Algebraic Sparse Modeling and Applications

BY B. MOURRAIN

July 13, 2014

Abstract

An overview of the reconstruction problem of sum of exponentials functions from truncated series is presented. We recall Prony's method for univariate problems, analyse the algebraic properties underlying this reconstruction problem and describe an extension of Prony's method for sparse modeling in several variables. Applications of this method are developed. A special attention is given to tensor decomposition problems.

Table of contents

1 Introduction . . . 2

1.1 Examples . . . 2

1.2 A general framework . . . 4

Reconstruction from Fourier coefficients. . . . 5

Reconstruction from values. . . . 5

1.3 Previous works . . . 6

2 Prony's method in one variable. . . . 6

3 Duality and Hankel operators . . . 7

3.1 Duality . . . 7

3.2 Artinian algebra . . . 9

3.3 Hankel operators . . . 10

3.4 Artinian Gorenstein algebra . . . 11

3.5 The support of σ . . . 14

3.6 The case of simple roots . . . 15

4 Flat extensions and decomposition algorithm . . . 16

4.1 Computing an orthogonal basis of $\mathcal{A}_\sigma$ . . . 19

4.2 Computing the support of σ . . . 21

4.3 Generalized Prony method . . . 21

Sparse reconstruction of sum of exponentials. . . . 21

Sparse reconstruction of sum of Dirac measures. . . . 22

Example. . . . 23

4.4 Numerical improvements . . . 23

5 Applications . . . 24

5.1 Vanishing ideal of points . . . 24

5.2 Signal reconstruction . . . 25

5.2.1 Wave identification . . . 26

5.2.2 Random exponential sum with complex parameters . . . 26

5.2.3 Recovering signal parameters. . . . 28

5.3 Sparse interpolation . . . 28

Reformulation. . . . 28

Example. 29

6 Tensor decomposition 29

6.1 Symmetric tensors 29
6.1.1 Sparse decomposition of tensors 29
6.1.2 Examples 31
6.1.3 Apolar ideals 32
6.2 Generalized decomposition 33
6.3 Decomposition of tensors 36
6.4 Geometry of tensor decomposition 37

Bibliography 38

1 Introduction

In many context of applications, it is nowadays possible to recover a huge amount of information on a phenomenon that we want to analyse. Sensors, scanners, etc. can produce a deluge of data, which in principle should be helpful for this analysis. But too much information may destroy the information. An important problem is to extract for this data, a structured representation which is simpler to manipulate and understand. Recovering this underlying structure can boil down to compute an explicit representation of a function in a given basis of a functional space. Classical interpolation problems can be used in this framework, as well as Fourrier decomposition. Usually, a “good” numerical approximation of the function as a linear combination of the function basis elements is sufficient. But the choice of the function basis is very important from this perspective. It can lead to a sparse representation which involve few non-zero coefficients or many coefficients. To illustrate this problem, consider for instance a linear function over an intervalle of $\mathbb{R}$. In the monomial basis it is represented by two coefficients. Its description as Fourier series involves an infinite sequence of (decreasing) Fourrier coefficients.

This raises two important problems:

- How to determine a good functional space, in which the functions we consider have a sparse representation with few non-zero coefficients, which exhibit the main characteristics of these functions.
- How to compute such a decomposition, using a small (if not minimal) amount of information or measurements.

These problems known as sparse modeling have in important impact in many domains such as Signal Processing, Image Analysis, Computer Vision, Statistics ...

Hereafter, we are going to study a specific reconstruction problem of truncated series, which allows to treat many other sparse modeling problems. With the multi-index notation: $\forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\forall \boldsymbol{u} \in \mathbb{C}^n$, $\alpha! = \prod_{i=1}^n \alpha_i!$, $\boldsymbol{u}^\alpha = \prod_{i=1}^n u_i^\alpha$ and $\boldsymbol{e}_\xi(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \xi^\alpha \boldsymbol{z}^\alpha = e^{\langle \xi, \boldsymbol{z} \rangle} = e^{\xi_1 z_1 + \cdots + \xi_n z_n}$, it can be stated as follows:

Problem 1. *(Reconstruction from truncated series)* Given the coefficients σ_α of the series

$$\sigma(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!}$$

for $|\alpha| \leqslant d$, recover r the number of terms, the points $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r \in \mathbb{C}^n$ and the polynomial coefficients $\omega_i(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$ such that

$$\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i(\boldsymbol{z}) \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}).$$

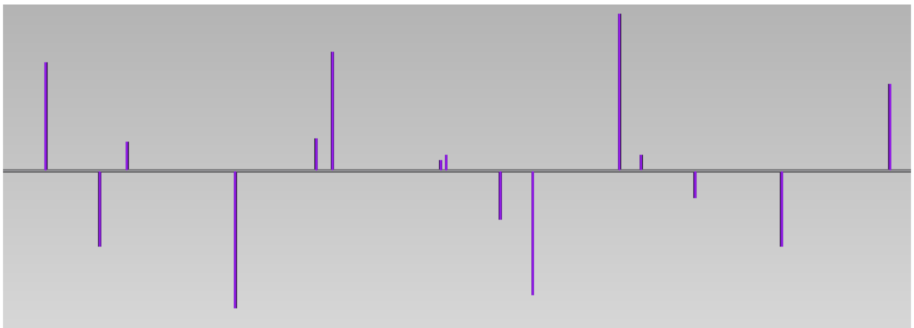
1.1 Examples

Let us show in some examples how this problem can appear.

Example 1. *(Train of spikes)* A train of spikes is a (complex) measure μ which is a weighted sum of Dirac measures

$$\mu = \sum_{i=1}^{r} \omega_i \delta_{\xi_i} \tag{1}$$

where $\xi_1, \ldots, \xi_r$ are pairwise distinct points of $\mathbb{C}$, δ_{ξ_i} is the Dirac measure at ξ_i and $\omega_i \in \mathbb{C} \setminus \{0\}$. It can represent for instance a sequence of impulsions over an interval of time, when $\xi_i \in \mathbb{R}$.



We want to recover

- the number r of points,
- the distinct points $\xi_1, \ldots, \xi_r \in \mathbb{C}$,
- the *weights* $w_i \in \mathbb{C} \setminus \{0\}$,

from measurements of these impulsions. Classical measurements used in Signal processing are the Fourier coefficients of the measure or its convolution with a function f:

$$\sigma_k = \frac{1}{2T} \int_{\frac{T}{2}}^{\frac{T}{2}} f(x)\, e^{-2i\pi\,(k\,x)}\, d\mu.$$

If μ is of the form (1), the generating series of σ_k is

$$\sigma(\boldsymbol{z}) = \sum_{k \in \mathbb{N}} \sigma_k \frac{z^k}{k!} = \sum_{i=1}^{r} w_i\, f(\xi_i)\, e^{\xi_i z}.$$

The decomposition problem consists in recovering the number r, the frequencies $\xi_1, \ldots, \xi_r \in \mathbb{C}$ and the weights $w_i\, f(\xi_i) \in \mathbb{C} \setminus \{0\}$ from the first (Fourier) coefficients $(\sigma_k)_{0 \leqslant k \leqslant 2r-1}$ of the series $\sigma(\boldsymbol{z})$.

Another problem, related to the previous one by Fourier transform, is the decomposition of an exponential polynomial from its values.

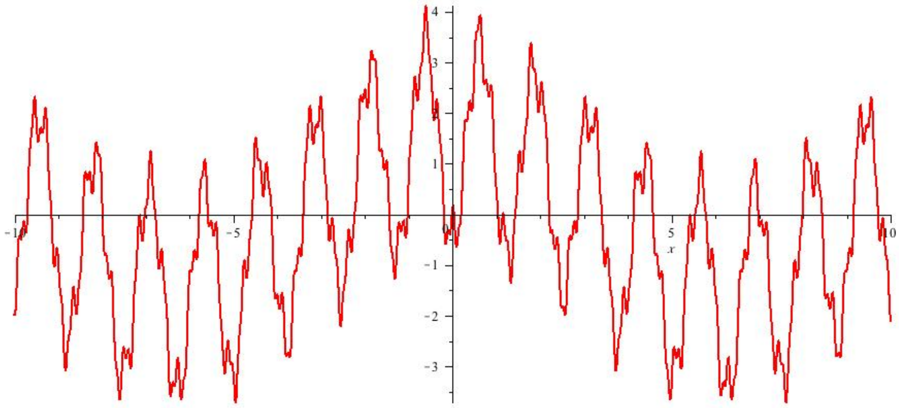
Example 2. *(Recovery of exponential polynomials from values)* Given a function $h \in C^\infty(\mathbb{C})$ of the form

$$x \in \mathbb{C} \mapsto h(x) = \sum_{i=1}^{r} a_i(x)\, e^{f_i\, x} \tag{2}$$

where $f_1, \ldots, f_r \in \mathbb{C}$ are pairwise distinct, $a_i(x) \in \mathbb{C}[x] \setminus \{0\}$, the problem consists in recovering

- the distinct *frequency vectors* $f_1, \ldots, f_r \in \mathbb{C}$,
- the *polynomial coefficients* $a_i(x) \in \mathbb{C}[x] \setminus \{0\}$,

This problem is sometimes called blind identification in signal processing [17]. We assume the "signal" h is the superposition of signals of a certain family, namely a product of a polynomial by an exponential and we want to find its decomposition. This may be useful to identify the number sources and some characteristics of these sources. Here is an example of a signal, which is the superposition of several "oscillations" with different frequencies.



This problem can also be reformulated into a truncated series reconstruction problem. By choosing an arithmetic progression of points (α) in $\mathbb{C}$, for instance $\mathbb{N}$, we can associate to h, the generating series:

$$\sigma(z) = \sum_{\alpha \in \mathbb{N}} h(\alpha) \frac{z^\alpha}{\alpha!} \in \mathbb{C}[[z]],$$

where $\mathbb{C}[[z]]$ is the ring of formal power series in z. If h is of the form (21), then

$$\sigma(z) = \sum_{i=1}^{r} \sum_{\alpha \in \mathbb{N}} a_i(\alpha)\, \xi_i^\alpha \frac{z^\alpha}{\alpha!} = \sum_{i=1}^{r} b_i(z)\, e^{\xi_i z}$$

where $\xi_i = e^{f_i}$ and $b_i(z)$ are polynomials in z, uniquely determined by a_i.

In practice, the evaluation of h may be marred by errors of measurements or by noise, the decomposition problem consists then in computing an approximate decomposition which satisfies

$$|\sigma(\mathbf{p}) - \sum_{i=1}^{r} a_i\,(\boldsymbol{p}) e^{\langle \mathbf{f}_i, \mathbf{p} \rangle}| < \epsilon, \forall \boldsymbol{p} \in P,$$

for some given tolerance ϵ and a set of points $P \subset \mathbb{C}$.

1.2 A general framework

The problem of truncated series can be considered in a general context that we describe now :

- Let $\mathfrak{F}$ be a functional space (in which "leaves the signals").
- Let $S_1, \ldots, S_n: \mathfrak{F} \rightarrow \mathfrak{F}$ be linear operators of $\mathfrak{F}$ which are commuting: $S_i \circ S_j = S_j \circ S_i$.
- Let $\Delta: h \in \mathfrak{F} \mapsto \Delta[h] \in \mathbb{C}$ be a linear functional on $\mathfrak{F}$.

The problem of decomposition of an element $h \in \mathfrak{F}$ can be restated in terms of its generating series:

Definition 3. *For $h \in \mathfrak{F}$, the generating series associated to h is*

$$\sigma_h(\boldsymbol{z}) \;=\; \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \Delta[S^\alpha(h)]\, \boldsymbol{z}^\alpha \tag{3}$$

where $S^\alpha = S_1^{\alpha_1} \circ \cdots \circ S_n^{\alpha_n}$.

The elements in $\mathfrak{F}$ that we are going to chose to decompose $h \in \mathfrak{F}$ are the eigenfunctions of the operators S_i. If E is an eigenfunction of S_i for the eigenvalue ξ_i, we easily check that

$$\sigma_E(\boldsymbol{z}) = \boldsymbol{e}_\xi(\boldsymbol{z})\, \Delta(E).$$

To find the sparse decomposition of the signal h, we consider the truncated generating series $\sigma_h(\boldsymbol{z})$. The following results shows that solving the truncated series problem for $\sigma_h(\boldsymbol{z})$ yields a solution of the sparse recovery problem.

Proposition 4. *Let $S_1, \ldots, S_n$ be linear operators of $\mathfrak{F}$ which are commuting: $S_i \circ S_j = S_j \circ S_i$. Let $E_1, \ldots, E_r$ be eigenfunctions of these operators: $S_j(E_i) = \xi_{i,j} E_i$ with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^n$ pairwise distinct. Let $\Delta: h \in \mathfrak{F} \mapsto \Delta[h] \in \mathbb{C}$ be a linear functional on $\mathfrak{F}$ such that $\Delta[E_i] = 1$.*

If $h = \sum_{i=1}^r \omega_i E_i$ then $\sigma_h(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \Delta[S^\alpha(h)] \frac{\boldsymbol{z}^\alpha}{\alpha!} = \sum_{i=1}^r \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$.

Proof. If $h = \sum_{i=1}^r \omega_i E_i$, then $S^\alpha(h) = \sum_{i=1}^r \omega_i S^\alpha(E_i) = \sum_{i=1}^r \omega_i \xi^\alpha E_i$ and we have

$$\sigma_h(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \Delta[S^\alpha(h)] \frac{\boldsymbol{z}^\alpha}{\alpha!} = \sum_{i=1}^r \sum_{\alpha \in \mathbb{N}^n} \omega_i \xi^\alpha \Delta[E_i] \frac{\boldsymbol{z}^\alpha}{\alpha!} = \sum_{i=1}^r \omega_i \sum_{\alpha \in \mathbb{N}^n} \xi^\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} = \sum_{i=1}^r \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z}).$$

□

The general decomposition problem consists in computing a decomposition of h as a weighted sum of eigenfunctions, from the first coefficients of its generating series σ_h.

If the map $\sigma: h \in \mathfrak{F} \mapsto \sigma_h(\boldsymbol{z}) \in \mathbb{C}[[\boldsymbol{z}]]$ is injective, then the solution $\sigma_h(\boldsymbol{z}) = \sum_{i=1}^r \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ of the truncated series problem yields the solution of this decomposition problem in $\mathfrak{F}$: $h = \sum_{i=1}^r \omega_i E_i$.

Let us illustrate how the previous examples fit with this framework.

Reconstruction from Fourier coefficients. In this problem, we take

- $\mathfrak{F}$ is the space of distributions on $\mathbb{C}$.
- $S: h(x) \mapsto e^{-2\pi \boldsymbol{i} x} h(x)$ is the multiplication by $e^{2\pi \boldsymbol{i} x}$.
- $\Delta: h(x) \mapsto \frac{1}{T} \int_{-\frac{T}{2}}^{-\frac{T}{2}} h(x)\, \mathrm{dx}$.

We easily check that for a Dirac measure δ_ξ at $\xi \in \left[-\frac{T}{2}, \frac{T}{2}\right]$, we have $S(\delta_\xi) = \xi\, \delta_\xi$, $\Delta[\delta_\xi] = \frac{1}{T}$ and for any $h \in \mathfrak{F}$, $\Delta[S^k(h)] = \frac{1}{T} \int_{-\frac{T}{2}}^{-\frac{T}{2}} h(x)\, e^{-2\pi \boldsymbol{i} k x}\, \mathrm{dx}$ is k^{th} Fourier coefficient of h.

Reconstruction from values. In this problem, we take

- $\mathfrak{F} = C^\infty(\mathbb{R})$,
- $S: h(x) \mapsto h\left(x + \frac{1}{T}\right)$ the shift operator by $\frac{1}{T}$ for $T \in \mathbb{R}_+$,
- $\Delta: h(x) \mapsto \Delta[h] = h(0)$ the evaluation at 0,

We have $S(e^{f_i x}) = \xi_i e^{f_i x}$, $\Delta[e^{f_i x}] = 1$ and $\Delta[S^k(h)] = h(\frac{k}{T})$. Thus the series $\sigma_h(z)$ is the series given in (3).

A more general context can be considered, replacing eigenfunctions by generalized eigenfunctions:

Theorem 5. *Let $S_1, \ldots, S_n$ be commuting operators of $\mathfrak{F}$. Let $E_{1,1}, \ldots, E_{1,\mu_1}, \ldots, E_{r,1}, \ldots, E_{r,\mu_r} \in \mathfrak{F}$ be generalized eigenfunctions of $S_1, \ldots, S_n$ such that for $i = 1, \ldots, r$, $j = 1, \ldots, n$, $k = 1, \ldots, \mu_i$,*

$$S_j(E_{i,k}) = \xi_{i,j} E_{i,k} + \sum_{k' < k} m_{i,j,k'} E_{i,k'}$$

with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^n$ pairwise distinct. Let Λ_0 be a linear functional on $C^\infty(\mathbb{C}^n)$ and $p_{i,1}, \ldots, p_{i,\mu_i} \in \mathbb{C}[\boldsymbol{x}]$ such that $\Lambda_0[p_{i,k}(S) E_{i,k'}] = \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{otherwise.} \end{cases}$ for $i = 1, \ldots, r$, $k, k' = 1, \ldots, \mu_i$.

If $h(\boldsymbol{x}) = \sum_{i=1}^r \sum_{k=1}^{\mu_i} \omega_{i,k} E_{i,k}(\boldsymbol{x})$, the generating series σ_h is:

$$\sigma_h(\boldsymbol{z}) \;=\; \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \Delta[S^\alpha(h)]\, \boldsymbol{z}^\alpha = \sum_{i=1}^r w_i(\boldsymbol{z})\; \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$$

where $w_i(\boldsymbol{z})$ uniquely determines the coefficients $\omega_{i;k}$: $\langle w_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) |\, p_{i,k}(\boldsymbol{x}) \rangle = \omega_{i,k}$.

Proof. □

1.3 Previous works

The approximation of functions by a linear combination of exponentials appears in many context. It is the basis of Fourier analysis, where infinite series are involved in the decomposition of these functions. For "nice" functions, the coefficients are decreasing exponentially and the series can be approximated by a finite sum of exponential functions. This problem of finding an approximation of a function by a finit sum of exponentials has a long history and many applications, in particular in signal processing [13], [23].

Many works have been developed in the one dimensional case ($n=1$), which refers to the well-known problem of *parameter estimation for exponential sums*. A first family of methods can be classified as Prony-type methods. It goes back to the work of Gaspard-Clair-François-Marie Riche de Prony in 1795 [9], who proposed to construct a recurrence relation of minimal size for the sequence $(h(k))_{k\in\mathbb{N}}$ when h is a linear combination of exponential functions. To take into account the problem of noisy data, the recurrence relation is be computed by minimization techniques [23][chap. 1]. Another type of methods is called Pencil-matrix. Instead of computing a recurrence relation, the generalized eigenvalues of a pencil of Hankel matrices are computed [23][chap. 1].

The survey paper [13] describes some of these minimization techniques implementing a variable projection algorithm and their applications in various domains, including antenna analysis with so-called MUSIC [25] or ESPRIT [24] methods.

In [5], another approach based on conjugate-eigenvalue computation and low rank Hankel matrix approximation is proposed. The extension of this method using controlled perturbations and called Approximate Prony Method is described in [21].

Another approach known as *compressive sensing* is considered for instance in [8] for one-dimensional problems. In this approach, a large dictionary of functions is chosen and a sparse combination with few non-zero coefficients is computed from some observation. This boils to find a sparse solution X of an underdetermined linear system $Y=AX$. Such a solution, which minimizes the L_0 "norm" can be computed by L_1 minimization, under some hypothesis.

In the sparse reconstruction problem we are considering, the Fourrier coefficients are chosen in a subset Ω of $\mathbb{Z}$, which is not necessarily of the form $[0,...,2r]$. If this set is "big enough" and random (here of size $\geqslant 4r$), it is shown in [8] that a sparse decomposition can be recovered by L_1 minimization.

Only recently the problem was studied in the multi-dimensional case like in [1], [22]. These methods project the problem in one dimension by sampling data along a line and recover the multivariate solution from projections along several directions. This approach is also used in sparse interpolation of black box polynomials. In the methods developed in [2] [27], and further improved in [12], the sparse polynomial is evaluated at points of the form $(\omega_1^k, ..., \omega_n^k)$ where ω_i are prime numbers or primitive roots of unity of coprime order.

2 Prony's method in one variable.

Gaspard Riche de Prony, mathematician and Engineer of the École Nationale des Ponts et Chaussées, was working on Hydraulics. To analyze the expansion of various gases, he proposed in [9] a method to fit a sum of exponentials to equally spaced data points in order to extend the model at intermediate points. We describe hereafter this method and will study later its extension to multivariate decomposition problems.

Let $h(x)=\sum_{i=1}^{r} w_i e^{f_i x}$ be a linear combination of exponentials with distinct frequencies $f_i\in\mathbb{C}$ and weights $w_i\in\mathbb{C}\setminus\{0\}$.

Prony's method performs as follows:

- Evaluate the function h at values on a grid of a certain step size $\frac{1}{T}\in\mathbb{R}_+$:

$$\sigma_k := h\left(\frac{k}{T}\right) = \sum_{i=1}^{r} w_i \left(e^{\frac{f_i}{T}}\right)^k = \sum_{i=1}^{r} w_i \xi_i^k$$

  for $k\in[0,...,2r-1]$ and $\xi_i = e^{\frac{f_i}{T}}$.

