Surfaces Parameterized in Degree 2: Algorithms and Applications

Franck Aries, Bernard Mourrain* and Jean-Pierre Técourt*

Abstract. In this paper, we study quadratic parameterization patches (generically Steiner patches) for modeling purposes. We give an overview of the properties and algorithmic of these surfaces, including classification, implicitization, and inversion. Our new approach for the classification is based on rational and certified computation, using effective algebraic geometry tools. We use resultant-based constructions to deduce the implicit equations of such patches. Exploiting the properties of these resultant matrices, we propose a new approach to solve the inversion problem. We show how it reduces to a linear algebra problem, for which we propose a stable and efficient algorithm including the treatment of singularities. The approach is illustrated by experiments on canopy models.

§1. Introduction

A classical representation of shapes is a set of points, connected by facets, called a linear mesh. Such a data structure has the enormous advantage of being easy to manipulate, but at the same time suffers from important drawbacks, such as

- the requirement of a huge amount of data to approximate correctly a surface, even for simple objects such as (natural) quadrics,
- delicate convergence problems in terms of local geometric invariants, as illustrated by the famous Schwartz lampion for instance (see [19] for a detailed discussion).

This is one of the motivating reasons for our interest in curved meshes. Let us illustrate this idea by a small exercise. Suppose that we want to represent a function σ from $[0,1]^2$ to $\mathbb{R}^3$, which derivatives are bounded, say by

xxx and xxx (eds.), pp. 1–4.

ISBN 0-9728482-x-x

*Partially supported by GAIA IST-2001-35512 and Aim@Shape NOE 506766

$M = 100$, within an error of at most $\epsilon < 10^{-2}$. Based on Taylor expansions, we easily check that a point-wise interpolation on a regular grid will require the storage of $N_0 = 3 \times \left(\frac{\epsilon}{M}\right)^{-2} = 3\,10^8$ values. A representation using the value of σ and its derivatives at regular points of a grid will store $3 \times (1+2)\,\left(2\,\frac{\epsilon}{M}\right)^{-1} = 4.5\,10^4$ values, for the same error bound. Similarly a representation using the value of σ, its first and second derivatives, at regular points of a grid will only store $3 \times (1+2+3)(6\,\frac{\epsilon}{M})^{-\frac{2}{3}} < 2.6\,10^3$ values, for the same error bound.

This shows that higher order models are more efficient to store geometric information than linear one. Investigating this idea, we will consider hereafter, patches of surfaces which are the images of a domain of the plane, by a map of the form

$$\overline{\sigma} : t = (t_1, t_2) \rightarrow \left(\frac{\overline{f}_1(t_1,t_2)}{\overline{f}_0(t_1,t_2)}, \frac{\overline{f}_2(t_1,t_2)}{\overline{f}_0(t_1,t_2)}, \frac{\overline{f}_3(t_1,t_2)}{\overline{f}_0(t_1,t_2)}\right), \tag{1}$$

where the functions f_i $(i = 0, \ldots, 3)$ are polynomials of degree ≤ 2.

Such a parameterization describes generically a surface, called a Steiner surface (we will give a precise definition below). Such surfaces are of interest in modeling, since they realize a compromise between the use of linear triangles (easy to implement) and surfaces of higher degree (difficult to manage) [20], [3]. Moreover, as we will see, it also includes the natural quadric surfaces.

In order to be able to deal effectively with such surfaces, we need to detect easily the type of surfaces being manipulated, to compute efficiently their intersections — for instance by exploiting the construction of their implicit equation — and to invert the map $\overline{\sigma}$ on the surface for the purpose of going from the three-dimensional space to the (two-dimensional) parameter space. Thus, the problems that we are going to address here, are:

- the effective classification of the parameterizations of degree 2 over $\mathbb{R}$ and $\mathbb{C}$,
- the determination of the implicit equation from a given parameterization,
- the computation of preimages of a given point on the surface.

Our new characterization of Steiner surfaces, exploits the previous classifications of [1], [8], [12]. Its novelty resides in the use of geometric information such as the number of real or complex points of special algebraic varieties. This characteristics can be easily computed and certified, through the use of the algebraic ingredients, that we describe. The implicitization techniques that we present, is related to classical resultant constructions [17], [4], [7]. However, the use of such tools to invert the parameterization map, even in the case of multiple preimages, is new. This allows us to develop a complete set of methods for the treatment of Steiner patches in geometric modeling. Preliminary experiments of canopy models illustrate the approach.

The paper is structured as follows. In the next section, we give the required definitions related to Steiner surfaces. In section 3, we describe the classification of Steiner surfaces. In the following section, we show how to compute the implicit equation of Steiner surfaces. Section 4 is dedicated to the inversion of the parameterization map. In the last section, we illustrate the methods by a modeling problem of canopy.

§2. Definitions

From a projective point of view, it is quite natural to consider instead of $\overline{\sigma}$, the parameterization from $\mathbb{P}^2$ to $\mathbb{P}^3$:

$$\mathbf{t} = (t_0, t_1, t_2) \xrightarrow{\sigma} (f_0(\mathbf{t}), f_1(\mathbf{t}), f_2(\mathbf{t}), f_3(\mathbf{t})), \tag{2}$$

where f_i $(i = 0, \dots, 3)$ are homogeneous functions of degree 2 in $\mathbf{t}$, which can also been written as

$$(f_0(\mathbf{t}), f_1(\mathbf{t}), f_2(\mathbf{t}), f_3(\mathbf{t})) = \begin{pmatrix} f_{0,0} & \cdots & f_{0,5} \\ f_{1,0} & \cdots & f_{1,5} \\ f_{2,0} & \cdots & f_{2,5} \\ f_{3,0} & \cdots & f_{3,5} \end{pmatrix} \begin{pmatrix} t_0^2 \\ t_1^2 \\ t_2^2 \\ t_0 t_1 \\ t_0 t_2 \\ t_1 t_2 \end{pmatrix}. \tag{3}$$

The map σ is defined on $\mathbb{P}^2$, except at the points where $f_i(\mathbf{t}) = 0$, $i = 0, \dots, 3$, which leads to the following definition:

Definition 1. *A point* $\mathbf{t} \in \mathbb{P}^2$ *is called a* base point *of a parameterization* σ *if* $\mathbf{t}$ *is a common root of the polynomials* f_i*, that is if* $f_i(\mathbf{t}) = 0$ *for* $i = 0, \dots, 3$.

The classical characterization of the degree of a variety and Bezout theorem yields the following results [16], [13].

Proposition 2. *Let* S *be a surface of* $\mathbb{P}^3$ *parametrized by a rational map* σ *of type* (2) *with deg* $f_i = d$ *then:*

$$deg(S) = d^2 - \sum_{b \in Z} \mu_b.$$

where $Z = \{\mathbf{t} \in \mathbb{P}^2 | f_0(\mathbf{t}) = f_1(\mathbf{t}) = f_2(\mathbf{t}) = f_3(\mathbf{t}) = 0\}$ *is the set of base points and* μ_b *is the multiplicity of intersection of two generic combinations of the polynomials* f_i $(i = 0, \dots, 3)$ *at the base point* b.

The class of surfaces that we are going to study, called Steiner surfaces, are defined as follows:

Definition 3. *A* Steiner surface *is a surface that admits a parameterization of type* (2) *such that:*

- *the four polynomials* $f_i(\mathbf{t})$ *of degree* 2 *are linearly independent,*
- *the four polynomials* $f_i(\mathbf{t})$ *have no base points.*

As Steiner surfaces have no base points, we deduce from proposition 2 that they are of degree 4.

