Symbolic-Numeric Sparse Polynomial Interpolation in Chebyshev Basis and Trigonometric Interpolation

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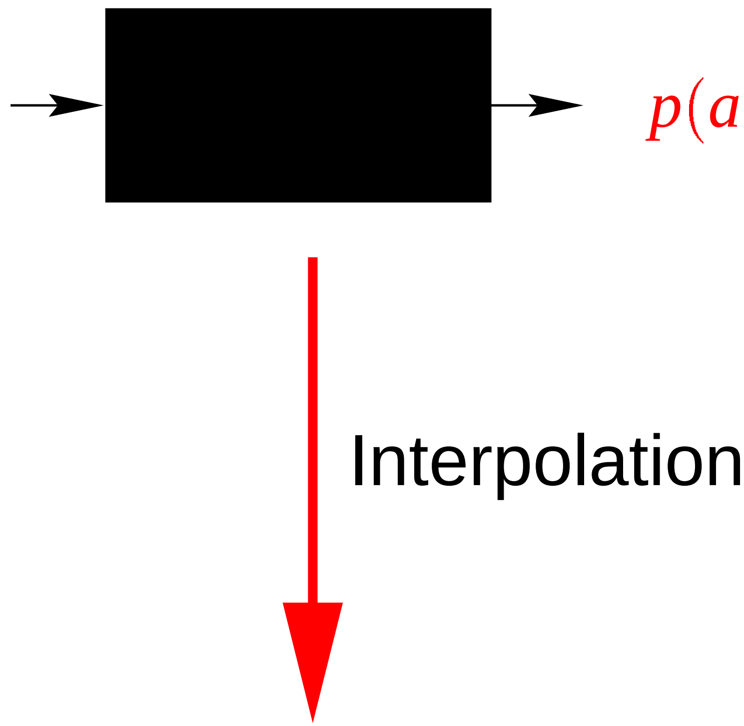
Black box polynomial interpolation

Black box polynomial p

$a_1, \ldots, a_n \in D$ $\quad$ $p(a_1, \ldots, a_n) \in D$



$$p(x_1, \ldots, x_n) = \sum_{j=1}^{t} c_j x_1^{d_{1,j}} \cdots x_n^{d_{n,j}} \in D[x_1, \ldots, x_n]$$

What if $p(x_1, \ldots, x_n)$ is sparse?

When $p(x_1,\ldots,x_n) = \sum_{j=1}^{t} c_j x_1^{d_{1,j}} \cdots x_n^{d_{n,j}}$ is sparse:

Exact arithmetic

- Zippel's probabilistic interpolation (1979)
- Ben-Or/Tiwari deterministic algorithm (1988)

Floating-point arithmetic

- Giesbrecht, Labahn, and Lee (2004)
  - Prony's method on the unit circle
  - Generalized eigenvalue reformulation

When $p(x) = \sum_{j=1}^{t} c_j T_{d_j}(x)$ is sparse in the Chebyshev basis:

$$T_0(x) = 1, \quad T_1(x) = x$$
$$k \geq 2: \quad T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

Exact arithmetic

Lakshman and Saunders (1995)

How about floating-point arithmetic?

Sparse Chebyshev Interpolation (Lakshman and Saunders 1995)

$$p(x) = 5.72T_{15}(x) + 2.11T_3(x) - 15.00T_{12}(x) + 0.99T_7(x)$$

- Pick $a > 1$: say $a = 2$.
  Evaluate $\alpha_0 = p(T_0(2)), \alpha_1 = p(T_1(2)), \alpha_2 = p(T_2(2)), \ldots$.
- Solve a $t \times t$ symmetric Hankel-plus-Toeplitz system:

$$\begin{bmatrix} 2\alpha_0 & 2\alpha_1 & \ldots & 2\alpha_{t-1} \\ 2\alpha_1 & \alpha_2+\alpha_0 & \ldots & \alpha_t+\alpha_{t-2} \\ \vdots & \vdots & \ddots & \vdots \\ 2\alpha_{t-1} & \alpha_t+\alpha_{t-2} & \ldots & \alpha_{2t-2}+\alpha_0 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{t-1} \end{bmatrix} = - \begin{bmatrix} 2\alpha_t \\ \alpha_{t+1}+\alpha_{t-1} \\ \vdots \\ \alpha_{2t-1}+\alpha_1 \end{bmatrix}$$

- Roots of $\Lambda(z) = T_t(z) + \lambda_{t-1}T_{t-1}(z) + \cdots + \lambda_0 = 0$ are non-zero Chebyshev terms in p at $x = a$:

$$\begin{matrix} 189750626, & 3650401, & 5042, & 26 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ T_{15}(2) & T_{12}(2) & T_7(2) & T_3(2) \end{matrix}$$

- Recover coefficients by solving a Vandermonde-like system.