- From these values, compute the polynomial

$$p(x)=\prod_{i=1}^{r}(x-\xi_i)=x^r-\sum_{j=0}^{r-1}p_j x^j,$$

which roots are $\xi_i=e^{\frac{f_i}{T}}$, $i=1,\ldots,r$ as follows. Since it satisfies the recurrence relations

$$\forall j\in[0,\ldots,r-1],\qquad \sum_{i=0}^{r-1}\sigma_{j+i}p_i-\sigma_{j+r}=\sum_{i=1}^{r}w_i\xi_i^j p(\xi_i)=0,$$

it is the unique solution of the system:

$$\begin{pmatrix}\sigma_0 & \sigma_1 & \cdots & \sigma_{r-1}\\ \sigma_1 & & ⋰ & \\ \vdots & ⋰ & & \vdots\\ & ⋰ & & \\ \sigma_{r-1} & & \cdots & \sigma_{2r-2}\end{pmatrix}\begin{pmatrix}p_0\\ p_1\\ \vdots\\ \vdots\\ p_{r-1}\end{pmatrix}=\begin{pmatrix}\sigma_r\\ \sigma_{r+1}\\ \vdots\\ \vdots\\ \sigma_{2r-1}\end{pmatrix}. \tag{4}$$

- Compute the roots $\xi_1,\ldots,\xi_r$ of the polynomial $p(x)$.
- To determine the weight coefficients $w_1,\ldots,w_r$, solve the following linear (Vandermonde) system:

$$\begin{pmatrix}1 & 1 & \cdots & 1\\ \xi_1 & \xi_2 & \cdots & \xi_r\\ \vdots & \vdots & & \vdots\\ \xi_1^{r-1} & \xi_2^{r-1} & \cdots & \xi_r^{r-1}\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ \vdots\\ w_r\end{pmatrix}=\begin{pmatrix}h_0\\ h_1\\ \vdots\\ h_{r-1}\end{pmatrix}.$$

This approach can be improved by computing the roots $\xi_1,\ldots,\xi_r$, directly as the generalized eigenvalues of a pencil of Hankel matrices. Namely, Equation (4) implies that

$$\overbrace{\begin{pmatrix}\sigma_0 & \sigma_1 & \cdots & \sigma_{r-1}\\ \sigma_1 & & ⋰ & \\ \vdots & ⋰ & & \vdots\\ & ⋰ & & \\ \sigma_{r-1} & & \cdots & \sigma_{2r-2}\end{pmatrix}}^{H_0}\overbrace{\begin{pmatrix}0 & & & p_0\\ 1 & \ddots & & p_1\\ & \ddots & \ddots & \vdots\\ & & \ddots\ 0 & \vdots\\ & & 1 & p_{r-1}\end{pmatrix}}^{C_x}=\overbrace{\begin{pmatrix}\sigma_1 & \sigma_2 & \cdots & \sigma_r\\ \sigma_2 & & ⋰ & \\ \vdots & ⋰ & & \vdots\\ & ⋰ & & \\ \sigma_r & & \cdots & \sigma_{2r-1}\end{pmatrix}}^{H_1}, \tag{5}$$

so that the generalized eigenvalues of the pencil (H_1, H_0) are the eigenvalues of the companion matrix C_p of $p(x)$, that is, its the roots $\xi_1,\ldots,\xi_r$. This variant of Prony's method is also called the *pencil method* in the literature.

3 Duality and Hankel operators

In this section, we consider polynomials and series with coefficients in a field $\mathbb{K}$ of characteristic 0. In the applications, we are going to take $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$.

3.1 Duality

In this section, we analyze the natural isomorphism between the ring of formal power series and the dual space of the ring of polynomials $R=\mathbb{K}[x_1,\ldots x_n]$. It is given by the following pairing:

$$\begin{aligned}\mathbb{K}[[z_1,\ldots,z_n]]\times\mathbb{K}[x_1,\ldots x_n] &\rightarrow \mathbb{K}\\ (\boldsymbol{z}^\alpha,\boldsymbol{x}^\beta) &\mapsto \langle\boldsymbol{z}^\alpha|\,\boldsymbol{x}^\beta\rangle=\begin{cases}\alpha! & \text{if } \alpha=\beta\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Namely, if $\Lambda \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}], \mathbb{K}) = R^*$ is an element of the dual of $\mathbb{K}[\boldsymbol{x}]$, it can be represented by the series:

$$\Lambda(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \Lambda(\boldsymbol{x}^{\alpha}) \frac{\boldsymbol{z}^{\alpha}}{\alpha!} \in \mathbb{K}[[z_1, \ldots, z_n]], \tag{6}$$

so that we have $\langle \Lambda(\boldsymbol{z}) | \, \boldsymbol{x}^{\alpha} \rangle = \Lambda(\boldsymbol{x}^{\alpha})$. This map $\Lambda \in R^* \mapsto \sum_{\alpha \in \mathbb{N}^n} \Lambda(\boldsymbol{x}^{\alpha}) \frac{\boldsymbol{z}^{\alpha}}{\alpha!} \in \mathbb{K}[[\boldsymbol{z}]]$ is an isomorphism and any series $\sigma(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_{\alpha} \frac{\boldsymbol{z}^{\alpha}}{\alpha!} \in \mathbb{K}[[\boldsymbol{z}]]$ can be interpreted as a linear form

$$p = \sum_{\alpha \in A \subset \mathbb{N}^n} p_{\alpha} \boldsymbol{x}^{\alpha} \in \mathbb{K}[\boldsymbol{x}] \mapsto \langle \sigma | \, p \rangle = \sum_{\alpha \in A \subset \mathbb{N}^n} p_{\alpha} \sigma_{\alpha}.$$

From now on, we identify the dual $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}], \mathbb{K})$ with $\mathbb{K}[[\boldsymbol{z}]]$. Using this identification, the dual basis of the monomial basis $(\boldsymbol{x}^{\alpha})_{\alpha \in \mathbb{N}^n}$ is $\left(\frac{\boldsymbol{z}^{\alpha}}{\alpha!}\right)_{\alpha \in \mathbb{N}^n}$. The coefficients $\sigma_{\alpha} = \langle \sigma | \, \boldsymbol{x}^{\alpha} \rangle$ are called the *moments* of σ.

In this identification, we can introduce new variables $\boldsymbol{y} = (y_1, \ldots, y_n)$ and replace $\frac{\boldsymbol{z}^{\alpha}}{\alpha!}$ by $\boldsymbol{y}^a$ so $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}], \mathbb{K})$ is identified with $\mathbb{K}[[y_1, \ldots, y_n]]$. This allows to extend the duality properties that we will use to a field $\mathbb{K}$ which is not of characteristic 0. But the relation with differential operators is less natural, that is why we assume that $\mathbb{K}$ is of characteristic 0 and use the identification (6).

If $\mathbb{K}$ is a subfield of a field $\mathbb{L}$, $\mathbb{K}[[\boldsymbol{z}]] \hookrightarrow \mathbb{L}[[\boldsymbol{z}]]$ and any element of $\mathbb{K}[\boldsymbol{x}]^*$ can be uniquely identified with an element of $\mathbb{L}[\boldsymbol{x}]^*$.

The truncation of an element $\sigma(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_{\alpha} \frac{\boldsymbol{z}^{\alpha}}{\alpha!} \in \mathbb{K}[[\boldsymbol{z}]]$ in degree $\leqslant d$ is $\sum_{|\alpha| \leqslant d} \sigma_{\alpha} \frac{\boldsymbol{z}^{\alpha}}{\alpha!}$. It is denoted $\sigma(\boldsymbol{z}) + ((\boldsymbol{z}))^{d+1}$, that is, the class of σ modulo the ideal $(z_1, \ldots, z_n)^{d+1} \subset \mathbb{K}[[\boldsymbol{z}]]$.

Among interesting elements of $\mathrm{Hom}(\mathbb{K}[\boldsymbol{x}], \mathbb{K}) \equiv \mathbb{K}[[\boldsymbol{z}]]$, we have the evaluations at points of $\mathbb{C}^n$:

Definition 6. *The evaluation at a point $\xi \in \mathbb{K}^n$ is:*

$$\begin{aligned} \boldsymbol{e}_{\xi} \colon \mathbb{K}[x_1, \ldots x_n] &\rightarrow \mathbb{K} \\ p(\boldsymbol{x}) &\mapsto p(\xi) \end{aligned}$$

which corresponds to the formal series:

$$\boldsymbol{e}_{\xi}(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \xi^{\alpha} \frac{\boldsymbol{z}^{\alpha}}{\alpha!} = e^{\langle \xi, \boldsymbol{z} \rangle}.$$

Using this formalism, the series $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ can be interpreted as a linear combination of evaluations at the points ξ_i which coefficients are ω_i, for $i = 1, \ldots, r$.

Notice that the product of $\boldsymbol{z}^{\alpha} \boldsymbol{e}_{\xi}(\boldsymbol{z})$ with a monomial $\boldsymbol{x}^{\alpha+\beta} \in \mathbb{C}[x_1, \ldots x_n]$ is given by

$$\left\langle \boldsymbol{z}^{\alpha} \boldsymbol{e}_{\xi}(\boldsymbol{z}) \,|\, \boldsymbol{x}^{\alpha+\beta} \right\rangle = \frac{(\alpha+\beta)!}{\beta!} \xi^{\beta} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \boldsymbol{x}^{\alpha+\beta}(\xi),$$

so that $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}) \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ can be seen as a sum of *polynomial differential operators* $\omega_i(\partial)$ "at" the points ξ_i, that we call *infinitesimal operators*: $\forall p \in \mathbb{C}[\boldsymbol{x}], \langle \sigma(\boldsymbol{z}) | \, p(\boldsymbol{x}) \rangle = \sum_{i=1}^{r} \omega_i(\partial) p(\xi)$.

Definition 7. *For any $\sigma(\boldsymbol{z}) \in \mathbb{K}[[\boldsymbol{z}]]$, the inner product associated to $\sigma(\boldsymbol{z})$ on $\mathbb{K}[\boldsymbol{x}]$ is*

$$\begin{aligned} \mathbb{K}[\boldsymbol{x}] \times \mathbb{K}[\boldsymbol{x}] &\rightarrow \mathbb{K} \\ (p(\boldsymbol{x}), q(\boldsymbol{x})) &\mapsto \langle p(\boldsymbol{x}), q(\boldsymbol{x}) \rangle_{\sigma} := \langle \sigma(\boldsymbol{z}) | \, p(\boldsymbol{x}) q(\boldsymbol{x}) \rangle. \end{aligned}$$

The dual space $\mathrm{Hom}(\mathbb{K}[\boldsymbol{x}], \mathbb{K}) \equiv \mathbb{K}[[\boldsymbol{z}]]$ has a natural structure of $\mathbb{K}[\boldsymbol{x}]$-module, defined as follows: $\forall \sigma(\boldsymbol{z}) \in \mathbb{K}[[\boldsymbol{z}]], \forall p(\boldsymbol{x}), q(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}]$,

$$\langle p(\boldsymbol{x}) \star \sigma(\boldsymbol{z}) \,|\, q(\boldsymbol{x}) \rangle = \langle \sigma(\boldsymbol{z}) \,|\, p(\boldsymbol{x}) q(\boldsymbol{x}) \rangle = \langle p(\boldsymbol{x}), q(\boldsymbol{x}) \rangle_{\sigma}.$$

We easily check that $\forall \sigma \in \mathbb{K}[[\boldsymbol{z}]], \forall p, q \in \mathbb{K}[\boldsymbol{x}]$, $(pq) \star \sigma = p \star (q \star \sigma)$.

Example 8. If $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$, with $\omega_i \in \mathbb{K}$ and $\xi_i \in \mathbb{K}^n$ and $p(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}]$, we have

$$p(\boldsymbol{x}) \star \sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, p(\xi_i) \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}). \tag{7}$$

An interesting property of this external product is that polynomials act as differentials on the series:

Lemma 9. $\forall p \in \mathbb{K}[\boldsymbol{x}], \forall \sigma \in \mathbb{K}[[\boldsymbol{z}]]$, $p(\boldsymbol{x}) \star \sigma(\boldsymbol{z}) = p(\partial_{z1}, \ldots, \partial_{z_n})(\sigma)$.

Proof. We first prove the relation for $p = x_i$ and $\sigma = \boldsymbol{z}^\alpha$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the exponent vector of x_i. $\forall \beta \in \mathbb{N}^n$, we have

$$\begin{aligned}\langle x_i \star \boldsymbol{z}^\alpha \mid \boldsymbol{x}^\beta \rangle &= \langle \boldsymbol{z}^\alpha \mid x_i\, \boldsymbol{x}^\beta \rangle = \alpha! \text{ if } \alpha = \beta + e_i \text{ and } 0 \text{ otherwise} \\ &= \alpha_i \langle \boldsymbol{z}^{\alpha - e_i} \mid \boldsymbol{x}^\beta \rangle.\end{aligned}$$

with the convention that $\boldsymbol{z}^{\alpha - e_i} = 0$ if $\alpha_i = 0$. This shows that $x_i \star \boldsymbol{z}^\alpha = \alpha_i\, \boldsymbol{z}^{\alpha - e_i} = \partial_{z_i}(\boldsymbol{z}^\alpha)$ as elements of $R^* \equiv \mathbb{K}[[\boldsymbol{z}]]$.

By transitivity and bilinearity of the product $\star$, we deduce that $\forall p \in \mathbb{K}[\boldsymbol{x}]$, $\forall \sigma \in \mathbb{K}[[\boldsymbol{z}]]$, $p(\boldsymbol{x}) \star \sigma(\boldsymbol{z}) = p(\partial_{z1}, \ldots, \partial_{z_n})(\sigma)$. ☐

For a subset $D \subset \mathbb{K}[[\boldsymbol{z}]]$, the *inverse system* generated by D is the vector space spanned by the elements $p(\boldsymbol{x}) \star \delta(\boldsymbol{z})$ for $\delta(\boldsymbol{z}) \in D$ and $p(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}]$. By Lemma 9, the inverse system of D is the space generated by the elements of D and all their derivative in the variables $\boldsymbol{z}$ at any order.

For an ideal $I \subset R = \mathbb{K}[\boldsymbol{x}]$, we denote by $I^\perp \subset \mathbb{K}[[\boldsymbol{z}]]$ the space of linear forms $\sigma \in \mathbb{K}[[\boldsymbol{z}]]$, such that $\forall p \in I$, $\langle \sigma(\boldsymbol{z}) \mid p(\boldsymbol{x}) \rangle = 0$.

Let $d \in \mathbb{N}$ and let $I_{\leqslant d}$ be the set of polynomials degree $\leqslant d$ in I. We denote by $I_{\leqslant d}^\perp \subset \mathbb{K}[[\boldsymbol{z}]]$, the set of linear forms σ such that $\forall p \in I_{\leqslant d}$, $\langle \sigma(\boldsymbol{z}) \mid p(\boldsymbol{x}) \rangle = 0$.

Lemma 10. $I_{\leqslant d}^\perp = I^\perp + ((\boldsymbol{z}))^{d+1}$.

This lemma says that an element of $I_{\leqslant d}^\perp$ is the truncation in degree $\leqslant d$ of an element of $I^\perp$.

3.2 Artinian algebra

In this section, we consider an ideal $I \subset \mathbb{K}[\boldsymbol{x}]$ and the associated quotient algebra $\mathcal{A} = \mathbb{K}[\boldsymbol{x}]/I$.

Definition 11. *The quotient algebra $\mathcal{A}$ is artinian if* $\dim_{\mathbb{K}}(\mathcal{A}) < \infty$.

Notice that if $\mathbb{K}$ is a subfield of a field $\mathbb{L}$, $\dim_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}]/I) = \dim_{\mathbb{L}}(\mathbb{L}[\boldsymbol{x}]/I_{\mathbb{L}}) = \dim_{\mathbb{L}} \mathcal{A} \otimes \mathbb{L}$ where $I_{\mathbb{L}}$ is the ideal of $\mathbb{L}[\boldsymbol{x}]$ generated by the element in I. Hereafter, we are going to assume that $\mathbb{K}$ is algebraically closed.

Theorem 12. *Let $\mathcal{A}$ be an artinian algebra of dimension r defined by an ideal I. Then we have a direct sum*

$$\mathcal{A} = \mathcal{A}_{\xi_1} \oplus \cdots \oplus \mathcal{A}_{\xi_{r'}},$$

where

- $\mathcal{V}(I) = \{\xi_1, \ldots, \xi_{r'}\} \subset \mathbb{K}^n$ *with* $r' \leqslant r$.
- $I = Q_1 \cap \cdots \cap Q_{r'}$ *is a minimal primary decomposition of I with Q_i $\boldsymbol{m}_{\xi_i}$-primary,*
- $\mathcal{A}_{\xi_i} \equiv \mathbb{K}[\boldsymbol{x}]/Q_i$ *and* $\mathcal{A}_{\xi_i} \cdot \mathcal{A}_{\xi_j} \equiv 0$ *if* $i \neq j$.

The dual $\mathcal{A}^* = \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$ of $\mathcal{A}$ is naturally identified with the sub-space

$$I^\perp = \{\Lambda \in \mathbb{K}[\boldsymbol{x}]^* = \mathbb{K}[[\boldsymbol{z}]] \mid \forall p \in I, \Lambda(p) = 0\}.$$

As I is stable by multiplication by the variables x_i, the orthogonal $I^\perp = \mathcal{A}^*$ is stable by the derivations $\frac{d}{\mathrm{d}z_i}$.

Proposition 13. *Let Q be a primary ideal for the maximal ideal $\mathfrak{m}_\xi$ of the point $\xi \in \mathbb{K}^n$ and let $\mathcal{A}_\xi = \mathbb{K}[\boldsymbol{x}]/Q$. Then there exists a vector space $D \subset \mathbb{K}[\boldsymbol{z}]$ stable by the derivations $\frac{d}{\mathrm{d}z_i}$ such that*

$$Q^\perp = \mathcal{A}_\xi^* = D \cdot \boldsymbol{e}_\xi(\boldsymbol{z}).$$

Theorem 14. *Let $\mathcal{A}$ be an artinian algebra of dimension r with $\mathcal{V}(I) = \{\xi_1, \ldots, \xi_{r'}\} \subset \mathbb{K}^n$. There exists vector spaces $D_i \subset \mathbb{K}[\boldsymbol{z}]$ stable by derivation of dimension μ_i with $\sum_{i=1}^{r'} \mu_i = r$, such that the elements of $\mathcal{A}^*$ are the elements $\Lambda \in \mathbb{K}[[\boldsymbol{z}]]$ of the form*

$$\Lambda(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}),$$

with $\omega_i(\boldsymbol{z}) \in D_i$.

Definition 15. *Let g be a polynomial in $\mathcal{A}$. The g-multiplication operator $\mathcal{M}_g$ is defined by*

$$\begin{array}{rccl} \mathcal{M}_g\colon & \mathcal{A} & \rightarrow & \mathcal{A} \\ & h & \mapsto & \mathcal{M}_g(h) = g\, h. \end{array}$$

The transpose application $\mathcal{M}_g^t$ of the g-multiplication operator $\mathcal{M}_g$ is defined by

$$\begin{array}{rccl} \mathcal{M}_g^t\colon & \mathcal{A}^* & \rightarrow & \mathcal{A}^* \\ & \Lambda & \mapsto & \mathcal{M}_g^t(\Lambda) = \Lambda \circ \mathcal{M}_g = g \star \Lambda. \end{array}$$

Let $\mathcal{B}$ be a monomial basis in $\mathcal{A}$ and $\mathcal{B}^*$ its dual basis in $\mathcal{A}^*$. As the matrix M_g^t of the transpose application $\mathcal{M}_g^t$ in the dual basis $\mathcal{B}^*$ in $\mathcal{A}^*$ is the transpose of the matrix M_g of the application $\mathcal{M}_g$ in the basis $\mathcal{B}$ in $\mathcal{A}$, the eigenvalues are the same for both matrices.

The main property we need is the following (see e.g. [11]):

Proposition 16. *Let I be an ideal of $R = \mathbb{K}[\boldsymbol{x}]$ and suppose that $\mathcal{V}(I) = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_r\}$. Then*

- *for all $g \in \mathcal{A}$, the eigenvalues of $\mathcal{M}_g$ and $\mathcal{M}_g^t$ are the evaluations at the polynomial g, namely $g(\boldsymbol{\xi}_1), \ldots, g(\boldsymbol{\xi}_r)$,*
- *The eigenvectors common to all $\mathcal{M}_g^t$ with $g \in \mathcal{A}$ are - up to a scalar - the evaluations $\mathbf{e}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{e}_{\boldsymbol{\xi}_r}$.*

Remark 17. *If $(\boldsymbol{x}^\beta)_{\beta \in B}$ is a basis of $\mathcal{A}$, then the coefficient vector of the evaluation*

$$\mathbf{e}_{\boldsymbol{\xi}_i} = \sum_{\beta \in B} \boldsymbol{\xi}_i^\beta \frac{\boldsymbol{z}^\beta}{\beta!} + \cdots$$

in the dual basis of $\mathcal{A}^*$ is $\left[\langle \boldsymbol{e}_{\xi_i} | \boldsymbol{x}^\beta \rangle\right]_{\beta \in B} = \left[\boldsymbol{\xi}_i^\beta\right]_{\beta \in B}$. The previous proposition says that if $[\mathcal{M}_g]$ is the matrix of $\mathcal{M}_g$ in the basis $(\boldsymbol{x}^\beta)_{\beta \in B}$ of $\mathcal{A}$, then

$$[\mathcal{M}_g]^t \left[\boldsymbol{\xi}_i^\beta\right]_{\beta \in B} = g(\xi_i) \left[\boldsymbol{\xi}_i^\beta\right]_{\beta \in B}$$

If moreover the basis $(\boldsymbol{x}^\beta)_{\beta \in B}$ contains the monomials $1, x_1, x_2, \ldots, x_n$, then the root ξ_i can be computed from the coefficient vector of any multiple $c\ \mathbf{e}_{\boldsymbol{\xi}_i}$, $c \in \mathbb{K} - \{0\}$ of the evaluation $\mathbf{e}_{\boldsymbol{\xi}_i}$, by taking the ratio of the coefficients of the monomials $x_1, \ldots, x_n$ by the coefficient of 1. Thus computing the common eigenvectors of all these matrices M_g^t yield the roots $\boldsymbol{\xi}_i$ $(i = 1, \ldots, r)$. In practice, it is sufficient to compute the common eigenvectors of $[\mathcal{M}_{x_1}]^t, \ldots, [\mathcal{M}_{x_n}]^t$ since we have $[\mathcal{M}_g]^t = g([\mathcal{M}_{x_1}]^t, \ldots, [\mathcal{M}_{x_n}]^t)$.