To analyze their geometry, we will work hereafter in the space of quadrics in $\mathbb{P}^2$, denoted $\mathcal{Q}$, and which is a projective space of dimension 5. The canonical basis of the underlying vector space is

$$\{t_0^2, t_1^2, t_2^2, t_0t_1, t_0t_2, t_1t_2\},$$

to which we associate the variables $\{x_0, \ldots, x_5\}$. A generic element of $\mathcal{Q}$ is thus of the form $x_0t_0^2 + x_1t_1^2 + \cdots + x_5t_1t_2$.

The vector space of quadratic polynomials is equipped with a natural scalar product defined as follows: For a multi-index $i = (i_0, i_1, i_2)$, let $|i| = i_0 + i_1 + i_2$, $(i)! = i_0!i_1!i_2!$, $t^i = t_0^{i_0}t_1^{i_1}t_2^{i_2}$ and $c(i) = |i|!/(i)!$ and for any quadratic forms $p = \sum_{|i|=2} \gamma(i;p)c(i)t^i$, $q = \sum_{|i|=2} \gamma(i;q)c(i)t^i$, let

$$\langle p, q\rangle = \sum_{|i|=2} c(i)\gamma(i;p)\gamma(i;q). \tag{4}$$

An interesting property of this scalar product is the following:

$$\langle p, \phi(u,v,w)\rangle = p(u,v,w). \tag{5}$$

where ϕ is the application:

$$(t_0, t_1, t_2) \xrightarrow{\phi} (t_0^2, t_1^2, t_2^2, 2\,t_0t_1, 2\,t_0t_2, 2\,t_1t_2) \tag{6}$$

See eg. [9] for more details on this scalar product.

Following a classical approach [16], we will consider the image of a parameterization of type 2, as the projection of the Veronese variety $\mathcal{V}$ of order 2, which is defined as the image of $\mathbb{P}^2$ by ϕ. By construction, ϕ is a bijective map between $\mathbb{P}^2$ and the Veronese surface $\mathcal{V}$, which is an algebraic subvariety of the space of quadrics $\mathcal{Q}$. Viewed in this space, the Veronese variety corresponds to the set of all double lines, or equivalently of quadratic forms

$$\begin{pmatrix} x_0 & x_3 & x_5 \\ x_3 & x_1 & x_4 \\ x_5 & x_4 & x_2 \end{pmatrix}$$

of rank 1. Thus the Veronese surface is also defined by the vanishing of the six 2×2 minors:

$$\begin{array}{c} x_0x_1 - x_3^2 = 0, x_0x_2 - x_5^2 = 0, x_1x_2 - x_4^2 = 0, \\ x_0x_4 - x_3x_5 = 0, x_1x_5 - x_3x_4 = 0, x_2x_3 - x_4x_5 = 0. \end{array} \tag{7}$$

The image of σ is the *linear projection* of $\mathcal{V}$ from $\mathcal{Q}$ onto $\mathbb{P}^3$ defined by the 4×6 matrix (3) of the coefficients of the f_i.

The space of quadrics span by f_0, f_1, f_2, f_3 is denoted $L = \langle f_0, f_1, f_2, f_3 \rangle$ and its orthogonal for the scalar product (4) by $N = L^\perp$.

The critical points for this projection will also play an important role: The set of points $\in \mathcal{Q}$, called the Chordal variety of the Veronese [16], and denoted by $\mathcal{C}$, is the union of all lines meeting $\mathcal{V}$ at least twice (counting multiplicities). As a Chordal variety of $\mathcal{V}$, it has twice the dimension of the Veronese that is 4 and it corresponds to the quadratic forms which are the sum of two quadratic forms of rank ≤ 1, that is of rank ≤ 2. These are the degenerated quadratic forms, such that their determinant vanishes:

$$x_0x_1x_2 - x_0x_4^2 - x_2x_3^2 - x_1x_5^2 + 2\,x_3x_4x_5 = 0.$$

In what follows, we will study $L \cap \mathcal{V}$, $N \cap \mathcal{C}$, $N \cap \mathcal{V}$ to classify the real and degenerate Steiner surfaces.

§3. Classification over $\mathbb{R}$ and $\mathbb{C}$

We will classify the parameterizations from $\mathbb{P}^2$ to $\mathbb{P}^3$:

$$\mathbf{t} = (t_0, t_1, t_2) \xrightarrow{p} (f_0(\mathbf{t}), f_1(\mathbf{t}), f_2(\mathbf{t}), f_3(\mathbf{t})).$$

Two parameterizations are equivalent if they are equal up to a change of the coordinates in the domain and the co-domain.
In other words, two parameterizations $\varphi(\mathbf{t})$ and $\psi(\mathbf{t})$ are equivalent if there exist two matrices $A \in GL(3, k)$ and $B \in GL(4, k)$ such that $\psi(\mathbf{t}) = B\varphi(A\,\mathbf{t})$.

$$\begin{tikzcd} (t_0, t_1, t_2) \arrow[r, "\psi(t)"] \arrow[d, "A"] & (f_0(\mathbf{t}), f_1(\mathbf{t}), f_2(\mathbf{t}), f_3(\mathbf{t})) \\ (u_0, u_1, u_2) \arrow[r, "\varphi(u)"] & (q_0(\mathbf{u}), q_1(\mathbf{u}), q_2(\mathbf{u}), q_3(\mathbf{u})) \arrow[u, "B"'] \end{tikzcd}$$

When we precise a field for the classification, it is the field k of the coefficients of the matrices A and B for the changes of coordinates.

Geometrically speaking, this classification boils down to classify the 3-pencils of conics up to an homography (bijective projective transformation) of $\mathbb{P}^2$. The classification over $\mathbb{C}$ has been known for more than a century, whereas the classification over $\mathbb{R}$ is far more recent. For recent works on the subject we mention [1] for a classification when the ground field is $\mathbb{C}$, [8] and [12] for a classification when the ground field is $\mathbb{R}$. In both cases, we recall the results and complete them from an algorithmic point of view.

Proposition 4 (in $\mathbb{C}$). *Up to homography with coefficients in $\mathbb{C}$, there are 4 types of Steiner surfaces.*

S_1	$(t_0t_1, t_2t_0, t_1t_2, t_0^2+t_1^2+t_2^2)$	$X^2Y^2+X^2Z^2+Y^2Z^2-XYZT$
S_2	$(t_0^2, t_1^2, t_2^2, t_0(t_1+t_2))$	$T^4-2T^2X(Y+Z)+X^2(Y-Z)^2$
S_3	$(t_0^2, t_1^2, t_1t_2, t_0t_1+t_2^2)$	$(TY-Z^2)^2-XY^3$
S_4	$(t_0^2, t_1^2, t_2^2, t_0t_1)$	$(XY-T^2)^2$

For a complete proof, see [1]. The case S_4 corresponds to a non-proper parameterization (when each point of the surface has at least two preimages).

