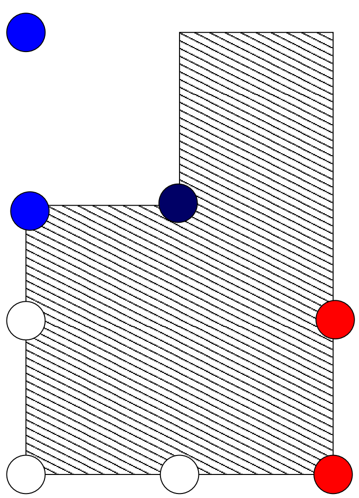

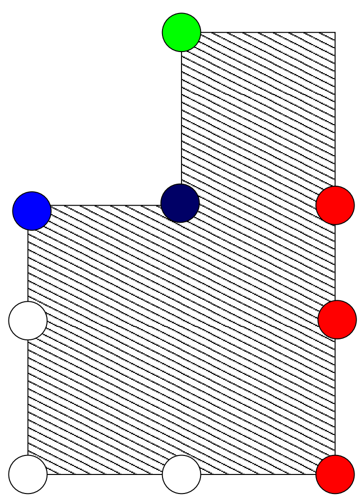

What to do if we do not refine the degree

An example : the `Lex` monomial order
The Problems :

- we do not have always *reduced* polynomials.
- we do not know in advance how to determine what polynomials will be reduced
- we must avoid dead lock (i.e. polynomial p depending on polynomial q and polynomial q depending on polynomial p).

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The solutions:

- put in stand by non fully reduced polynomials.
- at the end of each step determine what will be the polynomial set to consider.
- use a linear form from the first quadrant to avoid dead-locks.

Stability in practice

Parallel Robot with quaternion parametrization:
`http://www-sop.inria.fr/saga/POL/BASE/2.multipol/rbpl16.html` :

γ	time in s	Peak mem	average of $cond(M_i)$
Macaulay	632	17M	10^7
Dlex	3325	40M	10^7
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ε-Computations!!

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What remains to do :

- Find a suitable stopping criterion in positive dimension
- Use and optimize the infinitesimals for real life applications.