3.3 Hankel operators

The external product $\star$ allows us to define an Hankel operator as a multiplication operator by a dual element $\in \mathbb{K}[[\boldsymbol{z}]]$:

Definition 18. *The Hankel operator associated to an element $\sigma(\boldsymbol{z}) \in \mathbb{C}[[\boldsymbol{z}]]$ is*

$$\begin{array}{rcl} H_\sigma\colon \mathbb{K}[\boldsymbol{x}] & \rightarrow & \mathbb{K}[[\boldsymbol{z}]] \\ p(\boldsymbol{x}) & \mapsto & p(\boldsymbol{x}) \star \sigma(\boldsymbol{z}). \end{array}$$

Its kernel is denoted I_σ. We say that the series σ has a finite rank $r \in \mathbb{N}$ if $\operatorname{rank} H_\sigma = r < \infty$.

Example 19. If $\sigma = \boldsymbol{e}_\xi$ is the evaluation at a point $\xi \in \mathbb{C}^n$, then $H_{\boldsymbol{e}_\xi}\colon p \in \mathbb{K}[\boldsymbol{x}] \mapsto p(\xi)\ \boldsymbol{e}_\xi \in \mathbb{K}[[\boldsymbol{z}]]$

Remark 20. The matrix of the operator H_σ in the bases $(\boldsymbol{x}^\alpha)_{\alpha\in\mathbb{N}^n}$ and $\left(\frac{\boldsymbol{z}^\alpha}{\alpha!}\right)_{\alpha\in\mathbb{N}^n}$ is

$$[H_\sigma] \;=\; (\sigma_{\alpha+\beta})_{\alpha,\beta\in\mathbb{N}^n} = \left(\left\langle\sigma\,|\,\boldsymbol{x}^{\alpha+\beta}\right\rangle\right)_{\alpha,\beta\in\mathbb{N}^n}.$$

Definition 21. *For two vector spaces $V, V'\subset\mathbb{K}[\boldsymbol{x}]$ and $\sigma\in\langle V\cdot V'\rangle^*\subset\mathbb{K}[[\boldsymbol{z}]]$, we denote by $H_\sigma^{V,V'}$ the following map:*

$$\begin{array}{rcl} H_\sigma^{V,V'}: V & \to & V'^* = \hom_{\mathbb{C}}(V',\mathbb{C}) \\ p(\boldsymbol{x}) & \mapsto & p(\boldsymbol{x})\star\sigma(\boldsymbol{z})_{|V'}. \end{array}$$

It is called the truncated Hankel operator *on (V, V').*

When $V' = V$, The truncated Hankel operator is also denoted H_σ^V.

If $B = \{b_1, ..., b_r\}$ (resp. $B' = \{b'_1, ..., b'_r\}$) is a basis of V (resp. V'), then the matrix of the operator $H_\sigma^{V,V'}$ in B and the dual basis of B' is

$$\left[H_\sigma^{B,B'}\right] = (\langle\sigma\,|\,b_i\,b'_j\rangle)_{1\leqslant i\leqslant r, 1\leqslant j\leqslant r'}.$$

If B and B' are monomial sets, we obtain the so-called *truncated moment matrix* of σ:

$$\left[H_\sigma^{B,B'}\right] \;=\; (\sigma_{\beta+\beta'})_{\beta\in B,\beta'\in B'}$$

When $n = 1$, this matrix is a classical Hankel matrices, which entries depend only of the sum of the indices of the rows and columns. When $n\geqslant 2$, we have a similar family of structured matrices, which rows and columns are indexed by exponents in $\mathbb{N}^n$ and which entries depends on the sum of the row and column indices. These structured matrices called quasi-Hankel matrices have been studied for instance in [19].

3.4 Artinian Gorenstein algebra

In this section, we analyze the properties of artinian algebra that are obtained as quotient by the kernel $I_\sigma = \{p\in\mathbb{C}[\boldsymbol{x}] | p\star\sigma = 0\}$ of an Hankel operator H_σ. We assume that $\mathbb{K}$ is algebraically closed.

As $\forall p, q\in\mathbb{K}[\boldsymbol{x}]$, $pq\star\sigma = p\star(q\star\sigma)$, we easily check that I_σ is an ideal of $\mathbb{K}[\boldsymbol{x}]$, and we construct the quotient algebra $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$. By construction, $\mathcal{A}_\sigma^* = I_\sigma^\perp$ contains the elements $p\star\sigma$ for all $p\in\mathbb{K}[\boldsymbol{x}]$ and $\operatorname{im} H_\sigma\subset\mathcal{A}_\sigma^*$. The Hankel operator H_σ is a map from $\mathbb{K}[\boldsymbol{x}]$ into $\mathcal{A}_\sigma^*$:

$$0\to I_\sigma\to\mathbb{K}[\boldsymbol{x}]\xrightarrow{H_\sigma}\mathcal{A}_\sigma^*. \tag{8}$$

The variety defined by I_σ in $\mathbb{K}^n$ is denoted hereafter $\mathcal{V}_{\mathbb{K}}(I_\sigma)$ or simply $\mathcal{V}(I_\sigma)$ when $\mathbb{K}$ is algebraically closed.

A classical result states that a quotient algebra $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$ is finite dimensional, i.e. artinian iff $\mathcal{V}(I_\sigma)$ is finite, that is, I_σ defines a finite number of (isolated) points in $\mathbb{K}^n$.

The multiplicity of an isolated point ξ of $\mathcal{V}(I_\sigma)$ is the dimension over $\mathbb{K}$ of $\mathcal{A}_\sigma$ localized at ξ.

If $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i(\boldsymbol{z})\,\boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ then, by Lemma 9, the kernel I_σ is the set of polynomials $p\in\mathbb{K}[\boldsymbol{x}]$ such that $\forall q\in\mathbb{K}[\boldsymbol{x}]$, p is a solution of the following partial differential equation:

$$\sum_{i=1}^r \omega_i(\partial)(pq)(\xi_i) = 0.$$

Since $\forall p(\boldsymbol{x}), q(\boldsymbol{x})\in\mathbb{K}[\boldsymbol{x}]$, $\langle p(\boldsymbol{x}) + I_\sigma, q(\boldsymbol{x}) + I_\sigma\rangle_\sigma = \langle p(\boldsymbol{x}), q(\boldsymbol{x})\rangle_\sigma$, $\langle .,.\rangle_\sigma$ induces an inner product on $\mathcal{A}_\sigma$.

Theorem 22. *Let $\sigma(\boldsymbol{z})\in\mathbb{K}[[\boldsymbol{z}]]\setminus\{0\}$.*

- $\operatorname{rank} H_\sigma = \dim_{\mathbb{K}}(\mathcal{A}_\sigma) < \infty$*, if and only if,*

$$\sigma(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\,\boldsymbol{e}_{\xi_i}(\boldsymbol{z}) \tag{9}$$

with $\omega_i(\boldsymbol{z}) \in \mathbb{K}[\boldsymbol{z}] \setminus \{0\}$ and $\xi_i \in \mathbb{K}^n$ *pairwise distinct.*

- *If* $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ *with* $\omega_i(\boldsymbol{z}) \in \mathbb{K}[\boldsymbol{z}] \setminus \{0\}$*, then*
  - *the map* $\mathcal{H}_\sigma: \mathcal{A}_\sigma \rightarrow \mathcal{A}_\sigma^*$ *induced by* H_σ *is an isomorphism.*
  - *the inner product* $\langle .,. \rangle_\sigma$ *is non-degenerate on* $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$*.*
  - *the rank of* H_σ *is* $\sum_{i=1}^{r'} \mu_i$ *where* μ_i *is the dimension of the vector space spanned by* $\omega_i(\boldsymbol{z})$ *and all its derivatives* $\partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} \omega_i(\boldsymbol{z})$ *for* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$*;*
  - *the variety* $\mathcal{V}(I_\sigma)$ *is the set of points* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{r'} \in \mathbb{K}^n$*, with multiplicity* $\mu_1, \ldots, \mu_r'$*.*

Proof. By definition of I_σ and by the short exact sequence

$$0 \rightarrow I_\sigma \rightarrow \mathbb{K}[\boldsymbol{x}] \xrightarrow{H_\sigma} \mathrm{Im}(H_\sigma) \rightarrow 0, \tag{10}$$

we have $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma \sim \mathrm{Im}(H_\sigma)$. If rank $H_\sigma = \dim\, (\mathrm{Im}(H_\sigma)) = r < \infty$, then $\dim\, (\mathcal{A}_\sigma) = \dim\, (\mathbb{K}[\boldsymbol{x}]/I_\sigma) = r$ and $\mathcal{A}_\sigma$is an artinian algebra (of dimension r over $\mathbb{K}$). By Theorem 12, it can be decomposed as a direct sum of sub-algebras

$$\mathcal{A}_\sigma = \mathcal{A}_{\xi_1} \oplus \cdots \oplus \mathcal{A}_{\xi_{r'}}$$

where $\mathcal{V}_{\mathbb{K}}(I_\sigma) = \{\xi_1, \ldots, \xi_{r'}\}$ and $\mathcal{A}_{\xi_i}$ is a local algebra for the maximal ideal m_{ζ_i} defining the root $\xi_i \in \mathbb{K}^n$: $\mathcal{A}_{\xi_i} = \mathbb{K}[\boldsymbol{x}]/Q_i$ with Q_i an m_{ξ_i}-primary ideal of $\mathbb{K}[\boldsymbol{x}]$. Moreover, we have the minimal primary decomposition $I_\sigma = Q_1 \cap \cdots \cap Q_r$.

The series $\sigma(\boldsymbol{z})$ represents an element of the dual $\mathcal{A}_\sigma^* = I_\sigma^\perp$, which by Theorem 14 can be decomposed as

$$\sigma(z) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) \tag{11}$$

with $\omega_i(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$. The polynomial $\omega_i(\boldsymbol{z})$ cannot be zero, otherwise $Q_i \subset \ker\, H_\sigma = I_\sigma$. As $I_\sigma = Q_1 \cap \cdots \cap Q_r$, we deduce that $I_\sigma = Q_i$ and that $\sigma(\boldsymbol{z}) = \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) = 0$, which contradicts the hypothesis.

Conversely, if $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ with $\omega_i(\boldsymbol{z}) \in \mathbb{K}[\boldsymbol{z}] \setminus \{0\}$ and $\xi_i \in \mathbb{K}^n$ pairwise distinct, we easily check that I_σ contains $\cap_{i=1}^{r} \boldsymbol{m}_{\xi_i}^{d_i+1}$ where d_i is the degree of $\omega_i(\boldsymbol{z})$. Thus $\mathcal{V}(I_\sigma) \subset \{\xi_1, \ldots, \xi_r\}$.

The ideal I_σ contains in particular univariate polynomials in each variable x_i.Thus $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$ is of finite dimension over $\mathbb{K}$ and rank $H_\sigma < \infty$.

Let us assume now that $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ with $\omega_i(\boldsymbol{z}) \in \mathbb{K}[\boldsymbol{z}] \setminus \{0\}$ so that $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$ is of dimension r over $\mathbb{K}$.

As $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma \sim \mathrm{Im}(H_\sigma)$, H_σ induces an injection from $\mathcal{A}_\sigma$ into $\mathcal{A}_\sigma^*$ which is of dimension r. We deduce that H_σ induces an isomorphism between $\mathcal{A}_\sigma$ and $\mathcal{A}_\sigma^*$, and we have the short exact sequence:

$$0 \rightarrow I_\sigma \rightarrow \mathbb{K}[\boldsymbol{x}] \xrightarrow{H_\sigma} \mathcal{A}_\sigma^* \rightarrow 0.$$

This shows that $\mathcal{A}_\sigma^*$ is generated by elements $p \star \sigma$ for $p \in \mathbb{K}[\boldsymbol{x}]$, that is, $\mathcal{A}_\sigma^*$ is the inverse system generated by σ.

By definition of I_σ, if $p \in \mathbb{K}[\boldsymbol{x}]$ is such that $\forall q \in \mathbb{K}[\boldsymbol{x}]$

$$\langle p(\boldsymbol{x}), q(\boldsymbol{x}) \rangle_\sigma = \langle p \star \sigma(\boldsymbol{z}) \mid q(\boldsymbol{x}) \rangle = 0,$$

then $p \star \sigma(\boldsymbol{z}) = 0$ and $p \in I_\sigma$. We deduce that the inner product $\langle \cdot, \cdot \rangle_\sigma$ is non-generate on $\mathcal{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$.

By Theorem 14, $\sigma \in \mathcal{A}_\sigma^*$ has a decomposition of the form (9) which must coincides with the given one: $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$. Thus $\mathcal{A}_\sigma^* = \mathcal{A}_{\xi_1}^* \oplus \cdots \oplus \mathcal{A}_{\xi_{r'}}^*$ where $I_\sigma = Q_1 \cap \cdots \cap Q_{r'}$, $\mathcal{A}_{\xi_i}^* = Q_i^\perp$ is the inverse system generated by $\omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ for $i = 1, \ldots, r'$.

The dimension of $\mu_i = \dim \mathcal{A}_{\xi_i}^* = \dim \mathcal{A}_{\xi_i}$ of the inverse system $\mathcal{A}_{\xi_i}^*$ is the multiplicity of ξ_i; it is also the dimension of the vector space spanned by $\omega_i(\boldsymbol{z})$ and all its derivatives $\partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} \omega_i(\boldsymbol{z})$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We deduce that $\dim \mathcal{A}_\sigma = \dim \mathcal{A}_\sigma^* = r = \sum_{i=1}^{r'} \mu_i$.

As $I_\sigma = Q_1 \cap \cdots \cap Q_{r'}$, we deduce that $\mathcal{V}(I_\sigma) = \{\xi_1, \ldots, \xi_{r'}\}$, which concludes the proof of this theorem. □

Definition 23. *The rank of an element* $\sigma \in \mathbb{K}[[\boldsymbol{z}]]$ *is* $\dim_{\mathbb{K}}(\mathcal{A}_\sigma)$.

Definition 24. *The support of* σ *is* $\mathcal{V}(I_\sigma)$.

A special case of interest is when the roots are simple. We characterize it as follows:

Proposition 25. *Let* $\sigma(\boldsymbol{z}) \in \mathbb{K}[[\boldsymbol{z}]]$. *The following conditions are equivalent:*

1. $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$, *with* $\omega_i \in \mathbb{K} \setminus \{0\}$ *and* $\xi_i \in \mathbb{K}^n$ *pairwise distinct.*
2. *The rank of* H_σ *is* r *and the multiplicity of the points* $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$ *in* $\mathcal{V}(I_\sigma)$ *is* 1.
3. *A basis of* $\mathcal{A}_\sigma^*$ *is* $\mathbf{e}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{e}_{\boldsymbol{\xi}_r}$.

Proof. $1 \Rightarrow 2$. The dimension of the vector space spanned by $\omega_i \in \mathbb{K} \setminus \{0\}$ and its derivatives is 1. By Theorem 22, the rank $\mathcal{A}_\sigma$ is $r = \sum_{i=1}^{r} 1$ and the multiplicity of the roots $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$ in $\mathcal{V}(I_\sigma)$ is 1.

$2 \Rightarrow 3$. By Theorem 22, $\mathcal{A}_\sigma^*$ is the inverse system spanned by σ. As $\forall p \in \mathbb{K}[\boldsymbol{x}]$, $p \star \sigma = \sum_{i=1}^{r} \omega_i \, p(\xi_i) \, \boldsymbol{e}_{\xi_i}$, $\mathcal{A}_\sigma^*$ is in the vector space spanned by $\mathbf{e}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{e}_{\boldsymbol{\xi}_r}$. As $\dim(\mathcal{A}_\sigma^*) = r$, it is a basis.

$3 \Rightarrow 1$. As $\sigma \in \mathcal{A}_\sigma^*$, there exists $\omega_i \in \mathbb{K}$ such that $\sigma = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}$. If one of these coefficients ω_i vanishes that $\dim(\mathcal{A}_\sigma^*) < r$, which is contradicting point 3. Thus $\omega_i \in \mathbb{K} \setminus \{0\}$. □

In the case where all the coefficients of σ are in $\mathbb{R}$, we can consider the following notion of positivity:

Definition 26. *An element* $\sigma \in \mathbb{R}[[\boldsymbol{z}]] = \mathbb{R}[\boldsymbol{x}]^*$ *is positive if* $\forall p \in \mathbb{R}[\boldsymbol{x}], \langle p, p \rangle_\sigma = \langle \sigma \mid p^2 \rangle \geqslant 0$. *It is denoted* $\sigma \succcurlyeq 0$.

The positivity of σ induces the following property on its decomposition:

Proposition 27. *Let* $\sigma \in \mathbb{R}[[\boldsymbol{z}]]$ *of finite rank* $\sigma \succcurlyeq 0$ *iff*

$$\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$$

with $\omega_i > 0$, $\xi_i \in \mathbb{R}^n$.

Proof. If $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}$ with $\omega_i > 0$, $\xi_i \in \mathbb{R}^n$, then clearly $\forall p \in \mathbb{R}[\boldsymbol{x}]$,

$$\langle \sigma \mid p^2 \rangle = \sum_{i=1}^{r} \omega_i \, p^2(\xi_i) \geqslant 0$$

and $\sigma \succcurlyeq 0$.

Conversely suppose that $\forall p \in \mathbb{R}[\boldsymbol{x}]$, $\langle \sigma \mid p^2 \rangle \geqslant 0$. Then $p \in I_\sigma$ iff $\langle \sigma \mid p^2 \rangle = 0$. We check that I_σ is real radical: If $p^{2k} + \sum_j q_j^2 \in I_\sigma$ for some $k \in \mathbb{N}$, $p, q_j \in \mathbb{R}[\boldsymbol{x}]$ then

$$\left\langle \sigma \mid p^{2k} + \sum_j q_j^2 \right\rangle = \langle \sigma \mid p^{2k} \rangle + \sum_j \langle \sigma \mid q_j^2 \rangle = 0$$

which implies that $\langle \sigma \mid (p^k)^2 \rangle = 0$, $\langle \sigma \mid q_j^2 \rangle = 0$ and $p^k, q_j \in I_\sigma$. Let $k' = \lceil \frac{k}{2} \rceil$. We have $\langle \sigma \mid (p^{k'})^2 \rangle = 0$, which implies that $p^{k'} \in I_\sigma$. Iterating this reduction, we deduce that $p \in I_\sigma$. This shows that I_σ is real radical and $\mathcal{V}(I_\sigma) \subset \mathbb{R}^n$. By Proposition 25, we deduce that $\sigma = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}$ with $\omega_i \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{R}^n$. Let $p_i \in \mathbb{R}[\boldsymbol{x}]$ be interpolation polynomials at $\xi_i \in \mathbb{R}^n$: $p_i(\xi_i) = 1$, $p_i(\xi_j) = 0$ for $j \neq i$. Then $\langle \sigma \mid p_i^2 \rangle = \omega_i \in \mathbb{R}_+$. This proves that $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ with $\omega_i > 0$, $\xi_i \in \mathbb{R}^n$. □

3.5 The support of $\boldsymbol{\sigma}$

In this section, we consider the problem of computing the support $\{\xi_1, \ldots, \xi_r\}$ of a series $\sigma = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}) \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ from its hankel operator.

We recall classical results on the resolution of polynomial equations by eigenvalue and eigenvector computation. Hereafter, $\mathcal{A} = \mathbb{K}[\boldsymbol{x}]/I$ is the quotient algebra of $\mathbb{K}[\boldsymbol{x}]$ by any ideal I and $\mathcal{A}^* = \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$ is the dual of $\mathcal{A}$. It is naturally identified with the orthogonal $I^\perp = \{\Lambda \in \mathbb{K}[[\boldsymbol{z}]] \mid \forall p \in I, \langle \Lambda, p \rangle = 0\}$. In the reconstruction problem, we will take $I = I_\sigma$.

Coming back to our decomposition problem for which $\mathcal{A} = \mathcal{A}_\sigma \mathbb{K}[\boldsymbol{x}]/I_\sigma$, we observe that by projection and restriction, H_σ induces the map

$$\begin{array}{rcl}\mathcal{H}_\sigma\colon \mathcal{A}_\sigma & \rightarrow & \mathcal{A}_\sigma^* \\ p(\boldsymbol{x}) & \mapsto & p(\boldsymbol{x}) \star \sigma(\boldsymbol{z}).\end{array}$$

This map is a bijection, since we quotient H_σ through its kernel I_σ and restrict it onto its image $\mathcal{A}_\sigma^* = I_\sigma^\perp$.

Lemma 28. *For any $g \in \mathbb{K}[\boldsymbol{x}]$, we have*

$$\mathcal{H}_{g\star\sigma} = \mathcal{M}_g^t \circ \mathcal{H}_\sigma = \mathcal{H}_\sigma \circ \mathcal{M}_g. \tag{12}$$

Proof. This is a direct consequence of the definitions of $\mathcal{H}_{g\star\sigma}, \mathcal{H}_\sigma, \mathcal{M}_g^t$ and $\mathcal{M}_g$. □

If $(b_i)_{1\leqslant i\leqslant\delta}$ and $(b_i')_{1\leqslant i\leqslant\delta}$ are bases of $\mathcal{A}_\sigma$, then the matrix of $\mathcal{H}_\sigma$ in the basis $(b_i)_{1\leqslant i\leqslant\delta}$ and in the dual basis of $(b_i')_{1\leqslant i\leqslant\delta}$ is $[\mathcal{H}_\sigma] = (\langle\sigma|\ b_i(\boldsymbol{x}) b_i'(\boldsymbol{x})\rangle)_{1\leqslant i,j\leqslant\delta}$. In particular, if $(\boldsymbol{x}^\beta)_{\beta\in B}$ and $(\boldsymbol{x}^{\beta'})_{\beta'\in B'}$ are bases of $\mathcal{A}_\sigma$, its matrix in the corresponding bases is

$$[\mathcal{H}_\sigma] = (\langle\sigma\,|\,\boldsymbol{x}^{\beta+\beta'}\rangle)_{\beta\in B,\beta'\in B'} = \left(\sigma_{\beta+\beta'}\right)_{\beta\in B,\beta'\in B'} = H_\sigma^{B,B'}.$$

It is a submatrix of the (infinite) matrix $[H_\sigma]$. Conversely, we have the following property:

Lemma 29. *Let B, $B' \subset \mathbb{N}^n$ with $|B| = |B'|$. The matrix $\left[H_\sigma^{B,B'}\right] = \left(\sigma_{\beta+\beta'}\right)_{\beta\in B,\beta'\in B'}$ is invertible, if and only if, $(\boldsymbol{x}^\beta)_{\beta\in B}$ and $(\boldsymbol{x}^{\beta'})_{\beta'\in B'}$ are linearly independent in $\mathcal{A}_\sigma$.*

In particular, if $\dim \mathcal{A}_\sigma < +\infty$, $|B| = |B'| = \dim \mathcal{A}_\sigma$ and $H_\sigma^{B,B'}$ is invertible, then $(\boldsymbol{x}^\beta)_{\beta\in B}$ and $(\boldsymbol{x}^{\beta'})_{\beta'\in B'}$ are bases of $\mathcal{A}_\sigma$.