Proposition 5 (in $\mathbb{R}$). *Up to homography with coefficients in $\mathbb{R}$, there are 7 types of Steiner surfaces [8].*

Σ_1	$(t_0t_1, t_0t_2, t_1t_2, t_0^2+t_1^2+t_2^2)$	$X^2Y^2+X^2Z^2+Y^2Z^2-XYZT$
Σ_2	$(t_0t_1, t_0t_2, t_1t_2, t_0^2-t_1^2+t_2^2)$	$X^2Y^2-X^2Z^2+Y^2Z^2-XYZT$
Σ_3	$(t_0^2+t_1^2, t_1^2+t_2^2, t_0t_2, t_1t_2)$	$XY(Z^2+T^2)-X^2T^2-(Z^2+T^2)^2$
Σ_4	$(t_0^2-t_1^2, t_0t_1, t_1t_2, t_2^2)$	$XZ^2T-Y^2T^2+Z^4$
Σ_5	$(t_0t_1-t_1t_2, t_0^2, t_1^2, t_2^2)$	$(X^2-YT-ZT)^2-4YZT^2$
Σ_6	$(t_0^2, t_0t_2-t_1^2, t_1t_2, t_2^2)$	$(YT+Z^2)^2-XT^3$
Σ_7	$(t_0^2, t_1^2, t_2^2, t_0t_1)$	$(XY-T^2)^2$

We will work in the general case that will lead us to consider parameterizations with base points. If the parameterization admits one base point, we obtain surfaces of degree 3 (cubics) and of degree 2 (quadrics) if there are two base points. In the case of cubics, we have three classes:

Proposition 6 (in $\mathbb{R}$). *Up to homography with coefficients in $\mathbb{R}$, there are 3 types of parameterizations of type* (2) *of degree 3 [8].*

Σ_8	$(t_1^2-t_2^2, t_0t_1, t_0t_2, t_1t_2)$	$XYZ-Y^2T+Z^2T$
Σ_9	$(t_0t_1, t_0t_2, t_1^2, t_2^2)$	X^2T-YZ^2
Σ_{10}	$(t_0t_1, t_0t_2-t_1^2, t_1t_2, t_2^2)$	$YZT-XT^2+Z^3$

For quadrics, we know that the classification depends on the rank and the signature of the quadratic form and if we limit to the case of proper quadrics, we deduce that:

Proposition 7 (in $\mathbb{R}$). *Up to homography with coefficients in $\mathbb{R}$, there are 3 types of (proper) quadrics.*

Σ_{11}	$(2t_0t_2, 2t_1t_2, t_0^2+t_1^2-t_2^2, t_0^2+t_1^2+t_2^2)$	$X^2+Y^2+Z^2-T^2$
Σ_{12}	$(t_2^2, t_1t_2, t_0t_2, t_0t_1)$	$XT-YZ$
Σ_{13}	$(t_1^2, t_2^2, t_1t_2, t_0t_2)$	Z^2-XY

Based on these results, we give the following new characterization.

Theorem 8. *The projective surfaces parameterized by a polynomial map of degree 2 are characterized as follows:*

	Σ_i	$L \cap \mathcal{V}$	$N \cap \mathcal{C}$	$N \cap \mathcal{V}$
S_1	Σ_1	r, r, r, r	r, r, r	$\emptyset$
	Σ_2	c, c, c, c	r, r, r	
	Σ_3	r, r, c, c	r, c, c	
S_2	Σ_4	r^2, c, c	r^2, r	$\emptyset$
	Σ_5	r^2, r, r		
S_3	Σ_6	r, r^3	r^3	$\emptyset$
S_4	Σ_7	*dimension 1*	N	$\emptyset$
Cubics	Σ_8	r^2, c^2	r^2, r	p
	Σ_9	r^2, r^2	r^2, r	
	Σ_{10}	r^4	r^3	
Quadrics	Σ_{11}	r^2, c, c	N	p, p
	Σ_{12}	r^4		p, p
	Σ_{13}	*dimension 1*		p^2

where p means a simple point, p^i a point with multiplicity i, and r and c represent respectively a real and a complex (non-real) solution.

Proof. Since $\mathcal{V}$ and $\mathcal{C}$ are invariant by the action induced by $Gl(3, k)$ on $\mathcal{Q}$, the type of intersections of $L \cap \mathcal{V}$, $N \cap \mathcal{C}$, $N \cap \mathcal{V}$ over $\mathbb{C}$ (resp. $\mathbb{R}$) are the same for all the elements of a given orbit over $\mathbb{C}$ (resp. $\mathbb{R}$).

By property 5, the points $q = \phi(u, v, w)$ on the Veronese surface, orthogonal to the polynomials f_i, $i = 0, \ldots, 3$, *i.e.* are such that $\langle q, f_i \rangle = f(u, v, w) = 0$ for $i = 0, \ldots, 3$. In other words, the elements of $N \cap \mathcal{V}$ correspond to base-points.

We deduce from proposition 2, that the cubic surfaces correspond to the case of a parameterization with a single base point.

Similarly, the quadrics correspond to the case of a parameterization with two base points (2 distinct points or 1 double point). So according to the former reasoning, the case of 2 base points corresponds to an intersection between N and $\mathcal{V}$ made of two points.

An explicit construction for $\Sigma_{11}, \Sigma_{12}, \Sigma_{13}$ complete the justification of the last column of the table.

Knowing that the chordal variety of the Veronese corresponds to the union of lines that meet the Veronese at least twice (counted with multiplicity), we see that the condition that there are two base points implies the inclusion of N in the chordal variety $\mathcal{C}$.

By definition of $\mathcal{C}$, the intersection $N \cap \mathcal{C}$ corresponds to the set of lines in $\mathcal{Q}$, which intersect the center of projection N and twice $\mathcal{V}$. Thus they correspond to singular points of the image. Indeed, to each point of $N \cap \mathcal{C}$

corresponds a double line. See [12], [8] p.266-273. In particular, in the case S_4 which is a quadratic cone, counted twice, every point of N yields a singular point of S, so that $N \subset \mathcal{C}$. This justifies the second column of this table.

The entries of this table which are not yet determined, are obtained by an exhaustive treatment of the normal forms of the remaining cases. □

Notice that the subcases Σ_1, Σ_2, (resp. Σ_4, Σ_5) coincide in the classification of [12], but are separated according to the signature of the quadratic forms of $N \cap \mathcal{V}$ in [8]. In our approach, we are able to discriminate all the cases only by rational computation, as we will discuss in the next section.

The (proper) Steiner surfaces have all three singular lines D_1, D_2, D_3, corresponding to the points of $N \cap \mathcal{C}$. They are classified as follows:

Σ_i	D_1, D_2, D_3
Σ_1	real and not coplanar
Σ_2	real and not coplanar
Σ_3	one real and two conjugate complex and not coplanar
Σ_4	all real with one double
Σ_5	all real with one double
Σ_6	a triple line

See fig. 1 for a view of these different surfaces. They are obtained using a triangulation algorithm implemented by Lionel Deschamps in the AXEL library [6]. This algorithm keeps safe the topology of the surface and isolates the singularities in boxes. In particular, the singular locus is identified directly during this triangulation process.

3.1. Effective Classification

A natural question is to determine which class a given parameterization belongs to. The classification is based on the dimension, the degree and the number of real points in $N \cap \mathcal{V}$, $N \cap \mathcal{C}$, $L \cap \mathcal{V}$. Let us see how, we compute them effectively.

First, we compute $N \cap \mathcal{C}$. More precisely, we don't need to compute the solutions but to determine the number of real roots of $N \cap \mathcal{C}$. N is a projective line generated by two elements N_1 and N_2. We need to know when $\lambda_1 N_1 + \lambda_2 N_2$ is a degenerate quadratic form. Its determinant yields a homogeneous polynomial of degree 3 in λ_1 and λ_2. In order to compute its number of real roots, we test if $(\lambda_1, \lambda_2) = (0 : 1)$ is a root and compute its multiplicity. Then we reduce the problem to one variable by setting $(\lambda_1, \lambda_2) = (1 : t)$. To determine the number of real roots, we use Sturm sequences [21, 15].