Numerical Issues in Sparse Chebyshev Interpolation

☹ The Hankel-plus-Toeplitz system is often ill-conditioned....

$$\underbrace{\begin{bmatrix} 2\alpha_0 & 2\alpha_1 & \cdots & 2\alpha_{t-1} \\ 2\alpha_1 & \alpha_2+\alpha_0 & \cdots & \alpha_t+\alpha_{t-2} \\ \vdots & \vdots & \ddots & \vdots \\ 2\alpha_{t-1} & \alpha_t+\alpha_{t-2} & \cdots & \alpha_{2t-2}+\alpha_0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{t-1} \end{bmatrix} = - \begin{bmatrix} 2\alpha_t \\ \alpha_{t+1}+\alpha_{t-1} \\ \vdots \\ \alpha_{2t-1}+\alpha_1 \end{bmatrix}$$

☹ Root-finding for $T_t(z)+\lambda_{t-1}T_{t-1}(z)+\cdots+\lambda_0=0$ is not a good thing to do numerically: very sensitive to perturbations in λ_j.

☹ To recover coefficients c_j in $p(x)=\sum_{j=1}^{t} c_j T_{d_j}(x)$, need to solve a Vandermonde-like system that might be ill-conditioned.

Choose $a = \cos 2\pi/N, N \geq 2\deg p$, for evaluation: $\alpha_k = p(T_k(a))$

$$p(x) = \sum_{j=1}^{t} c_j T_{d_j}(x)$$

$$\underbrace{\begin{bmatrix} 2\alpha_0 & 2\alpha_1 & \ldots & 2\alpha_{t-1} \\ 2\alpha_1 & \alpha_2+\alpha_0 & \ldots & \alpha_t+\alpha_{t-2} \\ \vdots & \vdots & \ddots & \vdots \\ 2\alpha_{t-1} & \alpha_t+\alpha_{t-2} & \ldots & \alpha_{2t-2}+\alpha_0 \end{bmatrix}}_{\mathcal{A}} = \mathcal{W} \underbrace{\begin{bmatrix} 2c_1 & 0 & \ldots & 0 \\ 0 & 2c_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2c_t \end{bmatrix}}_{\mathcal{D}} \mathcal{W}^{\mathrm{Tr}}$$

$$\mathcal{W} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & \vdots \\ * & * & 2 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & 2^{t-2} \end{bmatrix} \begin{bmatrix} 1 & \ldots & 1 \\ T_{d_1}(a) & \ldots & T_{d_t}(a) \\ \vdots & \ddots & \vdots \\ T_{d_1}^{t-1}(a) & \ldots & T_{d_t}^{t-1}(a) \end{bmatrix} = \mathcal{L}\mathcal{V}$$

☺ Better for conditioning:

$$\frac{1}{2^{2(t-1)} \cdot t^2 \cdot \min_j |2c_j|} \leq \|\mathcal{A}^{-1}\| \leq \frac{K^2 \cdot t \cdot 2^{2(t-1)}}{\min \prod_{j \neq k} |T_{d_k}(a) - T_{d_j}(a)|^2 \cdot \min_j |2c_j|}$$

Choose $a = \cos 2\pi/N, N \geq 2\deg p$, for evaluation: $\alpha_k = p(T_k(a))$

$$p(x) = \sum_{j=1}^{t} c_j T_{d_j}(x)$$

All roots of $\Lambda(z) = T_t(z) + \lambda_{t-1} T_{t-1}(z) + \cdots + \lambda_0 = 0$ are in $(-1, 1)$.

☺ Root finding is easier: perturbing $\Lambda(z)$ as $\Lambda(z) + \varepsilon\Gamma(z)$ with $\Gamma(z) = \gamma_t z^t + \gamma_{t-1} z^{t-1} + \cdots + \gamma_0$ changes a root $T_{d_j}(a)$ by no more than

$$\frac{\varepsilon \cdot \sum_{k=0}^{t} |\gamma_k|}{|\prod_{j \neq k}(T_{d_k}(a) - T_{d_j}(a))|} + O(\varepsilon^2)$$

Choose $a = \cos 2\pi/N, N \geq 2\deg p$, for evaluation: $\alpha_k = p(T_k(a))$

$$p(x) = \sum_{j=1}^{t} c_j T_{d_j}(x)$$

$$\|\mathcal{W}^{-1}\| = \|\mathcal{V}^{-1}\| \|\mathcal{L}^{-1}\| \leq K \cdot \|\mathcal{V}^{-1}\|$$

☺ The condition of the Vandermonde-like system depends on the distribution of $T_{d_j}(a)$ and t.

Generalized eigenvalue reformulation of Prony's method

Golub, Milanfar, and Varah 1999

Sparse Chebyshev interpolation via generalized eigenvalue

$$\mathcal{A}_{\uparrow} = \begin{bmatrix} 2\alpha_1 & \alpha_2+\alpha_0 & \dots & \alpha_t+\alpha_{t-2} \\ 2\alpha_2 & \alpha_3+\alpha_1 & \dots & \alpha_{t+1}+\alpha_{t-3} \\ \vdots & \vdots & \ddots & \vdots \\ 2\alpha_t & \alpha_{t+1}+\alpha_{t-1} & \dots & \alpha_{2t-1}+\alpha_1 \end{bmatrix} \qquad \mathcal{A}_{\downarrow} = \begin{bmatrix} 2\alpha_1 & \alpha_2+\alpha_0 & \dots & \alpha_t+\alpha_{t-2} \\ 2\alpha_0 & 2\alpha_1 & \dots & 2\alpha_{t-1} \\ 2\alpha_1 & \alpha_2+\alpha_0 & \dots & \alpha_t+\alpha_{t-2} \\ \vdots & \vdots & & \vdots \\ 2\alpha_{t-2} & \alpha_{t-1}+\alpha_{t-3} & \dots & \alpha_{2t-3}+\alpha_1 \end{bmatrix}$$