Similarly, the matrix of $\mathcal{H}_{g\star\sigma}$ in the bases $(b_i)_{1\leqslant i\leqslant\delta}$ and $(b_i'^*)_{1\leqslant i\leqslant\delta}$ is

$$[\mathcal{H}_{g\star\sigma}] = (\langle\sigma(\boldsymbol{z})|\ g(\boldsymbol{x})\, b_i(\boldsymbol{x})\, b_i(\boldsymbol{x})\rangle)_{1\leqslant i,j\leqslant\delta}.$$

If $g = \boldsymbol{x}^\alpha$, the matrix $\mathcal{H}_{\boldsymbol{x}^\alpha\star\sigma}$ in the basis $(\boldsymbol{x}^\beta)_{\beta\in B}$ and the dual basis of $(\boldsymbol{x}^{\beta'})_{\beta'\in B'}$ is $\left(\sigma_{\alpha+\beta+\beta'}\right)_{\beta\in B,\beta'\in B'} = H_{\boldsymbol{x}^\alpha\star\sigma}^{B,B'}$.

From this relation (12) and Proposition 16, we have the following property.

Proposition 30. *If $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ with $\omega_i \in \mathbb{C}[\boldsymbol{z}] \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct, then*

- *for all $g \in \mathcal{A}$, the generalized eigenvalues of $(\mathcal{H}_{g\star\sigma}, \mathcal{H}_\sigma)$ are $g(\boldsymbol{\xi}_i)$ with multiplicity μ_i, $i = 1...r$,*
- *the generalized eigenvectors common to all $(\mathcal{H}_{g\star\sigma}, \mathcal{H}_\sigma)$ with $g \in \mathcal{A}$ are - up to a scalar - $\mathcal{H}_{\boldsymbol{\sigma}}^{-\mathbf{1}}(\boldsymbol{e}_{\boldsymbol{\xi}_1}), ..., \mathcal{H}_{\boldsymbol{\sigma}}^{-\mathbf{1}}(\boldsymbol{e}_{\boldsymbol{\xi}_r})$.*

Remark 31. *If we take $g = z_i$, then the eigenvalues are the i-th coordinates of the points $\boldsymbol{\xi}_j$.*

3.6 The case of simple roots

In this section, we assume that $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ with $\omega_i \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct.

By Proposition 25, $\{\boldsymbol{e}_{\xi_1}, ..., \boldsymbol{e}_{\xi_r}\}$ is a basis of $\mathcal{A}_\sigma^*$. We denote by $\{\mathbf{u}_{\boldsymbol{\xi}_1}, ..., \mathbf{u}_{\boldsymbol{\xi}_r}\}$ the basis of $\mathcal{A}_\sigma$, which is dual to $\{\boldsymbol{e}_{\xi_1}, ..., \boldsymbol{e}_{\xi_r}\}$, so that $\forall a(\boldsymbol{x}) \in \mathcal{A}$,

$$a(\boldsymbol{x}) \equiv \sum_{i=1}^r \langle \mathbf{e}_{\boldsymbol{\xi}_i}(\boldsymbol{z})|\, a(\boldsymbol{x})\rangle\, \boldsymbol{u}_{\xi_i}(\boldsymbol{x}) \equiv \sum_{i=1}^r a(\boldsymbol{\xi}_i)\, \boldsymbol{u}_{\xi_i}(\boldsymbol{x}). \tag{13}$$

From this formula, we easily verify that the polynomials $\mathbf{u}_{\boldsymbol{\xi}_1}, \mathbf{u}_{\boldsymbol{\xi}_2}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_r}$ are the *interpolation polynomials* at the points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_r$, and satisfy the following relations:

- $\mathbf{u}_{\boldsymbol{\xi}_i}(\boldsymbol{\xi}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathbf{u}_{\boldsymbol{\xi}_i}(\boldsymbol{x})^2 \equiv \mathbf{u}_{\boldsymbol{\xi}_i}(\boldsymbol{x})$.
- $\sum_{i=1}^r \mathbf{u}_{\boldsymbol{\xi}_i}(\boldsymbol{x}) \equiv 1$.

These relations and Proposition 16 imply the following result:

Corollary 32. *If $g \in \mathbb{C}[\boldsymbol{x}]$ is separating the roots $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$ (i.e. $g(\boldsymbol{\xi}_i) \neq g(\boldsymbol{\xi}_j)$ when $i \neq j$), then*

- *the operator $\mathcal{M}_g^t$ is diagonalizable and its eigenvalues are $g(\boldsymbol{\xi}_1), \ldots, g(\boldsymbol{\xi}_r)$,*
- *the corresponding eigenvectors of $\mathcal{M}_g$ are, up to a non-zero scalar, the interpolation polynomials $\mathbf{u}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_r}$.*
- *the corresponding eigenvectors of $\mathcal{M}_g^t$ are, up to a non-zero scalar, the evaluations $\mathbf{e}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{e}_{\boldsymbol{\xi}_r}$.*

In our context, we have the following property:

Proposition 33. *If $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ and $g \in \mathbb{C}[\boldsymbol{x}]$ is separating the roots $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$, then the generalized eigenvectors of $(\mathcal{H}_{g\star\sigma}, \mathcal{H}_\sigma)$ are, up to a non-zero scalar, the interpolation polynomials $\mathbf{u}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_r}$.*

Proof. By the relations (12) and Corollary 32, the eigenvectors $\mathbf{u}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_r}$ of $\mathcal{M}_g$ are the generalized eigenvectors of $(\mathcal{H}_{g\star\sigma}, \mathcal{H}_\sigma)$. □

Remark 34. If $\boldsymbol{v}_i(\boldsymbol{x})$ is a generalized eigenvector of $(\mathcal{H}_{x_j\star\sigma}, \mathcal{H}_\sigma)$ for the eigenvalue $\xi_{i,j}$, then by the previous proposition, it is a multiple of $\boldsymbol{u}_{\boldsymbol{\xi}_i}$ of the form $\boldsymbol{v}_i(\boldsymbol{x}) = \boldsymbol{v}_i(\xi_i)\, \boldsymbol{u}_i(\boldsymbol{x})$ since $\boldsymbol{u}_i(\xi_i) = 1$, and we have $\boldsymbol{u}_i(\boldsymbol{x}) = \frac{1}{\boldsymbol{v}_i(\xi_i)} \boldsymbol{v}_i(\boldsymbol{x})$.

Let us recall other relations between the structured matrices involved in this eigen problem, that will be useful to analyse the numerical behavior of the method. For more details, see e.g. [19].

Definition 35. *Let $\boldsymbol{z}^{\boldsymbol{\beta}} = (\boldsymbol{z}^{\beta_i})_{i=1,\ldots,r}$ be a family of monomials in $\mathcal{A}$. We define the $\boldsymbol{z}^{\boldsymbol{\beta}}$-Vandermonde matrix for the points $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r \in \mathbb{C}^n$ as*

$$V_{\boldsymbol{\beta}} = (\langle \mathbf{e}_{\boldsymbol{\xi}_j} | \boldsymbol{x}^{\boldsymbol{\beta}_i} \rangle)_{1 \le i,j \le r} = \left(\boldsymbol{\xi}_j^{\boldsymbol{\beta}_i}\right)_{1 \le i,j \le r}.$$

By remark 17, if $\boldsymbol{z}^{\boldsymbol{\beta}} = (\boldsymbol{z}^{\beta_i})_{i=1,\ldots,r}$ is a basis of $\mathcal{A}_\sigma$, then $V_{\boldsymbol{\beta}}$ is the matrix of coefficients of $\mathbf{e}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{e}_{\boldsymbol{\xi}_r}$ in the dual basis of $(\boldsymbol{z}^{\beta_i})_{i=1,\ldots,r}$ in $\mathcal{A}_\sigma^*$ and it is invertible. Conversely, we check that $V_{\boldsymbol{\beta}}$ invertible implies that $\boldsymbol{z}^{\beta_1}\ldots, \ldots, \boldsymbol{z}^{\beta_r}$ are linearly independent elements in $\mathcal{A}_\sigma$. Thus, they form a basis of $\mathcal{A}_\sigma$.

Proposition 36. *Suppose that $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ with $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r \in \mathbb{C}^n$ pairwise distinct and $\omega_1, \ldots, \omega_r \in \mathbb{C}$. Let $W = \operatorname{diag}(\omega_1, \ldots, \omega_r)$ be the diagonal matrix associated to the weights ω_i and let $D_g = \operatorname{diag}(g(\xi_1), \ldots, g(\xi_r))$ be the diagonal matrices associated to $g(\xi_1), \ldots, g(\xi_r)$. Then we have*

$$\begin{aligned} [\mathcal{H}_\sigma] &= V_{\boldsymbol{\beta}} W V_{\boldsymbol{\beta}}^t \\ [\mathcal{H}_{g\star\sigma}] &= V_{\boldsymbol{\beta}} W D_g V_{\boldsymbol{\beta}}^t = V_{\boldsymbol{\beta}} D_g W V_{\boldsymbol{\beta}}^t \\ [\mathcal{M}_g^t] &= V_{\boldsymbol{\beta}} D_g V_{\boldsymbol{\beta}}^{-1} \end{aligned}$$

Proof. If $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i \; \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ and $\boldsymbol{z}^{\boldsymbol{\beta}} = (\boldsymbol{z}^{\beta_i})_{i=1,\ldots,r}$ is a basis of $\mathcal{A}_\sigma$, then $[\mathcal{H}_{g\star\sigma}] = [\sum_{i=1}^n \omega_i g(\xi_i)\, \xi^{\beta_i+\beta_j}]_{i,j=1,\ldots,r}$. By an explicit computation, we check that $[\mathcal{H}_{g\star\sigma}] = V_{\boldsymbol{\beta}}^t W D_g V_{\boldsymbol{\beta}}$. Equation (12) implies that $[\mathcal{M}_g^t] = [\mathcal{H}_{g\star\sigma}][\mathcal{H}_\sigma]^{-1} = V_{\boldsymbol{\beta}}^t D_g V_{\boldsymbol{\beta}}^{-1}$. □

Proposition 37. *Let* $\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$. *The basis* $\{\mathbf{u}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_r}\}$ *is an orthogonal basis of* $\mathcal{A}_\sigma$ *for the inner product* $\langle ., . \rangle_\sigma$ *and satisfies* $\langle \mathbf{u}_{\boldsymbol{\xi}_i}, 1 \rangle_\sigma = \langle \sigma | \mathbf{u}_{\boldsymbol{\xi}_i} \rangle = \omega_i$ *for* $i = 1\ldots, r$.

Proof. For $i, j = 1 \ldots r$, we have $\langle \mathbf{u}_{\boldsymbol{\xi}_i}, \mathbf{u}_{\boldsymbol{\xi}_j} \rangle_\sigma = \langle \sigma | \, \mathbf{u}_{\boldsymbol{\xi}_i} \mathbf{u}_{\boldsymbol{\xi}_j} \rangle = \sum_{k=1}^{r} \omega_k \, \mathbf{u}_{\boldsymbol{\xi}_i}(\xi_k) \mathbf{u}_{\boldsymbol{\xi}_j}(\xi_k)$. Thus

$$\langle \mathbf{u}_{\boldsymbol{\xi}_i}, \mathbf{u}_{\boldsymbol{\xi}_j} \rangle_\sigma = \begin{cases} w_i \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

and $\{\mathbf{u}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_r}\}$ is an orthogonal basis of $\mathcal{A}_\sigma$. □

4 Flat extensions and decomposition algorithm

To solve the truncated series problem, we use a characterization of all possible extension of the truncated series that have finite rank.

For a vector space $V \subset \mathbb{C}[\boldsymbol{x}]$, we denote by V^+ the vector space $V^+ = V + x_1 V + \cdots + x_n V$. We denote by ∂V a vector space such that $V^+ = V \oplus \partial V$.

We says that V is *connected to* 1, if there exists an increasing sequence of vector spaces $V_0 \subset V_1 \subset \cdots \subset V_s = V$ such that $V_0 = \langle 1 \rangle$ and $V_{l+1} \subset V_l^+$. The *index* of an element $v \in V$ is the smallest l such that $v \in V_l$.

We say that a set of polynomials $B \subset \mathbb{C}[\boldsymbol{x}]$ is connected to 1 if the vector space $\langle B \rangle$ spanned by B is connected to 1. In particular, a monomial set $B = \{\boldsymbol{x}^{\beta_1}, \ldots, \boldsymbol{x}^{\beta_r}\}$ is connected to 1 if for all $m \in B$, either $m = 1$ or there exists $m' \in B$ and $i_0 \in [1, \ldots, n]$ such that $m = x_{i_0} m'$.

The truncated series problem is closely related to the notion of flat extension that we define now:

Definition 38. *For any matrix* H *which is a submatrix of another matrix* H', *we say that* H' *is a* flat extension *of* H *if* $\operatorname{rank} H = \operatorname{rank} H'$.

This flat extension property can be characterized as follows:

Proposition 39. *Let* H *be a submatrix of* H' *and* M, M', N *be matrices such that*

$$H' = \begin{pmatrix} H & M' \\ M^t & N \end{pmatrix}. \tag{14}$$

Then, H' *is a flat extension of* H *iff there exists matrices* P, P', *such that* H

$$M = H^t P, M' = H P', N = P^t H P'. \tag{15}$$

If H *and* H' *are symmetric, then one can take* $P = P'$.

Proof. Suppose that H' is a flat extension of H. As $\operatorname{rank} H' = \operatorname{rank} H$, we have $\operatorname{im} M' \subset \operatorname{im} H$ and there exists a matrix P' such that $M' = HP'$. Similarly, there exists P such that $M = H^t P$. We deduce that

$$\begin{pmatrix} \mathrm{Id} & 0 \\ -P^t & \mathrm{Id} \end{pmatrix} \begin{pmatrix} H & M' \\ M^t & N \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -P' \\ 0 & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & N - MP' \end{pmatrix},$$

which has the same rank as H. Thus $N - MP' = N - P^t H P' = 0$.

Conversely, if we have $M = H^t P, M' = H P', N = P^t H P'$, then

$$H' = \begin{pmatrix} H & H P' \\ P^t H & P^t H P' \end{pmatrix}$$

has clearly the same rank as H.

If H and H' are symmetric, then using $P' = P$ yields the same relations as above. □

We can now give the main result, which will allow us to recover the decomposition of σ. It generalized the sparse flat extension results of [16], [6], [3] to any vector space connected to 1:

Theorem 40. *Let V, $V' \subset \mathbb{K}[\boldsymbol{x}]$ be a vector space connected to 1 and $\sigma \in \left\langle V{\cdot}V' \right\rangle^*$. Let $B \subset V$, $B' \subset V'$ connected to 1 such that $B^+ \subset V$, $B'^+ \subset V'$ and $H_\sigma^{B,B'}$ invertible. Then the following points are equivalent:*

1. $\operatorname{rank} H_\sigma^{V,V'} = \operatorname{rank} H_\sigma^{B,B'}$,
2. *the operators* $M_i := \left(H_\sigma^{B,B'}\right)^{-1} H_{x_i \cdot \sigma}^{B,B'}$ *commute,*
3. *there is a unique extension $\tilde{\sigma} \in \mathbb{K}[[\boldsymbol{z}]]$ which coincides with σ on $\langle V \cdot V' \rangle$ and such that a basis of B is a basis of $\mathcal{A}_{\tilde{\sigma}}$. In this case, $I_{\tilde{\sigma}} = \left(\ker H_\sigma^{B,B'^+}\right)$.*

Proof. $1 \Rightarrow 2$. Let $r = \dim(B) = \dim(B') = \operatorname{rank} H_\sigma^{B,B'}$since $H_\sigma^{B,B'}$ is invertible. The condition $\operatorname{rank} H_\sigma^{B,B'} = \operatorname{rank} H_\sigma^{V,V'} = r$ implies that $\ker H_\sigma^{V,B'} = \ker H_\sigma^{V,V'}$. In particular we have

$$\begin{aligned} \kappa \in \ker H_\sigma^{V,V'} & \Leftrightarrow \forall v' \in V', \langle \sigma | \, k v' \rangle = 0 && (16)\\ & \Leftrightarrow \forall b' \in V', \langle \sigma | \, k b' \rangle = 0 && (17)\end{aligned}$$

Let $M_i := \left(H_\sigma^{B,B'}\right)^{-1} H_{x_i \star \sigma}^{B,B'}$. It is a linear operator of $\langle B \rangle$. As $H_{x_i \star \sigma}^{B,B'} = H_\sigma^{B,B'} \circ M_i$, we have $\forall b \in B$, $b' \in B'$

$$\langle \sigma | \, x_i \, b b' \rangle = \langle \sigma | \, M_i(b) \, b' \rangle$$

As $\operatorname{rank} H_\sigma^{V,V'} = \operatorname{rank} H_\sigma^{B^+,B'^+} = \operatorname{rank} H_\sigma^{B,B'} = r$, we also have $\forall j = 1, \ldots, n$,

$$\langle \sigma | \, x_i \, x_j b b' \rangle = \langle \sigma | \, x_i \, b x_j b' \rangle = \langle \sigma | \, x_j \, M_i(b) \, b' \rangle = \langle \sigma | \, M_j \circ M_i(b) \, b' \rangle.$$

We deduce that $\langle \sigma | \, M_j \circ M_i(b) \, b' \rangle = \langle \sigma | \, M_i \circ M_j(b) \, b' \rangle$ and the operators M_i, M_j commute: $M_j \circ M_i = M_i \circ M_j$.

$2 \Rightarrow 3$. Let us define the operator

$$\begin{aligned} \phi \colon \mathbb{K}[\boldsymbol{x}] & \rightarrow B\\ p & \mapsto p(M)(1) \end{aligned}$$

and the linear form

$$\begin{aligned} \tilde{\sigma} \colon \mathbb{K}[\boldsymbol{x}] & \rightarrow \mathbb{K}\\ p & \mapsto \langle \sigma | \, p(M)(1) \rangle \end{aligned}$$

We are going to show that $\tilde{\sigma}$ extends σ and that $I_{\tilde{\sigma}} = \left(\ker H_{\tilde{\sigma}}^{B,B'^+}\right)$. As the operators M_i commute, the operator obtained by substituting the variable x_i by M_i in a polynomial $p \in \mathbb{K}[\boldsymbol{x}]$ is well-defined and the kernel J of ϕ is an ideal of $\mathbb{K}[\boldsymbol{x}]$.

We first prove that $\tilde{\sigma}$ coincides with σ on $\langle V \cdot V' \rangle$.

Let us prove by induction on the index that $\forall v \in V$, $\forall b' \in B'$, $\langle \sigma | \, v b' \rangle = \langle \sigma | \, \phi(v) \, b' \rangle$. If v is of index 0, then $b = 1$ (up to a scalar) and $\phi(1) = 1$ so that the property is true.

Let us assume that the property is true for the elements of V of index $l - 1 \geqslant 0$ and let $v \in V$ of index l: there exists $v_i \in V$ of index $l - 1$ such that $v = \sum_i x_i \, v_i$. By induction hypothesis and the relations (16) and (17), we have $\forall b' \in B'$,

$$\begin{aligned} \langle \sigma | \, v b' \rangle &= \sum_i \langle \sigma | \, v_i \, x_i \, b' \rangle = \sum_i \langle \sigma | \, \phi(v_i) \, x_i \, b' \rangle = \sum_i \langle \sigma | \, M_i \circ \phi(v_i) \, b' \rangle\\ &= \left\langle \sigma | \left(\sum_i x_i \phi(v_i) \right) b' \right\rangle = \langle \sigma | \, \phi(v) \, b' \rangle. \end{aligned}$$

Using relations (16) and (17), we also have

$$\forall v \in V, \forall v' \in V', \langle \sigma | \, v v' \rangle = \langle \sigma | \, \phi(v) \, v' \rangle. \tag{18}$$

In a similar way, we prove that

$$\forall b \in B, \forall v' \in V', \langle \sigma | \, b v' \rangle = \langle \sigma | \, v'(M) \, (b) \rangle. \tag{19}$$

The property is true for $v'=1$. Let us assume that it is true for the elements of V' of index $l-1>0$ and let $v'\in V'$ be an element of index l. There exist $v_i'\in V'$ of index $l-1$ such that $v'=\sum_i x_i v_i'$. By induction hypothesis and the relations (16) and (17), we have $\forall v\in V$,

$$\begin{aligned}\langle\sigma|\, bv'\rangle &= \sum_i \langle\sigma|\, v_i' x_i b\rangle = \sum_i \langle\sigma|\, v_i' M_i(b)\rangle = \sum_i \langle\sigma|\, v_i'(M) M_i(b)\,\rangle\\ &= \left\langle\sigma|\left(\sum_i M_i\circ v_i'(M)\right)(b)\right\rangle = \langle\sigma|\, v'(M)\,(b)\rangle.\end{aligned}$$

By the relations (18) and (19), we have $\forall v\in V$, $\forall v'\in V'$,

$$\langle\sigma|\, vv'\rangle = \langle\sigma|\, v'\,v(M)(1)\rangle = \langle\sigma|\, v'(M)\circ v(M)(1)\rangle = \langle\sigma|\,\phi(vv')\rangle = \langle\tilde{\sigma}|\, vv'\rangle.$$

This shows that $\tilde{\sigma}$ coincides with σ on $\langle V\cdot V'\rangle$.