The other operation which has to be performed is the computation of the dimension and the number of real roots of $L \cap \mathcal{V}$, where $L = \langle f_0, f_1, f_2, f_3 \rangle$. It comes down to determine $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ so that $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ is a double line, that is satisfying the equations (7). The

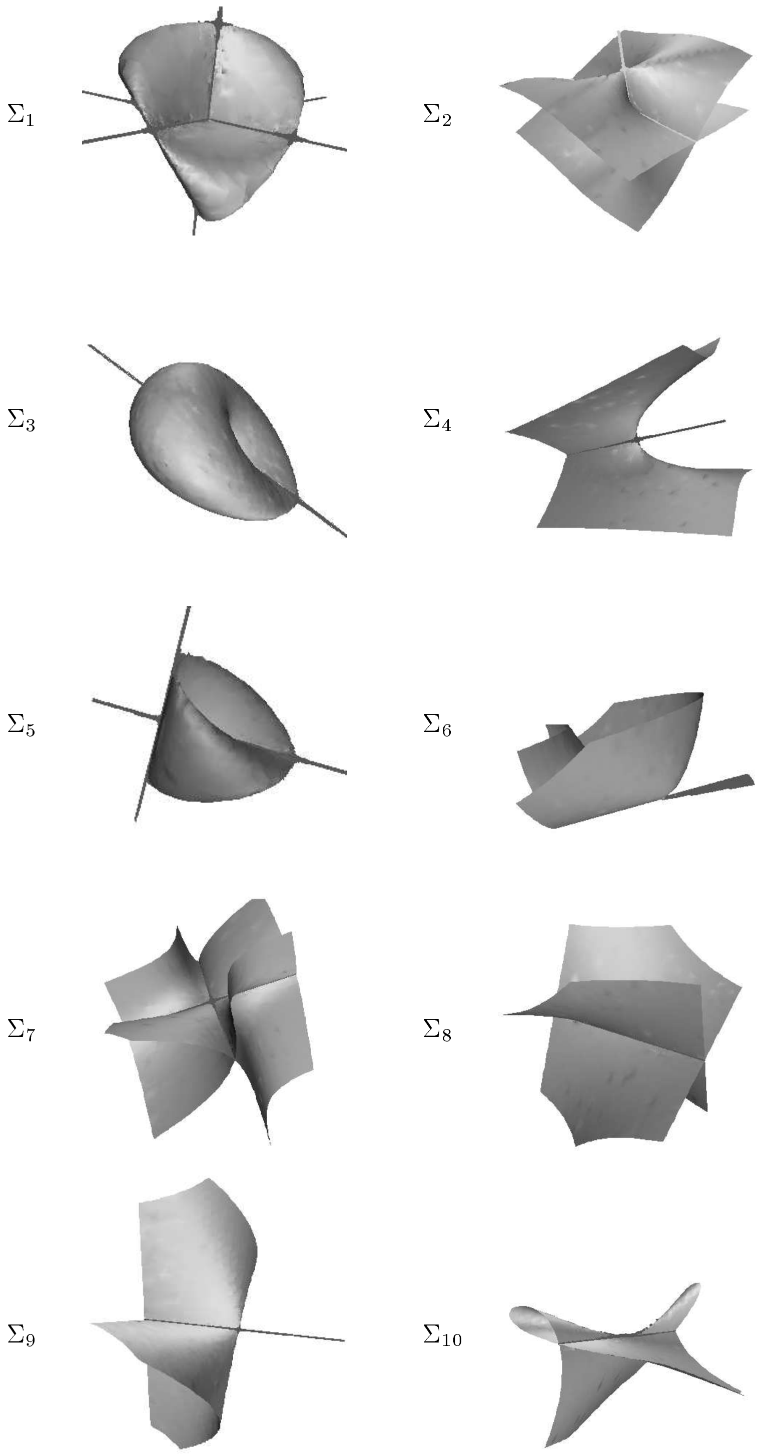

Figure 1. Views of the different type of "Steiner" surfaces.

dimension and multiplicity of the solutions of this algebraic problem can be done algorithmically via algebraic methods described in [10, 15, 13]. In particular, to compute the number of real and complex (non real) roots of $L \cap \mathcal{V}$, we use Hermite's theorem [15, 13]. It reduces our problem to the computation of the signature of a quadratic form, whose coefficients are rational functions of the coefficients of the f_i, $i = 0, \ldots, 3$. The use of such tools has the double advantage of being efficient and to certify the results, since we are dealing with rational computation.

Using these certified tools, we deduce easily from the previous table an algorithm which, for a given surface of type (2), classifies it after having computed the type of intersection of $L \cap \mathcal{V}$, $N \cap \mathcal{C}$ and $N \cap \mathcal{V}$.

§4. Implicit Equation of Steiner Surfaces

Our aim, in this section, is to compute the implicit equation of the image of a parameterization σ of type (2) with no base point, corresponding to Steiner surfaces. The variables of the domain are t_0, t_1, t_2 and the ones of the co-domain $\mathbf{y} = (y_0, y_1, y_2, y_3)$.

We are going to build a matrix $M_{\mathbf{u}}$, which entries depend on $\mathbf{u}$ and such that its determinant is the implicit equation of the surface. First from the parameterization, we deduce 4 equations in the unknowns $(t_0^2, t_1^2, t_2^2, t_0t_1, t_0t_2, t_1t_2, \lambda)$

$$\begin{cases} \lambda y_0 = f_0(\mathbf{t}) \\ \lambda y_1 = f_1(\mathbf{t}) \\ \lambda y_2 = f_2(\mathbf{t}) \\ \lambda y_3 = f_3(\mathbf{t}) \end{cases},$$

where $\mathbf{t} = (t_0, t_1, t_2)$. Consider first the tangent plane at $\mathbf{y} = \sigma(\mathbf{t})$ of equation (of degree 3):

$$T(\mathbf{t}, \mathbf{y}) = \det\left(\frac{\partial\sigma(\mathbf{t})}{\partial t_0}, \frac{\partial\sigma(\mathbf{t})}{\partial t_1}, \frac{\partial\sigma(\mathbf{t})}{\partial t_2}, \mathbf{y}\right) = 0. \tag{8}$$

We get 3 other equations from the following proposition:

Proposition 9. *If $\mathbf{y} = \sigma(\mathbf{t})$, then $\mathbf{t}$ is a critical point of the cubic defined by $T(\mathbf{t}, \mathbf{y}) = 0$.*

Proof. We have to verify that $\frac{\partial T(\mathbf{t},\mathbf{y})}{\partial t_i}$ equals 0 for $\mathbf{y} = \sigma(\mathbf{t})$ and $i = 0 \ldots 2$. As the function $L : \mathbf{t} \mapsto L(\mathbf{t}) = T(\mathbf{t}, \sigma(\mathbf{t}))$ is identically 0, so are the derivatives. For $i = 0 \ldots 2$, we have:

$$\frac{\partial T(\mathbf{t}, \mathbf{y})}{\partial t_i}\Big|_{\mathbf{y}=\sigma(\mathbf{t})} = \frac{\partial L(\mathbf{t})}{\partial t_i} + \det\left(\frac{\partial\sigma(\mathbf{t})}{\partial t_0}, \frac{\partial\sigma(\mathbf{t})}{\partial t_1}, \frac{\partial\sigma(\mathbf{t})}{\partial t_2}, \frac{\partial\sigma(\mathbf{t})}{\partial t_i}\right) = 0,$$

which proves that $\Delta_i = \frac{\partial T(\mathbf{t},\mathbf{y})}{\partial t_i}$ vanishes at $\mathbf{y} = \sigma(\mathbf{t})$. $\square$

We dispose now of 7 equations and considering the unknowns $(t_0^2, t_1^2, t_2^2, t_0t_1, t_0t_2, t_1t_2, \lambda)$, we deduce the linear system:

$$M_{\mathbf{y}}.\begin{pmatrix} t_0^2 \\ t_1^2 \\ t_2^2 \\ t_0t_1 \\ t_0t_2 \\ t_1t_2 \\ \lambda \end{pmatrix} = 0 \text{ with } M_{\mathbf{y}} = \begin{pmatrix} f_0 & -y_0 \\ f_1 & -y_1 \\ f_2 & -y_2 \\ f_3 & -y_3 \\ \Delta_0 & 0 \\ \Delta_1 & 0 \\ \Delta_2 & 0 \end{pmatrix}, \tag{9}$$

where the Δ_i correspond to the partial derivatives of $T(\mathbf{t}, \mathbf{u})$.