Non-zero Chebyshev terms $T_{d_1}(a)$, $T_{d_2}(a)$, …, $T_{d_t}(a)$ are solutions for z in the generalized eigenvalue system

$$\frac{1}{2}(\mathcal{A}_{\uparrow} + \mathcal{A}_{\downarrow})v = z\mathcal{A}v$$

Sparse Chebyshev interpolation via generalized eigenvalue

☺ Avoid solving the Hankel-plus-Toeplitz system.

☺ Avoid the root finding of polynomial

$$\Lambda(z) = T_t(z) + \lambda_{t-1} T_{t-1}(z) + \cdots + \lambda_0 = 0.$$

☺ The generalized eigenvalue problem has numerically more stable algorithms (even when $T_{d_j}(a)$ are not in $(-1,1)$.)

Sparse Chebyshev interpolation via generalized eigenvalue

$$p(x) = \sum_{j=1}^{t} c_j T_{d_j}(x)$$

- Choose $a = \cos 2\pi/N$, for $N \geq 2 \deg p$.
  Evaluate: $\alpha_k = p(T_k(a))$ for $k = 0, 1, \ldots, 2t-1$.
- Compute $T_{d_j}(a)$ by solving the corresponding generalized eigenvalue system

$$\frac{1}{2}(\mathcal{A}_{\uparrow} + \mathcal{A}_{\downarrow})v = z\mathcal{A}v$$

- Obtain d_j from values of $T_{d_j}(a)$ and integer roundings.
- Recover coefficients c_j.

The kth Chebyshev polynomial: $\cos k\theta$ in terms of $\cos\theta$:

$$\cos k\theta = T_k(\cos\theta)$$

Sparse cosine interpolation

$$g(\theta) = \sum_{j=1}^{t} A_j \cos h_j\theta, \qquad h_1 < h_2 < \cdots < h_t$$

In sparse Chebyshev interpolations:

- Choose $\phi = 2\pi/N$, for $N \geq 2h_t$.
  Evaluate: $\alpha_k = g(k\phi)$.
- $\cdots$

Sparse trigonometirc interpolations

$$f(\theta) = \underbrace{\sum_{j=1}^{t_1} A_j \cos h_j \theta}_{g_1(\theta)} + \underbrace{\sum_{j=1}^{t_2} B_j \sin k_j \theta}_{g_2(\theta)}.$$

- When $t_1 = t_2, h_j = k_j$: a Prony's variant (c.f. Hildebrand 1956)
- Interpolate its phase polynomial (c.f. Giesbrecht, Labahn, and Lee 2004)
- When $t_1 = t_2, h_j = k_j$ and $A_j \neq 0$: sparse cosine interpolation

$$g_1(\theta) = \frac{1}{2}(f(\theta) + f(-\theta))$$

Multivariate case (c.f. Giesbrecht, Labahn, and Lee 2004)

$$g(\theta_1, \ldots, \theta_n) = \sum_{j=1}^{t} A_{h_j} \cos(h_{j_1}\theta_1 + \ldots + h_{j_n}\theta_n), \; h_{j_i} \text{ are integers}$$

$$m_1 \geq 2 \max_{1\leq j\leq t}(h_{j_1}), \ldots, m_n \geq 2 \max_{1\leq j\leq t}(h_{j_n}), \quad m_i \text{ relatively prime}$$

$$m = m_1 \cdots m_n, \qquad \omega_i = 2\pi/m_i, \qquad \omega = 2\pi/m$$

Interpolate $\alpha_k = g(k\omega_1, \ldots, k\omega_n)$: each $\cos(h_{j_1}\theta_1 + \ldots + h_{j_n}\theta_n)$ is mapped to $\cos(h_{j_1}2\pi/m_1 + \ldots + h_{j_n}2\pi/m_n) = \cos(h_j 2\pi/m)$.

Then $(h_{j_1}, \ldots, h_{j_n}) \in \mathbb{Z}^n_{\geq 0}$ can be uniquely determined by the (reversed) Chinese remainder algorithm:

$$h_j = h_{j_1} \cdot \left(\frac{m}{m_1}\right) + \cdots + h_{j_n} \cdot \left(\frac{m}{m_n}\right)$$

Further developments

Sensitivity analysis and tests:

`http://scg.uwaterloo.ca/~ws2lee/software/sparsechebysev`

Randomize the distribution of $T_{d_j}(a)$:

Better condition with high probability?

Interpolating with an upper bound for the sparsity t:

An upper bound $T > t$ may still lead to a good result.

Connections to Fourier series?