We deduce from relation (18) that $\forall b\in B$, $\forall b'\in B'$, $\langle\sigma|\,(b-\phi(b))\,b'\rangle=0$ and $\phi(b)=b$ since $H_\sigma^{B,B'}$ is invertible. Therefore ϕ is the projection of $\mathbb{K}[\boldsymbol{x}]$ on B along its kernel J and we have the exact sequence

$$0\to J\to\mathbb{K}[\boldsymbol{x}]\xrightarrow{\phi}B\to 0.$$

Let $I_\sigma=\ker H_{\tilde{\sigma}}$ and $\mathcal{A}_\sigma=\mathbb{K}[\boldsymbol{x}]/I_\sigma$. As $J\subset I_\sigma$, we have $\dim_{\mathbb{K}}\mathcal{A}_\sigma\leqslant\dim_{\mathbb{K}}\mathbb{K}[\boldsymbol{x}]/J=\dim B=r$ and B is generating $\mathcal{A}_\sigma$. Since $\tilde{\sigma}$ coincides with σ on $\langle B\cdot B'\rangle$ and $H_\sigma^{B,B'}$ is invertible, a basis of the vector space $B\subset\mathbb{K}[\boldsymbol{x}]$ is free in $\mathcal{A}_\sigma$. This shows that $\dim_{\mathbb{K}}\mathcal{A}_\sigma=r$ and that $J=I_\sigma$.

Since B contains 1 and ϕ is the projection of $\mathbb{K}[\boldsymbol{x}]$ on B along $I_\sigma=J$, we check that I_σ is generated by the element $x_i b-\phi(x_i b)$ for $b\in B$, $i=1,\ldots,n$, that is by the elements of $\ker H_\sigma^{B^+,B'}$.

If there is another $\tilde{\sigma}'\in\mathbb{K}[[\boldsymbol{z}]]$ which coincides with σ on $\langle V\cdot V'\rangle$, then $J\subset I_{\tilde{\sigma}'}$ and $\forall p\in\mathbb{K}[\boldsymbol{x}]$, $\langle\tilde{\sigma}'|\,p\rangle=\langle\tilde{\sigma}'|\,\phi(p)\rangle=\langle\sigma|\,\phi(p)\rangle=\langle\tilde{\sigma}|\,p\rangle$, so that $\tilde{\sigma}'=\tilde{\sigma}$, which proves point 3.

$3\Rightarrow 1$. If $\tilde{\sigma}\in\mathbb{K}[[\boldsymbol{z}]]$ coincides with σ on $\langle V\cdot V'\rangle$ and B is a basis of $\mathcal{A}_{\tilde{\sigma}}$, then

$$r=\dim B=\operatorname{rank} H_{\tilde{\sigma}}\geqslant\operatorname{rank} H_{\tilde{\sigma}}^{V,V'}=\operatorname{rank} H_\sigma^{V,V'}\geqslant\operatorname{rank} H_\sigma^{B,B'}=r$$

which proves point 1. □

If only the low degree coefficients $\bar{\sigma}$ are known, this property can be used to find an extension of finite rank, by solving polynomial equations. We introduce a variable h_α for each unknown coefficients of σ. If we suppose that V (resp. V') is spanned by a set B (resp. B') of monomials connected to 1, to find if an extension σ of finite rank of $\bar{\sigma}$ is such that B is a basis of $\mathcal{A}_\sigma$, we use the flat extension constraint, which yields polynomial equations in the unknown h_α. As H_σ^B is a submatrix of $H_\sigma^{B^+}$, we have a decomposition of the form

$$H_\sigma^{B^+,B'^+}=\begin{pmatrix} H_\sigma^{B,B'} & H_\sigma^{\partial B,B'}\\ H_\sigma^{B,\partial B'} & H_\sigma^{\partial B,\partial B'}\end{pmatrix},$$

where the rows and columns of $H_\sigma^{B,B'}$ are indexed by the elements respectively in B' and B and the rows and columns of $H_\sigma^{\partial B,B'}$ are indexed by B' and a basis ∂B of a supplementary space of B in B^+.

By Proposition 39, the matrix $H_\sigma^{B^+,B'^+}$ has the same rank as $H_\sigma^{B,B'}$, if and only if, there exists a matrix P such that

$$H_\sigma^{B,\partial B}-H_\sigma^{B,B'}P=0, H_\sigma^{\partial B,\partial B'}-P^t H_\sigma^{B,B'}P=0. \tag{20}$$

In these relations, the coefficients of P can also be considered as variables and the system of equations (20) gives the condition that $H_\sigma^{B^+}$ is a flat extension of H_σ^B.

Each solution of this system gives a zero-dimensional ideal I_σ generated by ker $H_\sigma^{B^+}$ which uniquely defines all the coefficients of the series σ and such that $\operatorname{rank} H_\sigma=\operatorname{rank} H_\sigma^B$.

Remark 41. A basis of the kernel of $H_\sigma^{B^+,B'}$ is given by the columns of $\begin{pmatrix}-P\\ I\end{pmatrix}$, which represent polynomials of the form

$$\boldsymbol{x}^\alpha-\sum_{\beta\in B}p_{\alpha,\beta}\boldsymbol{x}^\beta$$

for $\alpha \in \partial B$. These polynomials are border relations which project monomials of ∂B on the vector space spanned by B, modulo $\ker H_\sigma^{B^+}$. Using Theorem 40 and the characterization of border bases in terms of commutation relations [18], [20], we can prove that they form a border basis of the ideal generated by $\ker H_\sigma^{B^+}$ iff $\operatorname{rank} H_\sigma^B = \operatorname{rank} H_\sigma^{B^+} = |B|$, or in other words, iff $H_\sigma^{B^+}$ has a flat extension. This condition is equivalent to the commutation property of formal multiplication operators.

4.1 Computing an orthogonal basis of $\mathcal{A}_\sigma$

In this section, we describe a new method to construct a basis B of $\mathcal{A}_\sigma$, from the knowledge of the first terms σ_α of $\sigma(\boldsymbol{z})$. We assume that $\sigma_0 \neq 0$.

To construct this basis B of $\mathcal{A}_\sigma$, we are going to define inductively vector spaces V_i as follows. Start with $V_0 = \langle 1 \rangle$ and compute a vector space L_l of maximal dimension in V_l^+ such that

- L_l is orthogonal to V_l: $\langle L_l, V_l \rangle_\sigma = 0$,
- $L_l \cap \ker H_\sigma^{V_l^+} = \{0\}$.

Then we define $V_{l+1} = V_l + L_l$. If there is no such L_l with dimension >0, this implies that $V_l^+ = V_l + K_l$ with $K_l \subset \ker H_\sigma^{V_l^+}$ and we stop. By theorem 40, a basis of V_l yields a basis of $\mathcal{A}_\sigma$.

Suppose that $b_1, \ldots, b_{r_i}$ is an orthogonal basis of V_i: $\langle b_i, b_j \rangle_\sigma = 0$ if $i \neq j$ and $\langle b_i, b_i \rangle_\sigma \neq 0$. Then L_i can be constructed as follows: Compute the vectors

$$b_{j,k} = x_k b_j - \sum_{i=1}^{r_l} \frac{\langle x_k b_j, b_i \rangle_\sigma}{\langle b_i, b_i \rangle_\sigma} b_i,$$

generating $V_i^\perp$ in V_i^+ and extract a maximal orthogonal family $b_{r_i+1}, \ldots, b_{r_{i+1}}$ for the inner product $\langle \cdot, \cdot \rangle_\sigma$, that form a basis of L_i. This can be done for instance by computing a QR decomposition of the matrix

$$[\langle b_{j,j}, b_{i',j'} \rangle_\sigma]_{1 \leqslant i,i' \leqslant r_i, 1 \leqslant j,j' \leqslant n}.$$

This leads to the following algorithm:

Algorithm 1

Input: the coefficients σ_α of a series $\sigma \in \mathbb{C}[[\boldsymbol{z}]]$ for $\alpha \in A \subset \mathbb{N}^n$ connected to 1.

- Let $B_0 := \{1\}$; s := 1; r := 1; $E = \langle \boldsymbol{z}^\alpha \rangle_{\alpha \in A}$;
- While $s > 0$ and $B^+ \subset E$ do
  - compute $b_{j,k} := x_k b_j - \sum_{i=1}^r \frac{\langle x_k b_j, b_i \rangle_\sigma}{\langle b_i, b_i \rangle_\sigma} b_i$ for $j = 1, \ldots, r$, $k = 1, \ldots, n$;
  - compute a maximal subset $\{b_{r+1}, \ldots, b_{r+s}\}$ of $\{b_{i,j}\}$ of orthogonal vectors for the inner product $\langle \cdot, \cdot \rangle_\sigma$;
  - $B_{s+1} := B \cup \{b_{r+1}, \ldots, b_{r+s}\}$; $r += s$;
- If $B^+ \not\subset E$ then return `failed`.

Output: `failed` or `success` with

- a basis $B = \{b_1, \ldots, b_r\}$ orthogonal for $\langle \cdot, \cdot \rangle_\sigma$.
- the relations $b_{j,k} := x_k b_j - \sum_{i=1}^r \frac{\langle x_k b_j, b_i \rangle_\sigma}{\langle b_i, b_i \rangle_\sigma} b_i$ for $j = 1, \ldots, r$, $k = 1, \ldots, n$.

In the main loop of this algorithm, when $x_k b_j \in \langle b_1, \ldots, b_r \rangle$ then $b_{j,k} = 0$.

Remark 42. If the polynomials b_i are at most of degree $d' < d$, then only the coefficients of $\sigma_h(\boldsymbol{z})$ of degree $\leqslant 2d' + 1$ are involved in this computation. In this case, the border basis and the decomposition of the series σ as a sum of exponential polynomials can be computed from coefficients first terms.

Remark 43. When all the coefficients are not known, ...

Theorem 44. *If Algorithm 1 outputs with success a set* $B = \{b_1, \ldots, b_r\}$ *and the relations* $b_{j,k} := x_k b_j - \sum_{i=1}^{r} \frac{\langle x_k b_j, b_i \rangle_\sigma}{\langle b_i, b_i \rangle_\sigma} b_i$, $j = 1 \ldots r$, $k = 1 \ldots n$, *then* σ *coincides on* $\langle B^+ \cdot B^+ \rangle$ *with a series* $\tilde{\sigma}$ *such that*

- $\operatorname{rank} H_{\tilde{\sigma}} = r$;
- *B is an orthogonal basis of $\mathcal{A}_{\tilde{\sigma}}$ for the inner product* $\langle .,. \rangle_{\tilde{\sigma}}$;
- *$I_{\tilde{\sigma}}$ is generated by* $x_k b_j - \sum_{i=1}^{r} \frac{\langle x_k b_j, b_i \rangle_\sigma}{\langle b_i, b_i \rangle_\sigma} b_i$ *for* $j = 1 \ldots r$, $k = 1 \ldots n$;
- *The matrix of multiplication by x_k in the basis B of $\mathcal{A}_{\tilde{\sigma}}$ is* $M_k := \left(\frac{\langle x_j b_i, b_k \rangle_\sigma}{\langle b_k, b_k \rangle_\sigma} \right)_{1 \leqslant i, j \leqslant r}$.

Proof. Let $V = \langle B \rangle$. By construction of B, V is connected to 1. A basis B' of $V^+ = \langle B^+ \rangle$ is formed by the elements of B and some of the polynomials $b_{j,k}$. Since Algorithm 1 stops with success, the matrix of $H_\sigma^{V^+}$ in this basis B' is of the form

$$H_\sigma^{B'} = \begin{pmatrix} H_\sigma^B & 0 \\ 0 & 0 \end{pmatrix}$$

where H_σ^B is diagonal and invertible matrix. The kernel of $H_\sigma^{B'}$ is generated by the polynomials $b_{j,k}$.

By Theorem 40, σ coincides on $\langle V^+ \cdot V^+ \rangle = \langle B' \cdot B' \rangle = \langle B^+ \cdot B^+ \rangle$ with a series $\tilde{\sigma}$ such that B is a basis of $\mathcal{A}_{\tilde{\sigma}} = \mathbb{C}[\boldsymbol{x}] / I_{\tilde{\sigma}}$ and $I_{\tilde{\sigma}} = \left(\ker H_{\tilde{\sigma}}^{V^+} \right) = (b_{j,k})_{j=1 \ldots r, k=1 \ldots n}$.

This shows that $\operatorname{rank} H_{\tilde{\sigma}} = \dim \mathcal{A}_{\tilde{\sigma}} = |B| = r$. By construction, B is orthogonal for the inner product $\langle .,. \rangle_\sigma$, which coincides with $\langle .,. \rangle_{\tilde{\sigma}}$ on $\langle B^+ \cdot B^+ \rangle$. Thus B is also an orthogonal basis of $\mathcal{A}_{\tilde{\sigma}}$ for the inner product $\langle .,. \rangle_{\tilde{\sigma}}$.

As $b_{j,k} \equiv 0$ in $\mathcal{A}_{\tilde{\sigma}}$, we have $x_k b_j \equiv \sum_{i=1}^{r} \frac{\langle x_k b_j, b_i \rangle_\sigma}{\langle b_i, b_i \rangle_\sigma} b_i$, which shows that the matrix of multiplication by x_k in the basis B of $\mathcal{A}_{\tilde{\sigma}}$ is $M_k = \left(\frac{\langle x_j b_i, b_k \rangle_\sigma}{\langle b_k, b_k \rangle_\sigma} \right)_{1 \leqslant i, j \leqslant r}$. □

4.2 Computing the support of $\boldsymbol{\sigma}$

This leads to the following algorithm to compute the decomposition, if a basis of $\mathcal{A}_\sigma$ is known.

Algorithm 2

Input: a basis $\{b_1, \ldots, b_r\}$ of $B \subset \mathbb{K}[\boldsymbol{x}]$, which is orthogonal for $\langle .,. \rangle_\sigma$.

- Take a generic linear form $l(\boldsymbol{x}) = l_1 x_1 + \cdots + l_n x_n$;
- Compute the matrices $M_j = \left(\frac{\langle x_j b_i, b_k \rangle_\sigma}{\langle b_k, b_k \rangle_\sigma} \right)_{1 \leqslant i, j \leqslant r}$;
- Compute the eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ of $[M_j]$;
- Compute $\xi_{i,j}$ such that $M_j \boldsymbol{v}_i - \xi_{i,j} \boldsymbol{v}_i = 0$ for $j = 1, \ldots, n$, $i = 1, \ldots, r$;
- Compute $\boldsymbol{u}_i(\boldsymbol{x}) = \frac{1}{\boldsymbol{v}_i(\xi_i)} \boldsymbol{v}_i(\boldsymbol{x})$ where $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n})$;
- Compute $\langle \sigma | \boldsymbol{u}_i \rangle = \omega_i$;

Output: a decomposition $\tilde{\sigma} = \sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ such that $\forall b, b' \in B^+$

$$\langle \sigma | b b' \rangle = \langle \tilde{\sigma} | b b' \rangle.$$

The step which compute the interpolation polynomial $\boldsymbol{u}_i(\boldsymbol{x})$ can be replace by the following step: Let E be the coefficient matrix of the eigenvectors $\boldsymbol{v}_1(\boldsymbol{x}), \ldots, \boldsymbol{v}_r(\boldsymbol{x})$ and U be the coefficient vector of 1 in the given basis of $\mathcal{A}_\sigma$. Solve the system EC=U and compute $\boldsymbol{u}_i(\boldsymbol{x}) = \frac{1}{C_i} \boldsymbol{v}_i(\boldsymbol{x})$.

4.3 Generalized Prony method

The previous algorithms, used together, give the following method which generalizes Prony's method to solve the reconstruction problems in several variables.

Algorithm 3

Input: the coefficients σ_α of a series $\sigma \in \mathbb{C}[[\boldsymbol{z}]]$ for $\alpha \in A \subset \mathbb{N}^n$ connected to 1.

- Apply Algorithm 1 to find an orthogonal basis of $B \subset \langle \boldsymbol{x}^A \rangle$ for $\langle ., . \rangle_\sigma$.
- If `success` then
  - apply Algorithm 2 to compute the decomposition $\tilde{\sigma} = \sum_{i=1}^r \omega_i\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$, which coincides with σ on $\langle B^+ \cdot B^+ \rangle$;
  - check that the decomposition $\tilde{\sigma}$ coincides with σ on A;

Output:

- `failed` or
- `success` with a decomposition $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ on A.

It can be applied in the following contexts:

Sparse reconstruction of sum of exponentials. Given a function $h \in C^\infty(\mathbb{C}^n)$ of the form

$$\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{C}^n \mapsto h(\boldsymbol{x}) = \sum_{i=1}^r a_i(\boldsymbol{x})\, e^{\langle \mathbf{f}_i, \mathbf{x} \rangle} \tag{21}$$

where $\mathbf{f}_1, ..., \mathbf{f}_r \in \mathbb{C}^n$ are pairwise distinct, $a_i(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}] \setminus \{0\}$ and $\forall \mathbf{f}, \mathbf{g} \in \mathbb{C}^n, \langle \mathbf{f}, \mathbf{g} \rangle = f_1\, g_1 + \cdots + f_n\, g_n$, the problem consists in recovering

- the distinct *frequency vectors* $\mathbf{f}_1, ..., \mathbf{f}_r \in \mathbb{C}^n$,
- the *polynomial coefficients* $a_i(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}] \setminus \{0\}$,

from evaluations of the function h at some points of $\mathbb{C}^n$. This corresponds to the following setting:

- $\mathfrak{F} = C^\infty(\mathbb{R}^n)$,
- S_i: $h(\boldsymbol{x}) \mapsto h(\boldsymbol{x} + e_i)$ the shift operators by the elements e_i of the canonical basis of $\mathbb{C}^n$ $(i = 1, ..., n)$,
- Δ: $h(x) \mapsto \Delta[h] = h(0, ..., 0)$ the evaluation at the origin.

Clearly, the shift operators S_i are commuting and we have $S_j\big(e^{\langle \boldsymbol{f}_i, \mathbf{x} \rangle}\big) = \xi_{i,j}\, e^{\langle \boldsymbol{f}_i, \mathbf{x} \rangle}$ with $\xi_{i,j} = e^{\langle \boldsymbol{f}_i, e_i \rangle}$ so that $E_i = e^{\langle \boldsymbol{f}_i, \mathbf{x} \rangle}$ is an eigenfunction of S_j for the eigenvalue $\xi_{i,j} = e^{\langle \boldsymbol{f}_i, e_i \rangle}$. Moreover, $\Lambda_0[S^\alpha(h)] = h(\alpha)$ and $\sigma_h(\boldsymbol{z})$ is the generating series (4) associated to h.

Notice that instead of the canonical basis, we can take any basis $v_1, ..., v_n$ of $\mathbb{C}^n$ and consider the shift operators S_i: $h(\boldsymbol{x}) \mapsto h(\boldsymbol{x} + v_i)$.

The linear functional Δ can also be replaced by any (non-zero) linear functional on $C^\infty(\mathbb{C}^n)$, for instance the integration over a compact domain Ω: $\Delta[h] = \int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x}$. This provides degrees of freedom in the application of the reconstruction method, which can be interesting for solving numerical issues. In particular, scaling variables can be used to reduce numerical overflows.

We easily check that $a_i(\boldsymbol{x})\; e^{\langle \mathbf{f}_i, \mathbf{x} \rangle}$ are generalized eigenfunctions of the operators S_i and if $h(\boldsymbol{x}) = \sum_{i=1}^r a_i(\boldsymbol{x})\, e^{\langle \mathbf{f}_i, \mathbf{x} \rangle}$, then its generating series are of the form $\sigma_h(\boldsymbol{z}) = \sum_{i=1}^r b_i(\boldsymbol{z})\, e_{\xi_i}(\boldsymbol{z})$ with $\xi_i = (e^{f_{i,1}}, ..., e^{f_{i,n}}) \in \mathbb{C}^n$ and $b_i(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$ polynomial uniquely determined by a_i.

To analyze the correspondance between the polynomials $a_i(\boldsymbol{x})$ and $b_i(\boldsymbol{z})$, we introduce the so-called Macaulay basis of $\mathbb{C}[\boldsymbol{x}]$: $\forall \beta \in \mathbb{N}^n$,

$$b_\beta(\boldsymbol{x}) = \binom{x_1}{\beta_1} \cdots \binom{x_n}{\beta_n}.$$

It has the following nice property: $\sum_{\alpha \in \mathbb{N}^n} b_\beta(\alpha)\, \boldsymbol{e}_\xi(\boldsymbol{z}) = \boldsymbol{z}^\beta\, \xi^{-\beta}\, \boldsymbol{e}_\xi(\boldsymbol{z})$ where $\boldsymbol{e}_\xi(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \xi^\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!}$.

Proposition 45. *Let $B_i \subset \mathbb{N}^n$, $\boldsymbol{f}_i \in \mathbb{C}^n$, $\omega_{i,\beta_i} \in \mathbb{C}$, for i=1,..,r, $\beta_i \in B_i$ and $h(\boldsymbol{x}) = \sum_{i=1}^r \sum_{\beta_i \in B_i} w_{i,\beta_i}\, b_{\beta_i}(\boldsymbol{x})\, e^{\langle \mathbf{f}_i, \mathbf{x} \rangle}$ then its generating series are*

$$\sigma_h(\boldsymbol{z}) = \sum_{i=1}^r \sum_{\beta \in B_i} w_{i,\beta}\, \boldsymbol{z}^\beta\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$$

Sparse reconstruction of sum of Dirac measures. We want to decompose a signal h as a weighted sum of Dirac measures, representing spikes: $h=\sum_{i=1}^{r}\omega_i\,\delta_{\xi_i}$. Consider the following space and linear maps:

- The space $\mathfrak{F}$ is the space of distributions on $\mathbb{C}^n$.
- The operator S_j: $\mathfrak{F}\rightarrow\mathfrak{F}$ is the operator of multiplication by $e^{\frac{2\pi\mathrm{i}}{T_j}x_j}$: $S_j(h)=e^{\frac{2\pi\mathrm{i}}{T_j}x_j}\,h$ for $T_j\in\mathbb{R}_+$. Clearly, the operators S_1, ..., S_n commute. Moreover, $\forall\,\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{C}$, $S_j(\delta_\xi)=e^{\frac{2\pi\mathrm{i}}{T_j}\xi_j}\delta_\xi$ and the Dirac measure δ_ξ is an eigenfunction of S_j for the eigenvalue $e^{\frac{2\pi\mathrm{i}}{T_j}\xi_j}$.
- The linear functional Δ is $\Delta\colon h\in\mathfrak{F}\mapsto\Delta[h]=\frac{1}{\prod_{j=1}^{n}T_j}\int_{\frac{T_1}{2}}^{\frac{T_1}{2}}\cdots\int_{\frac{T_n}{2}}^{\frac{T_n}{2}}h\,(\boldsymbol{x})d\boldsymbol{x}$.