Proposition 10. *If the parameterization has no base points,* $\det(M_{\mathbf{y}})$ *is the implicit equation of the Steiner surface associated to the parameterization* (1).

Proof. We know that the determinant of $M_{\mathbf{y}}$ is a multiple of the implicit equation, because this determinant is identically 0 for $\mathbf{y} = \sigma(\mathbf{t})$. The last three equations are linear forms in $\mathbf{y}$, so the degree in $\mathbf{y}$ of the determinant is at most 4. We know that the degree of the surface is 4, thus the determinant is either the implicit equation up to a non zero constant or 0. The action of $GL(3) \times GL(4)$ induces invertible transformations on the left and on the right of $M_{\mathbf{u}}$. Thus we just have to test on an element of each class of equivalence (in $\mathbb{C}$). We verify that it is not 0, so that we get the implicit equation. $\square$

Compared with the approach of [20], this construction has the advantage to produce an equation *of minimal degree in the coefficients of the* f_i.

§5. Inversion

For the search of the preimages by the map σ, we will need to work in the affine space. By a random change of coordinates, we can suppose that we search the preimages of a point $\tilde{\gamma} = (u_0, u_1, u_2, u_3)$ whose first coordinate u_0 is not 0 and can be assumed to be 1. So hereafter we will assume that $u_0 = 1$, $\gamma = (u_1, u_2, u_3) \in \mathbb{K}^3$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We are interested in computing the inverse image by σ of γ, that is the $\mathbf{t} \in \mathbb{P}^2$ such that $\sigma(\mathbf{t}) = \gamma$ satisfying

$$\begin{cases} u_1 = f_1(\mathbf{t})/f_0(\mathbf{t}) \\ u_2 = f_2(\mathbf{t})/f_0(\mathbf{t}) \\ u_3 = f_3(\mathbf{t})/f_0(\mathbf{t}) \end{cases}$$

Here are some notations: we denote by $\widehat{\mathbb{K}[\mathbf{t}]_2}$ the dual space of the vector space of polynomials with degree lower or equal to 2. If I is an ideal of the algebra $\mathbb{K}[\mathbf{u},\mathbf{t}]$, $\mathcal{A} = \mathbb{K}[\mathbf{u},\mathbf{t}]/I$ is the quotient algebra. Let $\mathcal{Z}(I)$ be the zero locus of I.

For any vector space E, for any elements $p_1,\ldots,p_s \in E$, we denote by $\langle p_1,\ldots,p_s\rangle$ the vector space span by these elements in E. We denote by $\widehat{E}$ the set of linear forms from E to $\mathbb{K}$.

The dual of the quotient algebra $\mathcal{A}$ is denoted by $\widehat{\mathcal{A}}$. It is isomorph to the vector space $I^{\perp}$ of all the linear forms vanishing on I.

5.2. Theoretical Aspect

In this subsection, we will explain the link between the rank of the matrix $M_{\mathbf{u}}$ and the number of preimages of $\mathbf{u}$. For any $\mathbf{u} = (u_1,u_2,u_3) \in \mathbb{K}^3$, we write $I_{\mathbf{u}}$, the ideal

$$(f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0) \subset \mathbb{K}[t_0,t_1,t_2] = \mathbb{K}[\mathbf{t}].$$

We denote $F_i = f_i - u_i f_0, i = 1,\ldots,3$ and $\mathcal{A}_{\mathbf{u}} = \mathbb{K}[\mathbf{t}]/I_{\mathbf{u}}$ the quotient algebra. By linear operations on the rows of the linear system (9), we get:

$$\begin{pmatrix} f_0 & -1 \\ f_1 - u_1 f_0 & 0 \\ f_2 - u_2 f_0 & 0 \\ f_3 - u_3 f_0 & 0 \\ \Delta_0 & 0 \\ \Delta_1 & 0 \\ \Delta_2 & 0 \end{pmatrix} \begin{pmatrix} t_0^2 \\ t_1^2 \\ t_2^2 \\ t_0 t_1 \\ t_0 t_2 \\ t_1 t_2 \\ \lambda \end{pmatrix} = 0.$$

Proposition 11. *The zeroes of* $(F_1,F_2,F_3,\Delta_0,\Delta_1,\Delta_2)$ *coincide with those of* $I_{\mathbf{u}}$.

Proof. We have to show that if $\alpha = (\alpha_0,\alpha_1,\alpha_2)$ is a zero of (F_1,F_2,F_3) then it is a zero of $(\Delta_0,\Delta_1,\Delta_2)$. Since we work in the projective space one of the α_i is different from 0, say α_0.

By linear operations on the rows of (8), we have:

$$T(\alpha) = \det(\frac{\partial F(\alpha)}{\partial t_0}, \frac{\partial F(\alpha)}{\partial t_1}, \frac{\partial F(\alpha)}{\partial t_2}) = 0 \tag{10}$$

with $F(\mathbf{t}) = (F_1,F_2,F_3)$. Euler's identity yields:

$$t_0 \frac{\partial F(\mathbf{t})}{\partial t_0} + t_1 \frac{\partial F(\mathbf{t})}{\partial t_1} + t_2 \frac{\partial F(\mathbf{t})}{\partial t_2} = 2F.$$

We deduce that:

$$T(\mathbf{t}) = \frac{1}{t_0} \det(2F - t_1 \frac{\partial F(\mathbf{t})}{\partial t_1} - t_2 \frac{\partial F(\mathbf{t})}{\partial t_2}, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2}) = \frac{1}{t_0} \det(2F, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2})$$

Differentiating with respect to t_0, t_1, t_2, we obtain :

$\Delta_0 = \frac{\partial T(\mathbf{t})}{\partial t_0} = \frac{-1}{t_0^2}\det(2F, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2}) + \frac{1}{t_0} det(2\frac{\partial F(\mathbf{t})}{\partial t_0}, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2}) + \frac{1}{t_0} det(2F, \frac{\partial^2 F(\mathbf{t})}{\partial t_0 \partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2}) + \frac{1}{t_0} det(2F, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial^2 F(\mathbf{t})}{\partial t_0 \partial t_2})$

$\Delta_1 = \frac{1}{t_0} det(2\frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2}) + \frac{1}{t_0} det(2F, \frac{\partial^2 F(\mathbf{t})}{\partial t_1^2}, \frac{\partial F(\mathbf{t})}{\partial t_2} + \frac{1}{t_0} det(2F, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial^2 F(\mathbf{t})}{\partial t_1 \partial t_2})$

$\Delta_1 = \frac{1}{t_0} det(2\frac{\partial F(\mathbf{t})}{\partial t_2}, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial F(\mathbf{t})}{\partial t_2}) + \frac{1}{t_0} det(2F, \frac{\partial^2 F(\mathbf{t})}{\partial t_1 \partial t_2}, \frac{\partial F(\mathbf{t})}{\partial t_2} + \frac{1}{t_0} det(2F, \frac{\partial F(\mathbf{t})}{\partial t_1}, \frac{\partial^2 F(\mathbf{t})}{\partial t_2^2})$

We deduce that $\alpha_0^2 \Delta_0(\alpha) = \alpha_0 \Delta_1(\alpha) = \alpha_0 \Delta_2(\alpha) = 0$. So as $\alpha_0 \neq 0$, α is a zero of $\Delta_0, \Delta_1, \Delta_2$.