Then $\Delta^\alpha(h):=\Delta[S^\alpha(h)]=\frac{1}{\prod_{j=1}^{n}T_j}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}}\cdots\int_{-\frac{T_n}{2}}^{\frac{T_n}{2}}e^{2\pi\mathrm{i}\sum_{j=1}^{n}\frac{\alpha_j x_j}{T_j}}h(\boldsymbol{x})d\boldsymbol{x}$ is the Fourier coefficient of index $\alpha\in\mathbb{N}^n$ of h. The problem of sparse recovery boils down to reconstruct the spikes at $\xi_1,\ldots,\xi_r\in\mathbb{C}^n$ and their weighs w_1, ..., w_r from the first Fourier coefficients of the signal. We apply the approach described in Section 4 to the first term of the series $\sigma_h(\boldsymbol{z})=\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\Delta^\alpha[h]\,\boldsymbol{z}^\alpha$.

Example. We consider the function $h(u_1,u_2)=2+3\cdot 2^{u_1}2^{u_2}-3^{u_1}$. Its associate generating series is $\sigma=\sum_{\alpha\in\mathbb{N}^2}h(\alpha)\frac{\boldsymbol{z}^\alpha}{\alpha!}$. Its (truncated) moment matrix is

$$H_\sigma^{[1,x_1,x_2,x_1^2,x_1x_2,x_2^2]}=\begin{bmatrix} h(0,0) & h(1,0) & h(0,1) & \cdots \\ h(1,0) & h(2,0) & h(1,1) & \cdots \\ h(0,1) & h(1,1) & h(0,2) & \cdots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{bmatrix}=\begin{bmatrix} 4 & 5 & 7 & 5 & 11 & 13\\ 5 & 5 & 11 & -1 & 17 & 23\\ 7 & 11 & 13 & 17 & 23 & 25\\ 5 & -1 & 17 & -31 & 23 & 41\\ 11 & 17 & 23 & 23 & 41 & 47\\ 13 & 23 & 25 & 41 & 47 & 49 \end{bmatrix}$$

We compute $B_0=\{1\}$, $B_1=\{1,x_1-\frac{5}{4},x_2-\frac{9}{4}x_1-4\}=\{b_0,b_1,b_2\}$.
We have modulo $\ker H_\sigma$:

$$\begin{aligned} x_1\,b_0 &\equiv \frac{5}{4}b_0+b_1\\ x_1\,b_1 &\equiv \sum_i\frac{\langle x_1\,b_1,b_i\rangle_\sigma}{\langle b_i,b_i\rangle_\sigma}=-\frac{5}{16}b_0-b_1+\frac{91}{20}b_2\\ x_1\,b_2 &\equiv \sum_i\frac{\langle x_1\,b_1,b_i\rangle_\sigma}{\langle b_i,b_i\rangle_\sigma}=\frac{96}{25}b_1+\frac{1}{5}b_2 \end{aligned}$$

The matrix of multiplication by x_1 in $B=\{b_0,b_1,b_2\}$ is

$$M_1=\begin{bmatrix} \frac{5}{4} & -\frac{5}{16} & 0\\ 1 & \frac{91}{20} & \frac{96}{25}\\ 0 & -1 & \frac{1}{5} \end{bmatrix}$$

Its eigenvalues are $[1,2,3]$ and the corresponding matrix of eigenvectors is

$$U:=\begin{bmatrix} 1/2 & 3/4 & -1/4\\ 2/5 & -9/5 & 7/5\\ -1/2 & 1 & -1/2 \end{bmatrix},$$

that is, the polynomials $U(x)=[2-\frac{1}{2}x_1-\frac{1}{2}x_2,-1+x_2,\frac{1}{2}x_1-\frac{1}{2}x_2]$.
By computing the Hankel matrix

$$H_\sigma^{[1,x_1,x_2],U}=\begin{bmatrix} 2 & 3 & -1\\ 2\times 1 & 3\times 2 & -1\times 3\\ 2\times 1 & 3\times 2 & -1\times 1 \end{bmatrix}$$

we deduce the weights $2,3,-1$ and the frequencies $(1,1),(2,2),(3,1)$, which corresponds to the decomposition $\sigma=e^{z_1+z_2}+3e^{2z_1+2z_2}-e^{2z_1+z_2}$ and $h(u_1,u_2)=2+3\cdot 2^{u_1+u_2}-3^{u_1}$.

5 Applications

5.1 Vanishing ideal of points

We consider here the problem of computing the vanishing ideal of a given set of points $\xi_1, \ldots, \xi_r \in \mathbb{C}^n$, that is, the ideal of polynomials that vanish at these points.

Let us take $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \sigma_\alpha \boldsymbol{z}^\alpha$. Its coefficients are $\sigma_\alpha = \frac{1}{r} \sum_{i=1}^r \xi_i^\alpha$ for $\alpha \in \mathbb{N}^n$. By Theorem 22, I_σ is the vanishing ideal of the points $\xi_1, \ldots, \xi_r \in \mathbb{C}^n$. By applying the method described in Section 4, we obtain

- a basis $B = \{b_1, \ldots, b_r\}$ of orthogonal polynomials for the inner product $\langle \cdot, \cdot \rangle_\sigma$,
- border polynomials $b_{i,j} := x_j b_i - \sum_{k=1}^r \frac{\langle x_j b_i, b_k \rangle_\sigma}{\langle b_k, b_k \rangle_\sigma} b_k$ $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n)$ generating the vanishing ideal I_σ.

Example 46. Let consider the points $\{(0, 0), (0, 1), (0, -1), (0, 1), (0, -1)\}$. The first terms of coefficients $\sigma_\alpha = \frac{1}{5} \sum_{i=1}^5 \xi_i^\alpha$ for $|\alpha| \leqslant 6$ gives the first terms of the series:

$$\sigma(\boldsymbol{z}) = 1 + \frac{2}{5} \frac{z_1^2}{2!} + \frac{2}{5} \frac{z_2^2}{2!} + \frac{2}{5} \frac{z_1^4}{4!} + \frac{2}{5} \frac{z_2^4}{4!} + \frac{2}{5} \frac{z_1^6}{6!} + \frac{2}{5} \frac{z_2^6}{6!} + \cdots$$

We detail the steps of the algorithm for constructing of an orthogonal basis of $\mathcal{A}_\sigma$ and generators of the vanishing ideal I_σ:

- $B_0 = \{1\}$,
- $B_1 = B_0^+ = \{1, x_1, x_2\}$,
- $B_1^+ = B_1 \cup \{x_1^2, x_1 x_2, x_2^2\}$. As $\langle x_1 x_2, b \rangle_\sigma = 0$ for all $b \in B_1^+$ and $\langle x_1^2, x_1^2 \rangle_\sigma = \langle x_2^2, x_2^2 \rangle_\sigma = \frac{2}{5}$ and $\langle x_1^2, 1 \rangle_\sigma = \langle x_2^2, 1 \rangle_\sigma = \frac{2}{5}$, the orthogonal basis computed at this step is
$$B_2 = \left\{ 1, x_1, x_2, x_1^2 - \frac{2}{5}, x_2^2 - \frac{2}{5} \right\}$$
- $B_2^+ = B_2 \cup \left\{ x_1^3 - \frac{1}{5} x_1, x_1^2 x_2 - \frac{1}{5} x_2, x_1 x_2, x_1 x_2^2 - \frac{2}{5} x_1, x_2^3 - \frac{2}{5} x_2 \right\}$. The vector space orthogonal to $\langle B_2 \rangle$ in $\langle B_2^+ \rangle$ is generated by the polynomials $b - \sum_{i=1}^5 \frac{\langle b, b_k \rangle_\sigma}{\langle b_k, b_k \rangle_\sigma} b_k$ for $\{b_1, \ldots, b_5\} = B_2$ and for $b \in \left\{ x_1^3 - \frac{1}{5} x_1, x_1^2 x_2 - \frac{1}{5} x_2, x_1 x_2, x_1 x_2^2 - \frac{2}{5} x_1, x_2^3 - \frac{2}{5} x_2 \right\}$.

  This yields the polynomials $\partial B_2 = \{x_1^3 - x_1, x_1^2 x_2, x_1 x_2, x_1 x_2^2, x_2^3 - x_2\}$. We check that these polynomials are orthogonal to all B_2^+, so that the algorithm stops and outputs the basis $B_2 = \left\{ 1, x_1, x_2, x_1^2 - \frac{2}{5}, x_2^2 - \frac{2}{5} \right\}$ of $\mathcal{A}_\sigma$ and the border polynomials ∂B_2, which give a generating family of the vanishing ideal I_σ of the points: $I_\sigma = (x_1^3 - x_1, x_1 x_2, x_2^3 - x_2)$.

Remark 47. The method can be extended to the computation of vanishing ideals of points with multiplicities. Instead of taking $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$, one can consider series of the form $\sigma(\boldsymbol{z}) = \sum_{i=1}^r \omega_i(\boldsymbol{z}) \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$, where $\omega_i(\boldsymbol{z})$ is a polynomial in $\mathbb{C}[\boldsymbol{z}]$, which prescribes the multiplicity of the point $\xi_i \in \mathbb{C}^n$. Notice that only multiple points with Gorenstein local rings can be prescribed in this way.

5.2 Signal reconstruction

We present some examples where we applied the Generalized Prony algorithm. It is implemented in MAPLE, and used in the following exemples with a precision of 16 digits. To give an idea of its accuracy, we compute errors at different steps. In order to compare with the results obtained in [22] we use similar formulas. So we compute the relative error of the frequencies with the following formula

$$\operatorname{err}(\boldsymbol{\xi}) = \max\left(\max_{j=1,\ldots,n} \left(\frac{\max_{i=1,\ldots,r} |Re(\xi_{i,j}) - Re(\tilde{\xi}_{i,j})|}{\max_{i=1,\ldots,r} |Re(\xi_{i,j})|} \right), \max_{j=1,\ldots,n} \left(\frac{\max_{i=1,\ldots,r} |Im(\xi_{i,j}) - Im(\tilde{\xi}_{i,j})|}{\max_{i=1,\ldots,r} |Im(\xi_{i,j})|} \right) \right),$$

where $Re(\tilde{\xi}_{i,j})$ and $Im(\tilde{\xi}_{i,j})$ are respectively the real and imaginary part of the j-th coordinate of the computed frequency $\tilde{\xi}_i$. Analogously the relative errors of the coefficients a_i is given by

$$\mathrm{err}(a) = \max\left(\frac{\max_{i=1,\ldots,r}|Re(a_i) - Re(\tilde{a}_i)|}{\max_{i=1,\ldots,r}|Re(a_i)|}, \frac{\max_{i=1,\ldots,r}|Im(a_i) - Im(\tilde{a}_i)|}{\max_{i=1,\ldots,r}|Im(a_i)|}\right),$$

where $Re(\tilde{a}_i)$ and $Im(\tilde{a}_i)$ are respectively the real and imaginary part of the computed coefficients $\tilde{a}_i$. Further we determine the relative of the evaluations by

$$\mathrm{err}(f(\mathbf{k})) = \max\left(\frac{\max_{\mathbf{k}\in E}|Re(f(\mathbf{k})) - Re(\tilde{f}(\mathbf{k}))|}{\max_{i=1,\ldots,r}|Re(f(\mathbf{k}))|}, \frac{\max_{i=1,\ldots,r}|Im(f(\mathbf{k})) - Im(\tilde{f}(\mathbf{k}))|}{\max_{i=1,\ldots,r}|Im(f(\mathbf{k}))|}\right),$$

where E is a subset of $\mathbb{N}^n$ and $Re(\tilde{f}(\mathbf{k}))$ and $Im(\tilde{f}(\mathbf{k}))$ are respectively the real and imaginary part of the computed evaluations $\tilde{f}(\mathbf{k})$.

5.2.1 Wave identification

We consider the following function

$$f(x,t) = e^{st+ip(x-ct)} \sum_{j=-100}^{100} \frac{1}{a^{|l|}} e^{ij(x-ct)},$$

with the following parameters $a=83$, $c=1$, $s=-0.7460264$, $p=0.81158387$ and $q=0.62944737$.

We apply our algorithm for exact sampled data. For this example we increase the number of terms r.

r	err($\boldsymbol{\xi}$)	err(a)	err($f(\mathbf{k})$)
5	5.1 e-2	1.1 e-4	1.4 e-4
7	1.4 e-2	2.3 e-8	4.3 e-8
9	2.0 e-1	1.3 e-4	4.1 e-4
11	4.1 e-2	5.9 e-4	3.2 e-3

Table 1. Results of Subsection 4

5.2.2 Random exponential sum with complex parameters

We consider the following bivariate exponential sum

$$f(x,y) = \sum_{i=1}^{r} a_i e^{b_i x + c_i y} \text{ with } a_i, b_i, c_i \in \mathbb{C}$$

with following parameters randomly generated

$$a = \begin{pmatrix} 0.6297120613009013 + 0.9555907644952910\,i \\ 0.9051083390162390 + 0.9483566987256117\,i \\ 0.1344470799676359 + 0.5459438888208842\,i \\ 0.8976760983341069 + 0.2829282544897074\,i \\ 0.8084292126653153 + 0.1055895968994213\,i \end{pmatrix},$$

$$b = \begin{pmatrix} 0.7942748595110241 + 0.9503025778650449\,i \\ 0.4856681357483844 + 0.7863631829683633\,i \\ 0.9480236092780866 + 0.9074208146852858\,i \\ 0.9611809261254034 + 0.4233260569737495\,i \\ 0.1644608200439973 + 0.1490486118546710\,i \end{pmatrix},$$

$$c = \begin{pmatrix} 0.6751604517606180 + 0.1777629540553305\,i \\ 0.9253133828023995 + 0.6523683323740055\,i \\ 0.8421467197514015 + 0.3943824791434848\,i \\ 0.04499744500270576 + 0.7382698187624178\,i \\ 0.6526258851734551 + 0.7525853279667668\,i \end{pmatrix}.$$

We apply our algorithm for exact sampled data and for noisy sampled data. The latter are obtained by adding a term of noise $f(\mathbf{k}) + 10^{-\delta} e_{\mathbf{k}}$ where $e_{\mathbf{k}}$ is uniformly distributed in $[-1,1]$. Exact data are denoted by $\delta = \infty$. For this example we do not use a transformation basis as the relative error of the evaluations is of order 10^{-11}.

δ	err($\boldsymbol{\xi}$)	err(a)	err($f(\mathbf{k})$)
∞	1.7 e-10	8.4 e-10	4.8 e-11
12	3.4 e-9	2.5 e-8	1.2 e-10
10	2.0 e-7	1.9 e-6	1.8 e-9
8	1.6 e-5	9.1 e-5	1.2 e-7
6	1.7 e-3	1.0 e-2	1.8 e-5

Table 2. Results of bivariate case in Subsection 5.2.2

We consider the following trivariate exponential sum

$$f(x,y,z)=\sum_{i=1}^{r} a_i\, e^{b_i x+c_i y+d_i z} \text{ with } a_i, b_i, c_i, d_i\in\mathbb{C}$$

with following parameters randomly generated

$$a=\begin{pmatrix} 0.1047954252652496+0.4547217545523811\,i\\ 0.1568681336284075+0.5182230436603275\,i\\ 0.5212168456149980+0.6460116628581090\,i\\ 0.7179908005195055+0.6564131478501410\,i\\ 0.8111961502377006+0.7943239638453534\,i\\ 0.8117855827824322+0.9635150636685853\,i \end{pmatrix},$$

$$b=\begin{pmatrix} 0.3931190462772609+0.4184634785721712\,i\\ 0.1799208408566254+0.5263383138981300\,i\\ 0.1405075874550185+0.4012726152013042\,i\\ 0.09180041856173574+0.06926175558649576\,i\\ 0.8188075194940048+0.7972971037703914\,i\\ 0.4337436737077925+0.8247521479822881\,i \end{pmatrix},$$

$$c=\begin{pmatrix} 0.9743824498915707+0.3427035451229425\,i\\ 0.02517738312329857+0.4898938852557034\,i\\ 0.4330181468437458+0.2041560344921151\,i\\ 0.2961443983624806+0.3749615452544262\,i\\ 0.6254138920063022+0.1140920180300905\,i\\ 0.6537226931542330+0.1738250417163629\,i \end{pmatrix},$$

$$d=\begin{pmatrix} 0.4243789027086317+0.6970767807829078\,i\\ 0.2737370378705848+0.3054258497378242\\ 0.7331009336066566+0.9733914171404590\,i\\ 0.06162345772717674+0.4193892222303290\,i\\ 0.9119253990398325+0.9338822445913956\,i\\ 0.9425978554821725+0.5469134831905478\,i \end{pmatrix}.$$

We apply our algorithm for exact sampled data and for noisy sampled data. The latter are obtained by adding a term of noise $f(\mathbf{k})+10^{-\delta}e_{\mathbf{k}}$ where $e_{\mathbf{k}}$ is uniformly distributed in $[-1,1]$. Exact data are denoted by $\delta=\infty$. For this example we do not use a transformation basis as the relative error of the evaluations is of order 10^{-8}.

δ	err($\boldsymbol{\xi}$)	err(a)	err($f(\mathbf{k})$)
∞	1.1 e-8	5.4 e-7	2.3 e-8
12	4.9 e-7	1.4 e-6	1.3 e-8
10	1.1 e-5	2.4 e-5	4.2 e-9
8	9.4 e-3	1.4 e-2	4.8 e-7
6	1.0 e-1	4.7 e-1	3.0 e-5

Table 3. Results of the trivariate case in Subsection 5.2.2

5.2.3 Recovering signal parameters.

We consider the following bivariate exponential sum taken from [22]

$$f(x,y)=\sum_{j=1}^{r} a_j\, e^{i(b_j x+c_j y)}$$

with following parameters generated

$$a=\begin{pmatrix}1+i\\ 2+3\,i\\ 5-6\,i\\ 0.2-i\\ 1+i\\ 2+3\,i\\ 5-6\,i\\ 0.2-i\end{pmatrix},\quad b=\begin{pmatrix}0.1\\ 0.19\\ 0.3\\ 0.35\\ -0.1\\ -0.19\\ -0.3\\ -0.3\end{pmatrix},\quad c=\begin{pmatrix}1.2\\ 1.3\\ 1.5\\ 0.3\\ 1.2\\ 0.35\\ -1.5\\ 0.3\end{pmatrix}.$$

We apply our algorithm for exact sampled data and for noisy sampled data. The latter are obtained by adding a term of noise $f(\mathbf{k})+10^{-\delta}e_{\mathbf{k}}$ where $e_{\mathbf{k}}$ is uniformly distributed in $[-1,1]$. Exact data are denoted by $\delta=\infty$. For this example we need transformation basis as coefficients in c are equal.

δ	err($\boldsymbol{\xi}$)	err(a)	err($f(\mathbf{k})$)
∞	7.9 e-6	1.3 e-4	1.6 e-5
12	1.1 e-5	1.8 e-4	6.3 e-5
10	3.7 e-4	6.5 e-4	1.4 e-5
8	5.9 e-2	8.4 e-2	7.8 e-5
6	4.0 e-1	1.7 e-1	5.0 e-5

Table 4. Results of Subsection 5.2.3

5.3 Sparse interpolation

The problem of sparse polynomial interpolation consists in recovering the monomials in the support of the polynomial and their non-zero coefficients, from the evaluation of the polynomial at some points. From an algorithmic point of view, methods which are sensitive to the number of terms in this decomposition are considered in this problem. We consider a sparse polynomial

$$p(x_1,\dots,x_n)=\sum_{a\in A} w_a\,\boldsymbol{x}^a \tag{22}$$

with the support $A\subset\mathbb{Z}^n$ of size r and $\omega_a\in\mathbb{C}\setminus\{0\}$. We want to recover the number r of terms, the support A and the coefficients ω_a.

Reformulation. Several algorithms have been proposed to solve this sparse polynomial reconstruction problem. In [2] or [27], the evaluations $h(k)=p(\omega_1^k,\dots,\omega_n^k)$ for conveniently chosen $\omega_1,\dots,\omega_n\in\mathbb{C}$ and $k=0,\dots,2r-1$ are used to recover the exponents $a\in A$ and the coefficients ω_a. By solving a Hankel system or applying Prony's method, the roots $\omega_1^{a_1}\cdots\omega_n^{a_n}$ and the coefficients ω_a are recovered. In the case of exact arithmetic as in [2] or [27], by choosing co-prime numbers $\omega_i\in\mathbb{Z}$, one can recover the exponents $a_1,\dots,a_n$ from the value $\omega_1^{a_1}\cdots\omega_n^{a_n}$. In the extension [12] of this method to approximate arithmetic, the values ω_j are roots of unity of the form $e^{\frac{2\pi \mathrm{i}}{m_j}}$ where the orders $m_j\in\mathbb{N}_+$ are coprime so that the exponents $a_1,\dots,a_n$ can also be recovered from $\omega_1^{a_1}\cdots\omega_n^{a_n}$.

We generalize this approach further, by transforming the sparse polynomial p into an exponential polynomial and by applying the method described in Section 4: We consider the exponential function

$$h(x_1,\dots,x_n)=p\big(e_1^{f_1x_1+g_1},\dots,e^{f_nx_n+g_n}\big)=\sum_{a\in A} w_a\,e^{g_1a_1+\cdots+g_na_n}\,e^{f_1a_1x_1+\cdots+f_na_nx_n},$$

for conveniently chosen $f_1, ..., f_n \in \mathbb{C} \setminus \{0\}$, $g_1, ..., g_n \in \mathbb{C}$. In particular, the values of $g_1, ..., g_n \in \mathbb{C}$ can be chosen (at random) so that $h(0, ..., 0) \neq 0$.

To recover the decomposition, we apply the method described in Section 4 to the first coefficients $\sigma_\alpha = h(\alpha_1, ..., \alpha_n) = p\big(e_1^{f_1\alpha_1 + g_1}, ..., e^{f_n\alpha_n + g_n}\big)$ with $|\alpha| \leqslant d$ of the series

$$\sigma_h(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} = \sum_{a \in A} w_a\, e^{g_1 a_1 + \cdots + g_n a_n}\, \boldsymbol{e}_{(f_1 a_1, \ldots, f_n a_n)}(\boldsymbol{z}).$$

This yields the points $(f_1 a_1, ..., f_n a_n)$ and the weights $w_a\, e^{g_1 a_1 + \cdots + g_n a_n}$ for $a \in A$, from which we deduce the decomposition (22).

Example. The new method that we describe can be applied to this case. We illustrate it on the following example

$$\begin{aligned} p(x, y, z) = &\ (0.79 + 0.08\, i)\, x\, y^2\, z + (0.51 + 0.93\, i)\, x^5\, y^{10}\, z^7 + (-0.25 - 0.09\, i) x^{30}\, y^{25}\, z^{12} \\ &+ (0.26 + 0.99\, i)\, x^{100}\, y^{40}\, z^3 + (-0.7 + 0.31\, i) x^{80}\, y^{60}\, z^{120}. \end{aligned}$$

We recover the $\zeta_{i,j}$ with a relative error of order 10^{-13}, the a_i with a relative error of 10^{-12} and the evaluations $h(\mathbf{k})$ with a relative error of order 10^{-12}.

6 Tensor decomposition

In this section, we consider more specifically the problem of tensor decomposition and analyze the decomposition method in this context.

6.1 Symmetric tensors

We consider an homogeneous polynomial $T(x_0, ..., x_n) \in S = \mathbb{C}[x_0, ..., x_n] = \mathbb{C}[\bar{\boldsymbol{x}}]$ of degree $d \in \mathbb{N}$. Hereafter, the $n+1$ variables of homogeneous polynomials will be denoted $\bar{\boldsymbol{x}} = (x_0, ..., x_n)$ and the ring of polynomials in these variables $S = \mathbb{C}[x_0, ..., x_n] = \mathbb{C}[\bar{\boldsymbol{x}}]$. The vector space spanned by the polynomials of degree d is denoted S_d. For a homogeneous ideal $I \subset S$, we denote by $I_d = I \cap S_d$ its degree d component. The dual space of S is $S^* = \mathbb{C}[[z_0, ..., z_n]] = \mathbb{C}[[\bar{\boldsymbol{z}}]]$.

The (symmetric) tensor decomposition problem consists, given $T \in S_d$, in finding the least number r of linear forms $l_1, ..., l_r$ and non-zero weights $\omega_1, ..., \omega_r \in \mathbb{C}$ such that

$$T(x_0, ..., x_n) = \sum_{i=1}^{r} \omega_i\, l_i(x_0, ..., x_n)^d. \tag{23}$$

Such a decomposition of T is sometimes called a *Waring decomposition* of T, after the work of the mathematician Edward Waring who studied a similar problem of decomposition of integers as a sum of powers of prime numbers.

Definition 48. *The minimal number r of terms in such a decomposition of $T \in S_d$ is called the* rank *of T.*

By setting $x_0 = 1$, this problem is equivalent to the decomposition of the polynomial $T(1, x_1, ..., x_n)$ of degree $\leqslant d$, as a sum of d-th power of polynomials of degree $\leqslant 1$ in the variables $\boldsymbol{x} = (x_1, ..., x_n)$.

We say that a Waring decomposition of $T \in S_d$ is an affine decomposition if for all the linear forms $l_i(x_0, x_1, ..., x_n) = \xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n$, we have $\xi_{i,0} = 1$ for $i = 1, ..., r$.

By a generic change of variables and by scaling the linear forms $l_i(\bar{\boldsymbol{x}})$, an affine decomposition can always be constructed over $\mathbb{C}$.

6.1.1 Sparse decomposition of tensors

The tensor decomposition problem can be transformed into an exponential polynomial decomposition problem, using the following inner-product:

Definition 49. *(Apolar product) For two polynomials* $f=\sum_{|\alpha|=d} f_\alpha\, \bar{\boldsymbol{x}}^\alpha$, $g=\sum_{|\alpha|=d} g_\alpha \bar{\boldsymbol{x}}^\alpha \in S_d$, *we define*

$$\langle f, g\rangle_d = \sum_{|\alpha|=d} f_\alpha\, g_\alpha \binom{d}{\alpha}^{-1},$$

where $\binom{d}{\alpha}=\frac{d!}{\alpha_0!\cdots\alpha_n!}$.

For any polynomial $f=\sum_{|\alpha|\leqslant d} f_\alpha\, \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]_{\leqslant d}$ of degree $\leqslant d$ in the variables $\boldsymbol{x}=(x_1,\ldots,x_n)$, let $f^h(x_0,\ldots,x_n)=\sum_{|\alpha|\leqslant d} f_\alpha\, x_0{}^{d-|\alpha|} x_1^{\alpha_1}\ldots x_n^{\alpha_n}\in S_d$ be its homogenization in x_0.

The apolar product defined on homogenous polynomials can be specialized for polynomials of degree $\leqslant d$: $\forall f=\sum_{|\alpha|\leqslant d} f_\alpha\, \boldsymbol{x}^\alpha$, $g=\sum_{|\alpha|\leqslant d} g_\alpha\, \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]_{\leqslant d}$,

$$\langle f, g\rangle_d = \langle f^h, g^h\rangle_d = \sum_{|\alpha|\leqslant d} f_\alpha\, g_\alpha \binom{d}{\bar{\alpha}}^{-1},$$

where $\bar{\alpha}=(\alpha_0,\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^{n+1}$ for $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$ with $|\alpha|\leqslant d$.

We have the following invariance property:

Lemma 50. *Let* $T, T'\in S_d$ *and* $G\in \mathrm{Sl}_{n+1}$ *a change of coordinates of determinant* 1. *Then*

$$\langle T, T'\rangle_d = \langle G{\cdot}T, G\cdot T'\rangle_d$$

where $G{\cdot}T$ *is the tensor* T *after the change of coordinates by* G.

Another interesting property of this apolar product is the following:

Lemma 51. *For any linear form* $l(\bar{\boldsymbol{x}})=l_0x_0+l_1x_1+\cdots+l_n x_n$ *and any homogeneous polynomial* $f(\boldsymbol{x})\in\mathbb{C}[\bar{\boldsymbol{x}}]_d$, *we have* $\langle f, l(\bar{\boldsymbol{x}})^d\rangle_d = f(l_0,l_1,\ldots,l_n)$.

Proof. We check the property for the monomials $\boldsymbol{x}^\alpha$ with $|\alpha|=d$ and deduce the lemma by linearity. □

Using the apolar product, we can associate to any $T\in S_d$ the element of $\mathbb{C}[\boldsymbol{x}]^*_{\leqslant d}$

$$T^*\colon f\in\mathbb{C}[\boldsymbol{x}]_{\leqslant d}\mapsto \langle T, f^h\rangle_d$$

Its associated element in the dual space $\mathbb{C}[[\boldsymbol{z}]]$ is denoted $T^*(\boldsymbol{z})$. It is in fact a polynomial in $\boldsymbol{z}$ of degree $\leqslant d$.

We can also associate to $T\in S_d$ the element of S_d^*

$$\bar{T}^*\colon f\in S_d\mapsto\langle T, f\rangle_d.$$

For $0\leqslant i\leqslant d$, let $H_T^{i,d-i}\colon S_i\to S_{d-i}^*$ be the truncated Hankel operator associated to $\bar{T}^*$: $\forall f\in S_i$

$$H_T^{i,d-i}(f)\colon g\in S_{d-i}\mapsto\langle T, fg\rangle_d\in\mathbb{C}.$$

Example 52. For $T:=\binom{d}{\alpha} x_0^{\alpha_0}\cdots x_n^{\alpha_n}$ with $\alpha_0+\cdots+\alpha_n=d$, we check that $T^*=\frac{1}{\alpha_1!\cdots\alpha_n!}z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, $\bar{T}^*=\frac{1}{\alpha_0!\cdots\alpha_n!}z_0^{\alpha_0}\cdots z_n^{\alpha_n}$.

Lemma 53. $T(\bar{\boldsymbol{x}})=\sum_{|\alpha|\leqslant d} T_{\bar{\alpha}}\binom{d}{\bar{\alpha}}\bar{\boldsymbol{x}}^{\bar{\alpha}}$ *iff* $T^*(\boldsymbol{z})=\sum_{|\alpha|\leqslant d} T_{\bar{\alpha}}\frac{\boldsymbol{z}^\alpha}{\alpha!}\in\mathbb{C}[\boldsymbol{z}]$.

Proof. If $T=\sum_{|\alpha|=d} T_\alpha\binom{d}{\alpha}\bar{\boldsymbol{x}}^\alpha$, then for $\alpha\in\mathbb{N}^n$ with $|\alpha|\leqslant d$, we have

$$T^*(\boldsymbol{x}^\alpha)=\left\langle T, x_0^{d-|\alpha|}\boldsymbol{x}^\alpha\right\rangle_d=\langle T,\bar{\boldsymbol{x}}^{\bar{\alpha}}\rangle_d=T_{\bar{\alpha}},$$

which proves that $T^*(\boldsymbol{z})=\sum_{|\alpha|\leqslant d} T^*(\boldsymbol{x}^\alpha)\frac{\boldsymbol{z}^\alpha}{\alpha!}=\sum_{|\alpha|\leqslant d} T_{\bar{\alpha}}\frac{\boldsymbol{z}^\alpha}{\alpha!}\in\mathbb{C}[\boldsymbol{z}]$. □

This construction allows us to reformulate the problem of decomposition of T as a truncated series problem:

Proposition 54. *A tensor* $T(\bar{\boldsymbol{x}}) = \sum_{|\alpha| \leqslant d} T_{\bar{\alpha}} \binom{d}{\bar{\alpha}} \bar{\boldsymbol{x}}^{\bar{\alpha}}$ *has an affine Waring decomposition*

$$T(\bar{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d \tag{24}$$

with $\omega_1, \ldots, \omega_r \in \mathbb{C} \setminus \{0\}$, $\xi_1, \ldots, \xi_r \in \mathbb{C}^n$ *iff* $T^*(\boldsymbol{z}) = \sum_{|\alpha| \leqslant d} T_{\bar{\alpha}} \frac{\boldsymbol{z}^\alpha}{\alpha!}$ *coincides with the series* $\sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ *up to degree* d.

Proof. By Lemma 53, if $T(\bar{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d = \sum_{i=1}^{r} \omega_i \sum_{|\alpha| \leqslant d} \binom{d}{\bar{\alpha}} \xi_i^\alpha \bar{\boldsymbol{x}}^{\bar{\alpha}}$, then

$$T^*(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \sum_{|\alpha| \leqslant d} \xi_i^\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!}$$

which coincides with the series $\sigma(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} = \sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$ up to degree d. The reverse implication is also true, by Lemma 53. □

In other words, if $T \in S_d$ has an affine decomposition of the form (23), then T^* coincides on $\mathbb{C}[\boldsymbol{x}]_{\leqslant d}$ with the element of the dual space $\mathbb{C}[[\boldsymbol{z}]]$ of $\mathbb{C}[\boldsymbol{x}]$:

$$\sigma(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\xi_i}(\boldsymbol{z}).$$

We will write it $T^*(\boldsymbol{z}) = \sigma(\boldsymbol{z}) + ((\boldsymbol{z}))^{d+1}$.

The method described in the previous section to recover a sparse decomposition of a truncated series can therefore be applied here. If the number of terms r is small enough compared to the size of the sequence moments, this yields directly a decomposition of the tensor T.

6.1.2 Examples

Example 55. Let $T(x_{0,}, x_1) = x_0^4 + 12 x_0^3 x_1 + 6 x_0^2 x_1^2 + 12 x_0 x_1^3 + x_1^4$. Then we have

$$T^*(\boldsymbol{z}) = 1 + 3 z_1 + \frac{z_1^2}{2} + 3 \frac{z_1^3}{3!} + \frac{z_1^4}{4!}.$$

We apply Prony's method and compute the kernel of

$$H = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 3 \end{pmatrix}.$$

It contains $(-1, 0, 1)$ which corresponds to the polynomial $x^2 - 1$. The roots of this polynomial are ± 1. Thus the decomposition is of the form $T(x_0, x_1) = a (x_0 + x_1)^4 + b (x_0 - x_1)^4$. By expansion and identification of the coefficients, we find that $a + b = 1, 4a - 4b = 12$. This yields $a = 2, b = -1$ and the decomposition is

$$T(x_{0,}, x_1) = 2 (x_0 + x_1)^4 - (x_0 - x_1)^4.$$

Example 56. Let

$T(x_0, x_1, x_2) = 3 x_0^4 + 16 x_0^3 x_1 + 24 x_0^2 x_1^2 - 8 x_0 x_1^3 - 32 x_1^4 + 24 x_0^3 x_2 + 120 x_0^2 x_1 x_2 + 192 x_0 x_1^2 x_2 + 88 x_1^3 x_2 + 72 x_0^2 x_2^2 + 264 x_0 x_1 x_2^2 + 240 x_1^2 x_2^2 + 96 x_0 x_2^3 + 184 x_1 x_2^3 + 48 x_2^4.$

The associated dual element is

$T^*(\boldsymbol{z}) = 3 + 4 z_1 + 6 z_2 + 4 \frac{z_1^2}{2!} + 10 z_1 z_2 + 12 \frac{z_2^2}{2!} - 2 \frac{z_1^3}{3!} + 16 \frac{z_1^2 z_2}{2!} + 22 \frac{z_1 z_2^2}{2!} + 24 \frac{z_2^3}{3!} - 32 \frac{z_1^4}{4!} + 22 \frac{z_1^3 z_2}{3!} + 40 \frac{z_1^2 z_2^2}{2!2!} + 46 \frac{z_1 z_2^3}{3!} + 48 \frac{z_2^4}{4!}.$

It coincides with the first terms of the generating series of example 47. The application of the generalized Prony method yields the points $(1,1),(2,2),(3,1)$ and the weights 2, 3, -1. Therefore, the decomposition of T is

$$T(x_0,x_1,x_2)=2\,(x_0+x_1+x_2)^4+3\,(x_0+2x_1+2x_2)^4-\big(x_0+3x_1+x_2\big)^4.$$

6.1.3 Apolar ideals

We have seen that if $T\in S_d$ has a rank r, then after a generic of coordinates, it has an affine Waring decomposition of the form (24). Using the apolarity property, Proposition 54, the polynomials p of degree $\leqslant d$ which vanish at the points $\xi_1, \ldots, \xi_r \in \mathbb{C}^n$ satisfies the equation $\langle T^*|\, p\rangle = 0$. Their homogenization p^h in degree d satisfy $\langle T, p^h\rangle_d=\langle T^*|\, p\rangle=0$.

This naturally leads us to the study of ideals $I\subset S$ which are the homogenization of the vanishing ideal of a set of points and such that $\langle T, I_d\rangle_d=0$. First we introduce the notion of apolar ideal associated to a tensor:

Definition 57. *Let $T\in S^d$. We define the apolar ideal of T as the homogeneous ideal of S generated by S^{d+1} and by the polynomials $g\in S^i$ $(0\le i\le d)$ such that $\langle gh, T\rangle_d=0$ for all $h\in S^{d-i}$. It is denoted $(T^{\perp})$ and called the ideal apolar to T.*

Remark 58. By definition, $(T^{\perp})_i=\ker H_T^{i,d-i}$ for $0\leqslant i\leqslant d$.

Example 59. For $T:=x_0^{\alpha_0}\cdots x_n^{\alpha_n}$ with $\alpha_0+\cdots+\alpha_n=d$, we check that $(T^{\perp})=(x_0^{\alpha_0+1},\ldots,x_n^{\alpha_n+1})$.

This apolar ideal provides a simple characterization of the ideals which degree-d component is apolar to T:

Lemma 60. *For any ideal $I\subset S$, $\langle I_d, T\rangle_d=0$ if and only if $I\subset(T^{\perp})$.*

Proof. Clearly, if $I\subset(f^{\perp})$ then $I_d\subset(f^{\perp})_d$ so that $\langle T, I_d\rangle_d=0$.

Let us prove the reverse inclusion. By definition of the apolar ideal $J:=(f^{\perp})$, we have J_i: $S_k=J_{i-k}, \forall 0\le i\le d, 0\le k\le i$. We also have I_d: $S_k\supset I_{d-k}, \forall 0\le k\le d$. The hypothesis $\langle T, I_d\rangle_d=0$ implies that $I_d\subset J_d$. We deduce that $I_i\subset J_i, \forall 0\le i\le d$. Since $J_{d+1}=S_{d+1}$, we have the inclusion $I\subset J=(f^{\perp})$. $\square$

For any homogeneous ideal $I\subset S$, let $h_{S/I}$: $n\in\mathbb{N}\mapsto\dim S_n/I_n$ be the Hilbert function of S/I. We have the following simple relation.

Lemma 61. *If $I\subset(T^{\perp})$, then for $n\in\mathbb{N}$, $h_{S/I}(n)\geqslant h_{S/(T^{\perp})}(n)$.*

The tensor decomposition problem can then be reformulated in terms of apolarity as follows via the well known Apolarity Lemma (cf. [15, Lemma 1.15]).

Proposition 62. *Let $T\in S^d$ be a symmetric tensor. The following are equivalent:*

- *T has a decomposition of size $\leqslant r$,*
- *there exits an ideal $I\subset S$ such that $I\subset(T^{\perp})$ with I is saturated, defining r simple points.*

Proof. Suppose that T has a decomposition of size r: $T(\bar{\boldsymbol{x}})=\sum_{i=1}^r \omega_i\ (\xi_{i,0}x_0+\xi_{i,1}x_1+\cdots+\xi_{i,n}x_n)^d$. Then consider the homogeneous ideal I of polynomials vanishing at the points $\bar{\xi_i}=[\xi_{i,0}:\cdots:\xi_{\underline{i},n}]\in\mathbb{P}^n$, $i=1,\ldots,s$. By apolarity, for all $g\in I_d$,

$$\langle T, g\rangle_d=\sum_1^s \omega_i\, g(\bar{\xi_i})=0.$$

so that I is a saturated ideal, defining r simple points and with $I\subset(T^{\perp})$.

Conversely, suppose that I is an ideal of S satisfying (a), (b), (c). By a change of variables, we can assume that the simple points of $\mathbb{P}^n$ defined by I are of the form $\bar{\xi_i}=[1:\xi_{i,1}:\cdots:\xi_{i,n}]\in\mathbb{P}^n$.

The ideal I' of R obtained by substituting $x_0 = 1$ in I is the vanishing ideal of the points $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^n$. As $\langle T, I_d \rangle_d = 0$, we have $\langle T^* | I'_{\leqslant d} \rangle = 0$ and by Lemma 10, T^* is the truncation in degree $\leqslant d$ of an element of $(I')^{\perp}$.

As $(I')^{\perp}$ is the vector space spanned by $\boldsymbol{e}_{\xi_1}, \ldots, \boldsymbol{e}_{\xi_r}$, there exists $\omega_1, \ldots, \omega_r \in \mathbb{C}$ such that

$$T^*(\boldsymbol{z}) = \sum_{i=1}^{r} \omega_i \, \boldsymbol{e}_{\xi_i} + ((\boldsymbol{z}))^{d+1}.$$

By Proposition 54, this implies that

$$T(\bar{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i \; (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$$

and T has a decomposition of size $\leqslant r$. □

This proposition provides a characterization for the decomposition of a tensor, which is independent of any choice of coordinates.

6.2 Generalized decomposition

Definition 63. *A generalized affine decomposition of size r of $T \in S^d$ is a decomposition of the form*

$$T^*(\boldsymbol{z}) = \sum_{i=1}^{m} \omega_i(\boldsymbol{z}) \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) + ((\boldsymbol{z}))^{d+1} \tag{25}$$

where $\xi_i \in \mathbb{C}^n$ and $\omega_i(\boldsymbol{z})$ are polynomials in $\mathbb{C}[\boldsymbol{z}]$, such that the sume of the dimension r_i of the vector spaces spanned by $\partial_z^{\alpha}(\omega_i(\boldsymbol{z}) \boldsymbol{e}_{\xi_i}(\boldsymbol{z}))$ for $\alpha \in \mathbb{N}^n$ is r.

This decomposition generalizes the affine Waring decomposition of Definition ?, since when $\omega_i(\boldsymbol{z}) = \boldsymbol{\omega}_i \in \mathbb{C}$ are constant polynomials, we have the decomposition

$$T^*(\boldsymbol{z}) = \sum_{i=1}^{m} \omega_i \, \boldsymbol{e}_{\xi_i} + ((\boldsymbol{z}))^{d+1} \text{ iff } T(\bar{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i \, (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d.$$

Definition 64. *The minimal r such that T has a generalized decomposition of size r is called the* generalized rank *of T. It is denoted $r_g(T)$.*

Notice that the size of a generalized affine decomposition does not depend on the coordinate system and is invariant by a generic affine transformation.

This allows us to check that the generalized rank is sub-additive:

Lemma 65. *For any $T, T' \in S_d$, $r_g(T + T') \leqslant r_g(T) + r_g(T')$.*

Proof. If after a generic change of variables, each dual element T^* and T'^* has a decomposition of the form (25) of size respectively $r_g(T)$ and $r_g(T')$, then there is a (generic) change of variables for which, both have such a decomposition. The sum of these decompositions yields a generalized decomposition of size $\leqslant r_g(T) + r_g(T')$ for the tensor $T + T'$, which proves that $r_g(T + T') \leqslant r_g(T) + r_g(T')$. □

Example 66. Let $T(x_0, x_1) = x_0 x_1^{d-1} \in \mathbb{C}[x_0, x_1]$. After the change of variables $x_{0'} = x_1$, $x_1' = x_0$, we obtain the tensor $\tilde{T}(x_0, x_1) = x_0^{d-1} x_1$ such that

$$\tilde{T}^*(\boldsymbol{z}) = z_1 = z_1 \, \boldsymbol{e}_0(\boldsymbol{z}) + ((\boldsymbol{z}))^{d+1}.$$

Thus the generalized rank of $x_0 x_1^{d-1}$ is 2 (since it cannot be 1). Notice that

$$T = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon d} ((x_1 + \varepsilon x_0)^d - x_1^d)$$

that is a limit of tensors $T_\varepsilon = \frac{1}{\varepsilon d}((x_1 + \varepsilon x_0)^d - x_1^d)$ of rank 2.