□

We recall here a classical result [10] on polynomial systems with a finite number of complex solutions, that we will need hereafter:

Proposition 12. *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal such that $\mathcal{Z}(I)$ is finite, then* $\dim_{\mathbb{K}} \mathcal{A} = \dim_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n]/I)$ *is the number of roots of $\mathcal{Z}(I)$ (counted with multiplicity).*

Proposition 11 shows that the ideals $I_{\mathbf{u}}$ and $(F_1, F_2, F_3, \Delta_0, \Delta_1, \Delta_2)$ define the same solutions. Here is the main result, which precises this point:

Proposition 13. *For any $p \in \mathbb{K}^3$, we have* $\dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = \dim(\mathrm{Ker}(M_{\mathbf{u}}))$.

Proof. We clearly have:

$$\dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = \dim_{\mathbb{K}} \widehat{\mathcal{A}_{\mathbf{u}}} = \dim I_{\mathbf{u}}^{\perp} \leq \dim(\mathrm{Ker}(M_{\mathbf{u}})).$$

From this result and the fact that $\dim(\mathrm{Ker}(M_{\mathbf{u}})) \leq 3$ (by hypothesis the polynomials f_i are linearly independent), we deduce that $\dim \mathcal{A}_{\mathbf{u}} \leq 3$.

Consequently we have one of the following possibilities:

$$\begin{array}{lcl} \dim(\mathrm{Ker}(M_{\mathbf{u}})) = 0 & \Longrightarrow & \dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = 0 \\ \dim(\mathrm{Ker}(M_{\mathbf{u}})) = 1 & \Longrightarrow & \dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} \leq 1 \\ \dim(\mathrm{Ker}(M_{\mathbf{u}})) = 2 & \Longrightarrow & \dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} \leq 2 \\ \dim(\mathrm{Ker}(M_{\mathbf{u}})) = 3 & \Longrightarrow & \dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} \leq 3 \end{array}$$

First, if $\dim \mathcal{A}_{\mathbf{u}} = 0$, we deduce from proposition 12 that $\mathcal{Z}(I_{\mathbf{u}}) = \emptyset$ (i.e $\mathbf{u}$ does not belong to S), thus that $\det(M_{\mathbf{u}}) \neq 0$ and $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 0$.

Secondly, we show that $\dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = 1$ implies $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 1$. Consider the comatrix $\tilde{M}_{\mathbf{u}}$ of the matrix $M_{\mathbf{u}}$ at $\mathbf{u} = \sigma(\mathbf{t})$. An explicit computation yields the following form:

$$\tilde{M}_{\mathbf{u}} = D_3 \times \begin{pmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & t_0 \\ . & . & . & . & . & . & . & t_1 \\ . & . & . & . & . & . & . & t_2 \end{pmatrix}.$$

where D_3 is the product of the equations of the three double lines, which image by σ is the singular locus of the surface. When $\mathbf{u}$ has only one preimage $\mathbf{t}$, the point is not on a double line. Thus $D_3 \neq 0$. Since one of the coordinate t_i is not zero, it implies that the corresponding cofactor of $M(\mathbf{u})$ is not zero. Therefore, the rank of the matrix is at least 6, and exactly 6, because the point has a preimage. Consequently, we have the equivalence between $\dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = 1$ and $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 1$.

Next, we have to show that $\dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = 2$ implies $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 2$. In this case, $\mathbf{u}$ is on one and only one double line. Explicit symbolic computation with the canonical representatives of the different classes of equivalence and generic points on the corresponding double lines allow us to deduce that $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 2$.

For the last case $\dim_{\mathbb{K}} \mathcal{A}_{\mathbf{u}} = 3$, the only possibility is $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 3$, which concludes the proof of the theorem. □

5.3. Effective Aspect

The aim is to determine by effective means the preimages. Consider a point $\mathbf{u} = (u_1, u_2, u_3)$ on the surface. We are looking for its different preimages. There exists an affine subspace of $\mathbb{P}^2$, which contains all those preimages. We choose at random such affine subspace (generically it will suit). For an affine subspace, containing all the preimages induces some properties that can be checked during the algorithm and if a property is not checked, we choose a new random subspace. Hereafter we assume that the subspace is such that $t_0 = 1$ and we search the two other coordinates of the preimages.

Hereafter, to simplify the notation, we denote by f_i the affine polynomial $f_i(1, t_1, t_2)$. We will work in the quotient algebra $\mathcal{A}_{\mathbf{u}} = \mathbb{K}[\mathbf{t}]/I_{\mathbf{u}}$ where $I_{\mathbf{u}} = (f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2)$. In this perspective, next proposal is essential:

Proposition 14. *A linear form Λ which vanishes on $\langle f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2\rangle$ is the restriction to polynomials of degree 2 of a linear form $\overline{\Lambda}$ which vanishes on $I = (f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2)$ i.e $\overline{\Lambda} \in I^{\perp}$.*

Proof. Consider the restriction $I^{\perp} \xrightarrow{\pi} \widehat{\mathbb{K}[\mathbf{t}]_2}$, where $\mathbb{K}[\mathbf{t}]_2$ is the set of polynomials of degree ≤ 2, in $\mathbf{t}$ π is injective and $\mathrm{Im}(\pi) \subset \langle f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2\rangle^{\perp}$. Let us prove that

$$\mathrm{Im}(\pi) = \langle f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2\rangle^{\perp}.$$

Since π is injective, $\dim(\mathrm{Im}(\pi)) = \dim I^{\perp} = \dim A_{\mathbf{u}}$ and by definition,

$$\dim\langle f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2\rangle^{\perp} = \dim \mathrm{Ker}(M_{\mathbf{u}}).$$

So by proposition 12, the dimensions are equal. □

Let us see how to compute effectively the preimages. First we build $M_{\mathbf{u}}$, deduce $\dim \mathrm{Ker}(M_{\mathbf{u}})$ and thus the number of preimages of $\mathbf{u}$. We distinguish three cases:

• If $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 1$, $\mathbf{u}$ has only one preimage t=(t_1, t_2). Assume Λ a vector of the kernel of $M_{\mathbf{u}}$. As the kernel is of dimension 1, Λ and $(1, t_1^2, t_2^2, t_1t_2, t_2, t_1)$ are collinear. We deduce that :

$$(1, t_1^2, t_2^2, t_1t_2, t_2, t_1) = \frac{1}{\Lambda[1]}\Lambda.$$

Thus $t_1 = \frac{\Lambda[t_1]}{\Lambda[1]}$ and $t_2 = \frac{\Lambda[t_2]}{\Lambda[1]}$ where $\Lambda[t_2]$ (resp. $\Lambda[t_1]$) corresponds to the 5^{th} (resp. 6^{th}) coordinate of Λ. If $\Lambda[1] = 0$ the affine subspace must be changed, we take a new one and restart.

• If $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 2$, $\mathbf{u}$ has two preimages. Consider 2 vectors Λ_1 and Λ_2 forming a basis of the kernel of $M_{\mathbf{u}}$. By transposition of $M_{\mathbf{u}}.\Lambda_i = 0$, we can interpret Λ_i as a linear form vanishing over $\langle f_1 - u_1 f_0, f_2 - u_2 f_0, f_3 - u_3 f_0, \Delta_0, \Delta_1, \Delta_2\rangle$. From proposition 14, (Λ_1,Λ_2) are the restrictions of $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$. The independence of the Λ_i implies that $(\overline{\Lambda}_1, \overline{\Lambda}_2)$ is a basis of $I^{\perp}$. We know that $\dim A_{\mathbf{u}} = \dim \mathrm{Ker}(M_{\mathbf{u}}) = 2$ thus $(1, t_1)$ or $(1, t_2)$ is a basis of $\mathcal{A} = \mathbb{K}[\mathbf{t}]/I$. We deduce that one of the following minors is not zero:

$$\begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) \\ \Lambda_1(t_1) & \Lambda_2(t_1) \end{pmatrix} \qquad \begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) \\ \Lambda_1(t_2) & \Lambda_2(t_2) \end{pmatrix}.$$

(if both minors vanish, the map must be changed, we take a new one and restart). We consider the matrix of multiplication by t_2 in the quotient algebra $\mathcal{A}$: $\mathcal{A} \xrightarrow{M_{t_2}} \mathcal{A}$, which induces the transposed map: $\widehat{\mathcal{A}} \xrightarrow{(M_{t_2})^T} \widehat{\mathcal{A}}$. Coordinate-wise speaking, we have

$$(M_{t_2})^T \begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) \\ \Lambda_1(t_1) & \Lambda_2(t_1) \end{pmatrix} = \begin{pmatrix} \Lambda_1(t_2) & \Lambda_2(t_2) \\ \Lambda_1(t_1t_2) & \Lambda_2(t_1t_2) \end{pmatrix}.$$

By construction the coefficients of Λ_i correspond to the coordinates of the linear form in the dual basis $(1^*, t_1^*, t_2^*, (t_1t_2)^*, t_1^{2*}, t_2^{2*})$ that is to say that the vector Λ_i is of the form:

$$\begin{pmatrix} \Lambda_i(1) \\ \Lambda_i(t_1) \\ \Lambda_i(t_2) \\ \Lambda_i(t_1t_2) \\ \Lambda_i(t_1^2) \\ \Lambda_i(t_2^2) \end{pmatrix}$$

We extract the 2×2 matrices from the matrix made up of the two vectors Λ_1 and Λ_2

$$R_1 = \begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) \\ \Lambda_1(t_1) & \Lambda_2(t_1) \end{pmatrix}, R_2 = \begin{pmatrix} \Lambda_1(t_2) & \Lambda_{t_2}(1) \\ \Lambda_1(t_1 t_2) & \Lambda_2(t_1 t_2) \end{pmatrix}$$

with $R_2 = (M_{t_2})^T\ R_1$, so that $R_2 - z\, R_1 = ((M_{t_2})^T - z\, Id)\ R_1$. If R_1 is invertible, the generalized eigenvalues of (R_2, R_1) correspond to the eigenvalues of the multiplication operator by t_2 in the quotient algebra. Those eigenvalues are the $t_2(\xi)$, $\xi \in \mathcal{Z}(I)$ [5], [18], [13]. Thus computing these eigenvalues, we deduce the second coordinate of the different zeroes of the ideal I. For the first coordinate, we consider two cases: if both values $t_2(\xi)$ are different, we get interested in the generalized eigenvectors associated with the different eigenvalues. These eigenvectors are multiples of the evaluations in the zeroes of the ideal [18], [13]; thus looking at the coordinate along t_1^* knowing that the coordinate along 1^* of an evaluation is 1, we deduce the wanted value of t_1. If both values $t_2(\xi)$ are equal, we work with the multiplication operator by t_1 and deduce the two values $t_1(\xi)$, and similarly the other coordinate.

The reasoning we have made is valid if R_1 is invertible. If it is not the case, we just need to exchange t_1 with t_2 because as we have seen, one of the two matrices

$$\begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) \\ \Lambda_1(t_1) & \Lambda_2(t_1) \end{pmatrix}, \begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) \\ \Lambda_1(t_2) & \Lambda_2(t_2) \end{pmatrix}$$

is invertible.

• Finally, if $\dim(\mathrm{Ker}(M_{\mathbf{u}})) = 3$, we construct a matrix with three independent vectors of the kernel:

$$\begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) & \Lambda_3(1) \\ \Lambda_1(t_1) & \Lambda_2(t_1) & \Lambda_3(t_1) \\ \Lambda_1(t_2) & \Lambda_2(t_2) & \Lambda_3(t_2) \\ \Lambda_1(t_1^2) & \Lambda_2(t_1^2) & \Lambda_3(t_1^2) \\ \Lambda_1(t_1 t_2) & \Lambda_2(t_1 t_2) & \Lambda_3(t_1 t_2) \\ \Lambda_1(t_2^2) & \Lambda_2(t_2^2) & \Lambda_3(t_2^2) \end{pmatrix}$$

By hypothesis $\dim\mathcal{A}_{\mathbf{u}} = \mathbb{K}[\mathbf{t}]/I_{\mathbf{u}} = 3$, thus $(1, t_1, t_2)$, $(1, t_1, t_1^2)$ or $(1, t_2, t_2^2)$ is a basis of the quotient space. We compute the 3×3 minors to find a base. (If the three minors vanish, then the map must be changed, we take a new one and restart). If $(1, t_1, t_2)$ is a base, we work similarly to the case of dimension 2. We consider the two matrices:

$$N_1 = \begin{pmatrix} \Lambda_1(1) & \Lambda_2(1) & \Lambda_3(1) \\ \Lambda_1(t_1) & \Lambda_2(t_1) & \Lambda_3(t_1) \\ \Lambda_1(t_2) & \Lambda_2(t_2) & \Lambda_3(t_2) \end{pmatrix}$$

$$N_2 = \begin{pmatrix} \Lambda_1(t_1) & \Lambda_2(t_1) & \Lambda_3(t_1) \\ \Lambda_1(t_1^2) & \Lambda_2(t_1^2) & \Lambda_3(t_1^2) \\ \Lambda_1(t_1 t_2) & \Lambda_2(t_1 t_2) & \Lambda_3(t_1 t_2) \end{pmatrix}$$

As $N_2 = (M_{t_1})^T N_1$ where $M_{\mathbf{t}_1}$ is the matrix of multiplication by t_1 in $\mathcal{A}_{\mathbf{u}}$, by computing the generalized eigenvectors of (N_1, N_2) we can conclude, because this computation yields the coordinates of the roots as the coefficients of 1^*, t_1^* and t_2^*.

If $(1, t_1, t_2)$ is not a basis but $(1, t_1, t_1^2)$ is a base, by linear combinations of the Λ_i, we may assume that Λ_1, Λ_2, Λ_3 correspond to the dual basis $(1^*, t_1^*, t_1^{2*})$. We read the decomposition of t_2 directly on the matrix along the basis $(1, t_1, t_1^2)$, we have: $t_2 = \Lambda_1(t_2) + \Lambda_2(t_2)t_1 + \Lambda_3(t_2)t_1^2$. The points (t_1, t_2) we want, have the property $F_i(1, t_1, t_2) = 0$, for $i = 1, \ldots, 3$. Thus computing the gcd of the three polynomials, substituting t_2 with its expression in t_1 and looking for the roots of an univariate polynomial, we deduce for each value of t_1, the corresponding value of t_2.

§6. Applications in Agricultural Research

Many studies in agricultural research need realistic description of the canopy. For example, the geometrical shape of plants is involved in phenomena such as competition or interception of solar radiation.

Recent models based on a botanical description of plants take into account the precise shape of plant organs as well as their geometrical organization in the three dimensional space [11], [14].

For each organ θ, its surface is defined by a parametric equation $\mathbf{u} = \sigma_\theta(\mathbf{t})$ where $\mathbf{t}$ is a point of domain $\mathcal{D}$ of the parametric plane and $\mathbf{u}$ a point of $\mathbb{R}^3$.

The geometrical plant model is too specific to be used in classical programs, so it is transformed by dividing each surface of the geometric model in standardized patches.

In many studies, good results are obtained using planar triangles. But, it can also lead to a prohibitive number of triangles when a high precision is needed. Our experiments show for instance that a torus, could be approximated by 128 patches of Steiner surfaces, for an error bound of one degree on the angles between normal vectors. But using a regular triangulation of 8000 linear triangles, we obtain an even higher error on the normal vectors.

To obtain accurate reflected ray in simulation, the normal of the surface has to be closed to the normal of the fitted patch. If so, quadratically parameterizable patches are a compromise between planar and bicubic patches. A middle approximation of a maize leaf needs about twenty triangles or only one Steiner patch. Accurate approximations need thousands of triangles or ten Steiner patches. We present in Figure 2 an artificial random generated maize canopy. Each leaf is approximated by one Steiner

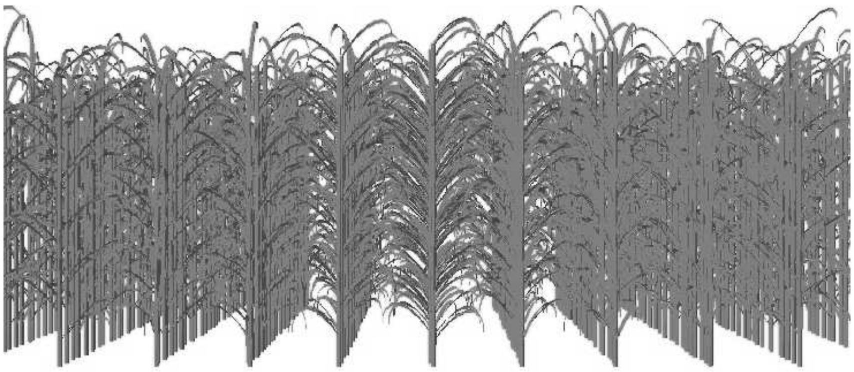

Figure 2. Maize canopy

patch, each stem by a cone. This model can be used for characterization of the microclimat at the level of the canopy, for computation of the radiations reaching each organ into photosynthetic estimation, or for characterization of radiative responses of the canopy for remote sensing applications [2], [14], based on ray-tracing techniques.

For ray-tracing, the main problem is to compute many ray-patch intersections. An answer is based on resultant matrix constructions and implicit equations. However mathematics have to be completed, together with a geometric optimization step, in order to avoid unnecessary intersection computation. This work, which lies beyond the scope of the paper, is not yet performed.

To build the model, we need an algorithm to test the intersection of two patches. Intersections of two such Steiner surfaces are the image by σ of planar curves of degree eight. The intersection of two patches is associated to a piece of this curve. Its effective computation is performed by using the implicit equation of one of the patches and by substituting the other parameterization in it. A fast and certified algorithm for the arrangement of such curves requires the manipulation of algebraic numbers. In the experimentations, we consider control approximate computation, which still require some improvement to be fully certified.

We used in this study rational surfaces with no base points. Using one or two base point rational surfaces give implicit surfaces of degree three or two. We can also get more flexible patches using conic arcs instead of triangles for the definition of the domain in parametric plane. A deeper analysis of the abilities for Steiner patches to give an accurate approximation of the surface, regarding the difficulties and the reliability of each sort of computations, is still needed. This experimentation however reveals the potential of such representations.

§7. References

1. Apery, F., *Models of the real projective plane.* Vieweg, 1987.
2. Aries, F., *Modélisation surfacique d'un couvert végétal pour l'étude du rayonnement.* PhD thesis, Nantes, 1997.
3. Aries, F. and R. Senoussi, Approximation de surfaces paramétriques par des carreaux rationnels du second degré en lancer de rayons. *Revue Internationale de CFAO et d'Informatique Graphique*, 1997.
4. Aries, F. and R. Senoussi, An Implicitization Algorithm for Rational Surfaces with no Base Points. *J. Symb. Comput.*, 31:357–365, 2001.
5. Auzinger, W. and H. J. Stetter, An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations. In *Proc. Intern. Conf. on Numerical Math.*, volume 86 of *Int. Series of Numerical Math*, pages 12–30. Birkhäuser Verlag, 1988.
6. AXEL library, `http://www-sop.inria.fr/galaad/logiciels/axel/`.
7. Busé, L., M. Elkadi, and B. Mourrain, Using projection operators in computer aided geometric design. In *Topics in Algebraic Geometry and Geometric Modeling,*, pages 321–342. Contemporary Mathematics, 2003.
8. Coffman, A., J. Schwartz, and C. Stanton, The algebra and geometry of steiner and other quadratically parametrizable surfaces. *Computer Aided Geometric Design*, 13:257–286, 1996.
9. Comon, P. and B. Mourrain, Decomposition of quantics in sums of power of linear forms. *Signal Processing*, 53(2):93–107, 1996. Special issue on High-Order Statistics.
10. Cox, D., J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Undergraduate Texts in Mathematics. Springer Verlag, New York, 1992.
11. de Reffye, P., C. Edelin, J. Françon, M. Jaeger, and C. Puech, Plant models faithful to botanical structure and development. *Computer Graphics*, 22:151–158, 1988.
12. Degen, W. L. F., The types of triangular bézier surfaces. *The Mathematics of Surfaces VI*, pages p. 153–170, 1996. G.Mullineux, ed., Oxford, 1996.
13. Elkadi, M., and B. Mourrain, *Introduction à la résolution des systèmes d'équations algébriques*, 2003. Notes de cours, Univ. de Nice (310 p.).
14. España, M .L., F. Baret, F. Aries, M. Chelle, B. Andrieu, and L. Prévot, Modeling maize canopy 3d architecture. application to reflectance simulation. *Ecological Modelling*, 122:25–43, 1999.

15. Gonzalez-Vega, L., F. Rouillier, and M.F. Roy, *Symbolic Recipes for Polynomial System Solving.* Some Tapas of Computer Algebra. Springer, 1997.
16. Harris, J., *Algebraic Geometry, a first course*, volume 133 of *Graduate Texts in Math.* New-York, Springer-Verlag, 1992.
17. Jouanolou, J. P., Formes d'inertie et résultant: un formulaire. *Adv. in Math.*, 126(2):119–250, 1997.
18. Mourrain, B., Computing isolated polynomial roots by matrix methods. *J. of Symbolic Computation, Special Issue on Symbolic-Numeric Algebra for Polynomials*, 26(6):715–738, Dec. 1998.
19. Nitsche, Johannes C. C., *Lectures On Minimal Surfaces*, volume 1. Cambridge University Press, 1989.
20. Sederberg, T. W. and D.C. Anderson, Steiner surface patches. *IEEE Comput. Graphics Appl.*, 5(5):23–36, 1985.
21. von zur Gathen, J. and J. Gerhard, *Modern computer algebra.* Cambridge University Press, New York, 1999.

Aries Franck, INRA,
Site Agroparc, 84914 Avignon Cedex 9, France
franck@avignon.inra.fr

Mourrain Bernard, GALAAD, INRIA,
BP 93, 06902 Sophia Antipolis, France
Bernard.Mourrain@sophia.inria.fr

Técourt Jean-Pierre, GALAAD, INRIA,
BP 93, 06902 Sophia Antipolis, France
Jean-Pierre.Tecourt@sophia.inria.fr