Example 67. The polynomial $T = x_0^3 x_1 + x_0^3 x_2$ is a sum of two tensors $T_1 = x_0^3 x_1$ and $T_2 = x_0^3 x_2$. From the Example 66, we have $r_g(T_1) = r_g(T_2) = 2$. The sub-additivity of the rank implies that $r_g(T) \leqslant 4$. Moreover, we are in a case of a polynomial of border rank 4 and rank 7 (as described in [4, Theorem 44]). In this case $r_g(f) = 4 = r_\sigma(f) < r(f) = 7$.

Example 68. For a monomial $T = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$, we have

$$T^* = \frac{1}{d!} z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

The inverse system spanned by $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is of dimension $(\alpha_1 + 1) \times \cdots \times (\alpha_n + 1)$. Assuming that $\alpha_0 = \max_i \alpha_i$, the previous decomposition is a generalized decomposition of minimal size (according to Corollary ? and Example ?). Therefore we have

$$r_g(x_0^{\alpha_0} \cdots x_n^{\alpha_n}) = \frac{\prod_{i=0}^n (\alpha_i + 1)}{\max_i (\alpha_i + 1)}.$$

In order to relate the generalized rank to a usual rank, we introduce the following definition:

Definition 69. *A tensor $T \in S_d$ has a flat extension of rank r if there exists $u \in S_1$ and $\tilde{T} \in S_{m+m'}$ with $m = \max\left\{r, \lceil \frac{d}{2} \rceil\right\}, m' = \max\left\{r - 1, \lfloor \frac{d}{2} \rfloor\right\}$ such that*

- rank $H_{\tilde{T}}^{m,m'} = r$,
- $u^{m+m'-d} \star \tilde{T} = T$.

This definition says that by a change of variables such that $u = x_0$, T^* is the truncation in degree $\leqslant d$ of $\tilde{T}^* \in (\mathbb{K}[\boldsymbol{x}] \leqslant m + m')^*$ and rank $H_{\tilde{T}}^{m,m'} = r$.

Lemma 70. *Let $d \geq r$ and $E \subset S_d$ such that S_d / E is of dimension r. Then for a generic change of coordinates $g \in PGL(n+1)$, $S_d / g \cdot E$ has a monomial basis of the form $x_0 B$ with $B \subset S_{d-1}$. Moreover, $\underline{B}$ is connected to 1.*

Proof. Let $\succ$ be the lexicographic ordering such that $x_0 \succ \cdots \succ x_n$. By [10][Theorem 15.20, p. 351], after a generic change of coordinates, the initial J of the ideal $I = (E)$ for $\succ$ is Borel fixed. That is, if $x_i \boldsymbol{x}^\alpha \in J$ then $x_j \boldsymbol{x}^\alpha \in J$ for $j > i$.

To prove that there exists a subset B of monomials of degree $d - 1$ such that $x_0 B$ is a basis of S_d / I_d, we show that $J_d + x_0 S_{d-1} = S_d$. Let $J_d' = (J_d + x_0 S_{d-1}) / x_0 S_{d-1}$, $S_d' = S_d / x_0 S_{d-1} = \mathbb{K}[x_1, \ldots, , x_n]_d$ and $L = (J : x_0)$. Then we have the exact sequence

$$0 \rightarrow S_{d-1} / L_{d-1} \xrightarrow{\mu_{x_0}} S_d / J_d \rightarrow S_d' / J_d',$$

where μ_{x_0} is the multiplication by x_0. Let us denote by $s_k = \dim S_k$ and $q(k) = s_k - r$ for $k \in \mathbb{N}$. Suppose that $\dim S_d' / J_d' > 0$, then $\dim L_{d-1} > s_{d-1} - r = q(d-1)$. As $d \geq r$ and r is the Gotzmann regularity of q, by [14, (2.10), p. 66] we have $\dim S_1 L_{d-1} > q(d)$. As J is Borel fixed, i.e. $x_0 p \in J$ implies $x_i p \in J$ for $i \geq 0$, we have $S_1 L_{d-1} \subset J$, so that $\dim J_d \geq \dim S_1 L_{d-1} > q(d) = s_d - r$. This implies that $\dim S_d / J_d = \dim S_d / I_d = \dim S_d / E < r$, which contradicts the hypothesis on E. Thus $J_d + x_0 S_{d-1} = S_d$.

Let B' be the complement of J_d in the set of monomials of degree d. The sum $S_d = J_d + x_0 S_{d-1}$ shows that $B' = x_0 B$ for some subset B of monomials of degree $d - 1$.

As J_d is Borel fixed and different from S_d, its complement B' contains x_0^d. Similarly we check that if $x_0^{\alpha_0} \cdots x_n^{\alpha_n} \in B'$ with $\alpha_1 = \cdots = \alpha_{k-1} = 0$ and $\alpha_k \neq 0$ then $x_0^{\alpha_0+1} x_k^{\alpha_k-1} x_{k+1}^{\alpha_{k+1}} \cdots x_n^{\alpha_n} \in B'$. This shows that $\underline{B'} = \underline{B}$ is connected to 1. ☐

A generalized decomposition can be characterized algebraically in a way similar to the classical decomposition and by this flat extension property:

Proposition 71. *Let $T \in S^d$ be a symmetric tensor. The following points are equivalent:*

1. *T has a generalized decomposition of size $\leqslant r$,*
2. *There exits an ideal $I \subset S$ such that $I \subset (T^{\perp})$ and I is saturated, zero dimensional, of degree r,*
3. *T has a flat extension of rank $\leqslant r$.*

Proof. $2 \Rightarrow 1$. Supposed that I is an ideal of S such that $I \subset (T^{\perp})$ with I is saturated, zero dimensional, of degree r. By a change of variables, we can assume that the points of $\mathbb{P}^n$ defined by I are of the form $\bar{\xi}_i = [1{:}\, \xi_{i,1}{:}\, \cdots{:}\, \xi_{i,n}] \in \mathbb{P}^n$.

By dehomogenization (setting $x_0 = 1$), we obtained an ideal I' such that $\mathcal{A} = R/I'$ is of dimension r. It is defining the points $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^n$ with multiplicity r_i.

As $\langle T, I_d \rangle_d = 0$, we have $\langle T^* | I'_{\leqslant d} \rangle = 0$ and by Lemma 10, T^* is the truncation in degree $\leqslant d$ of an element of $(I')^{\perp}$. By Theorem 14, T^* is the truncation in degree $\leqslant d$ of an element of $(I')^{\perp} = \mathcal{A}^*$ of the form

$$\sigma(\boldsymbol{z}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{z}) \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$$

for some points $\xi_i \in \mathbb{C}^n$ and some polynomials $\omega_i(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$, $i = 1 \ldots r'$.

Moreover the inverse system spanned by σ is included in $(I')^{\perp} = \mathcal{A}^*$ and its dimension is $\leqslant r = \dim(\mathcal{A})$. This shows that the sum of the dimensions r_i of the vector spaces spanned by $\partial_z^{\alpha}(\omega_i(\boldsymbol{z})\boldsymbol{e}_{\xi_i}(\boldsymbol{z}))$ for $\alpha \in \mathbb{N}^n$ is $\leqslant r$ and T has a generalized decomposition of size $\leqslant r$.

$1 \Rightarrow 3$. If $T \in S_d$ has a generalized decomposition of size $\leqslant r$, then after a change of coordinates, we have $T^*(\boldsymbol{z}) = \sum_{i=1}^m \omega_i(\boldsymbol{z})\boldsymbol{e}_{\xi_i}(\boldsymbol{z}) + ((\boldsymbol{z}))^{d+1}$ with $\xi_i \in \mathbb{C}^n$ and $\omega_i(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$ such that the sum of the dimensions r_i of the vector spaces spanned by $\partial_z^{\alpha}(\omega_i(\boldsymbol{z})\boldsymbol{e}_{\xi_i}(\boldsymbol{z}))$ for $\alpha \in \mathbb{N}^n$ is r.

Let $\sigma = \sum_{i=1}^m \omega_i(\boldsymbol{z})\boldsymbol{e}_{\xi_i}$ and let $I'_{\sigma} = \ker H_{\sigma}$ be the associated ideal in $R = \mathbb{C}[\boldsymbol{x}]$. By Theorem 22, $\mathcal{A}_{\sigma} = R/I'_{\sigma}$ is artinian of dimension r. Let $m = \max\left\{r, \lceil \frac{d}{2} \rceil\right\}, m' = \max\left\{r - 1, \lfloor \frac{d}{2} \rfloor\right\}$ and $\tilde{T} \in S_{m+m'}$ be the unique element such that $\tilde{T}^* = \sum_{i=1}^m \omega_i(\boldsymbol{z})\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z}) + ((\boldsymbol{z}))^{m+m'+1}$. Then

$$\operatorname{rank} H_{\tilde{T}}^{m,m'} = \operatorname{rank} H_{\sigma}^{R_{\leqslant m}, R_{\leqslant m'}} \leqslant \operatorname{rank} H_{\sigma} \leqslant r$$

and $\tilde{T}^*$ truncated in degree $\leqslant d$ coincides is σ truncated in degree $\leqslant d$, that is, T. Thus $\tilde{T}$ is a flat extension of rank $\leqslant r$ of T. This proves point 1.

$3 \Rightarrow 2$. Suppose that $\tilde{T}$ is a flat extension of rank $\leqslant r$ of T. Let $E := \ker(H_{\tilde{T}}^{m,m'})$ and $F := \operatorname{Ker}(H_{\tilde{T}}^{m',m})$ and $k \leq r$ the rank of $H_{\tilde{T}}^{i,d-i}$. As $H_{\tilde{T}}^{m',m} = (H_{\tilde{T}}^{m,m'})^t$, the quotients S_m/E and $S_{m'}/F$ are of dimension k. By Lemma 70, after a generic change of coordinates we may assume that there exists a family B (resp. B') of k monomials of S_m (resp. $S_{m'}$) such that $x_0\, B$ (resp. $x_0\, B'$) is a basis of S_m/E (resp. $S_{m'}/F$) and that

$$\underline{B} \subset R_{\leqslant m-1} \,(\text{resp. } \underline{B}' \subset R_{\leqslant m'-1})$$

are connected to 1. Notice then that

$$H_{\tilde{T}^*}^{\underline{B}, \underline{B'}} = H_{\tilde{T}}^{x_0 B, x_0 B'}$$

is an invertible matrix of size $k \times k$. As the monomials of $\underline{B}$ (resp. B') are in $R_{\leqslant m-1}$ (resp. $R_{\leqslant m'-1}$), the sets of $\underline{B}^+$ (resp. $\underline{B}'^+$) is a subset of $R_{\leqslant m}$ (resp. $R_{\leqslant m'}$) and

$$k = \operatorname{rank} H_{\tilde{\underline{T}}^*}^{\underline{B}', \underline{B}} = \operatorname{rank} H_{\tilde{T}^*}^{R_{\leqslant m}, R_{\leqslant m'}} = \operatorname{rank} H_{\tilde{T}}^{m,m'}.$$

By Theorem 40, there exists a linear form $\sigma \in R^*$ which extends $\tilde{T}^*$ such that $\dim(R/I_{\sigma}) = k$ where $I_{\sigma} = \ker H_{\sigma}$. By Theorem 22, there exists $\xi_i \in \mathbb{C}^n$ and $\omega_i(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$ such that

$$\tilde{T}^*(\boldsymbol{z}) = \sum_{i=1}^m \omega_i(\boldsymbol{z})\boldsymbol{e}_{\xi_i}(\boldsymbol{z}) + ((\boldsymbol{z}))^{d+1}$$

with the sum of the dimensions r_i of the vector spaces spanned by $\partial_z^{\alpha}(\omega_i(\boldsymbol{z})\boldsymbol{e}_{\xi_i}(\boldsymbol{z}))$ for $\alpha \in \mathbb{N}^n$ $\leqslant r$.

This shows that $\tilde{T}$ has a generalized rank $\leqslant r$. As T is the truncation of $\tilde{T}^*$ in degree $\leqslant d$, it also has a generalized rank $\leqslant r$. This proves point 2. $\square$

Remark 72. The generalized rank of a tensor is related to the additive decomposition of binary forms and coincides with the scheme length introduced in [15]. It coincides also with the *cactus rank* used in [7].

6.3 Decomposition of tensors

The flat extension property leads to the following algorithm to compute the decomposition.

Algorithm 4

Input: $T \in S_d$.

- Make a generic change of variables and substitute $x_0 = 1$;
- Apply the Generalized Prony Algorithm 3 to the sequence T^*;
- If `success` then stop and output the affine decomposition of T;
- Otherwise set $r := r_0 + 1$ where r_0 is the maximal size of an orthogonal basis in Algorithm 3;
- While not `success` do
  - Choose a monomial set B of size $r+1$, connected to 1;
  - Solve the commutation relations for the matrices associated to B;
  - Apply Generalized Prony method to reconstruct the decomposition;
  - If the roots are simple then stop with `success` := true else set $r := r+1$;

Output: The decomposition $T = \sum_{i=1}^{r} \omega_i \, (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$

Example 73. We consider the following ternary cubic:

$$T(x_0, x_1, x_2) := x_0^2 x_1 + x_0 x_2^2.$$

We set $x_0 = 1$. The matrix of the truncated Hankel operator in degree $\leqslant 3$ is

$$\begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} & h_{2,2,0} & h_{1,3,0} \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} & h_{1,3,0} & h_{0,4,0} \\ 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} & h_{2,2,0} & h_{5,0,0} & h_{4,1,0} & h_{3,2,0} & h_{2,3,0} \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} & h_{1,3,0} & h_{4,1,0} & h_{3,2,0} & h_{2,3,0} & h_{1,4,0} \\ \frac{1}{3} & 0 & 0 & h_{2,2,0} & h_{1,3,0} & h_{0,4,0} & h_{3,2,0} & h_{2,3,0} & h_{1,4,0} & h_{0,5,0} \\ 0 & h_{4,0,0} & h_{3,1,0} & h_{5,0,0} & h_{4,1,0} & h_{3,2,0} & h_{6,0,0} & h_{5,1,0} & h_{4,2,0} & h_{3,3,0} \\ 0 & h_{3,1,0} & h_{2,2,0} & h_{4,1,0} & h_{3,2,0} & h_{2,3,0} & h_{5,1,0} & h_{4,2,0} & h_{3,3,0} & h_{2,4,0} \\ 0 & h_{2,2,0} & h_{1,3,0} & h_{3,2,0} & h_{2,3,0} & h_{1,4,0} & h_{4,2,0} & h_{3,3,0} & h_{2,4,0} & h_{1,5,0} \\ 0 & h_{1,3,0} & h_{0,4,0} & h_{2,3,0} & h_{1,4,0} & h_{0,5,0} & h_{3,3,0} & h_{2,4,0} & h_{1,5,0} & h_{0,6,0} \end{bmatrix},$$

where the $h_{i,j,k}$ are the *unknown moments*.

Starting with the rank 3, we compute the commutation relations and solve them using algebraic solvers. The rank is increased until a solution is found. This happens for $r = 5$ and $B = \langle 1, x_1, x_2, x_1^2, x_2^2 \rangle$. We compute the matrices H_0, H_1 and H_2 corresponding to $B, x_1 B, x_2 B$:

$$\begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} \end{bmatrix}, \begin{bmatrix} 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} \\ 0 & h_{4,0,0} & h_{3,1,0} & h_{5,0,0} & h_{4,1,0} \\ 0 & h_{3,1,0} & h_{2,2,0} & h_{4,1,0} & h_{3,2,0} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} \\ 1/3 & 0 & 0 & h_{2,2,0} & h_{1,3,0} \\ 0 & h_{3,1,0} & h_{2,2,0} & h_{4,1,0} & h_{3,2,0} \\ 0 & h_{2,2,0} & h_{1,3,0} & h_{3,2,0} & h_{2,3,0} \end{bmatrix}.$$

If we form the matrix equation

$$M_i\,M_j - M_j\,M_i = H_0^{-1}H_1H_0^{-1}H_2 - H_0^{-1}H_2H_0^{-1}H_1 = 0,$$

then we have a system of 8 non-trivial equations in 8 unknowns. The unknowns are

$$\{h_{4,0,0}, h_{3,1,0}, h_{2,2,0}, h_{1,3,0}, h_{5,0,0}, h_{4,1,0}, h_{3,2,0}, h_{2,3,0}\}.$$

It turns out that the system is not zero dimensional, and that we can choose (randomly) the values of five of these variables. We take $h_{4,0,0}=0$, $h_{3,1,0}=2$, $h_{2,2,0}=2$, $h_{1,3,0}=0$, $h_{5,0,0}=0$, $h_{4,1,0}=1$, $h_{3,2,0}=1$, $h_{2,3,0}=1$. The matrices of multplication by x_1, x_2 are repectively:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 6 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 6 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/2 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1/2 & 1/2 \end{bmatrix}.$$

The eigenvectors of M_1^t are

$[1.0, 3.013881550, 3.805799635, 9.083481997, 11.47022931]$,
$[1.0, 0.8835710415 + 1.470912711\,i, -0.7256326469 - 0.1728474157\,i, -1.382886418 + 2.599311752\,i, -0.3869045326 - 1.220065254\,i]$,
$[1.0, 0.8835710415 - 1.470912711\,i, -0.7256326469 + 0.1728474157\,i, -1.382886418 - 2.599311752\,i, -0.3869045326 + 1.220065254\,i]$,
$[1.0, -1.890511816 + 0.6949025162\,i, -0.9272671689 + 1.404829646\,i, 3.091145419 - 2.627442836\,i, 0.7767898835 - 3.300207334\,i]$,
$[1.0, -1.890511816 - 0.6949025162\,i, -0.9272671689 - 1.404829646\,i, 3.091145419 + 2.627442836\,i, 0.7767898835 + 3.300207334\,i]$

The second and third coordinates of these eigenvectors are the coordinates of the points ξ_i of the exponentials. We recover the weights by solving a linear system. This yields the decomposition:

$$\begin{aligned} T :=\ & 0.008034037278\,(x_0 + 3.013881550\,x_1 + 3.805799635\,x_2)^3 \\ & +(0.03600262077 - 0.03860608482\,i)\,(x_0 + (0.8835710415 + 1.470912711\,i)\,x_1 - (0.7256326469 + 0.1728474157\,i)\,x_2)^3 \\ & +(0.03600262077 + 0.03860608482\,i)\,(x_0 + (0.8835710415 - 1.470912711\,i)\,x_1 - (0.7256326469 - 0.1728474157\,i)\,x_2)^3 \\ & -(0.04001963941 - 0.01395131919\,i)\,(x_0 - (1.890511816 - 0.6949025162\,i)\,x_1 - (0.9272671689 - 1.404829646\,i)\,x_2)^3 \\ & -(0.04001963941 + 0.01395131919\,i)\,(x_0 - (1.890511816 + 0.6949025162\,i)\,x_1 - (0.9272671689 + 1.404829646\,i)\,x_2)^3 \end{aligned}$$

This shows that the rank of T is 5, which is the maximal rank of a ternary cubic.

6.4 Geometry of tensor decomposition

The set $\mathrm{Hilb}_r(\mathbb{P}^n)$ of zero-dimensional saturated ideals of S of degree r is known as the Hilbert scheme of r points. It has a structure of Scheme and can be defined by quadratic equations of degree 2 in the Plücker coordinates of the Grassmannian $\mathrm{Gr}_r(S_r^*) \subset \mathbb{P}(\wedge^r S_r^*)$.

Tensor decomposition is related to the following incidence variety:

$$\mathcal{W}_r = \{(T, I) \in \mathbb{P}(S_d) \times \mathrm{Hilb}_r(\mathbb{P}^n) |\ \langle T, I_d \rangle_d = 0\}.$$

By Proposition 71, its projection on the first component $\mathbb{P}(S_d)$ is the set $\mathcal{G}_r = \pi_1(\mathcal{W}_r)$ of tensors with a generalized rank $\leqslant r$. Let $\mathcal{K}_r = \overline{\mathcal{G}_r}$ be the closure of $\mathcal{G}_r$ in $\mathbb{P}(S_d)$.

The following example from W. Buczyńska and J. Buczyński [26] shows that $\mathcal{G}_r$ is not necessarily closed.

Example 74. The following polynomial

$$T = x_0^2\,x_2 + 6\,x_1^2\,x_3 - 3\,(x_0 + x_1)^2\,x_4.$$

is the limit of tensors of rank $\leqslant 5$:

$$T_\epsilon = (x_0 + \epsilon\,x_2)^3 + 6\,(x_1 + \epsilon\,x_3)^3 - 3\,(x_0 + x_1 + \epsilon\,x_4)^3 + 3\,(x_0 + 2\,x_1)^3 - (x_0 + 3\,x_1)^3$$

We easily check that $\lim_{\epsilon \to 0} \frac{1}{3\,\epsilon}\,T_\epsilon = T$. But its generalized rank is not $\leqslant 5$.

An explicit computation of $(T^\perp)$ yields the following Hilbert function for $h_{R/(T^\perp)} = [1, 5, 5, 1, 0, \ldots]$. Let us prove, by contradiction, that there is no saturated ideal $I \subset (T^\perp)$ of degree ≤ 5.

Suppose on the contrary that I is such an ideal. Then $h_{R/I}(n) \geq h_{R/(T^\perp)}(n)$ for all $n \in \mathbb{N}$. As $h_{R/I}(n)$ is an increasing function of $n \in \mathbb{N}$ with $h_{R/(T^\perp)}(n) \leq h_{R/I}(n) \leq 5$, we deduce that $h_{R/I} = [1, 5, 5, 5, \ldots]$.

This shows that $I_1=\{0\}$ and $I_2=(T^{\perp})_2$. As I is saturated, $(I_2\colon (x_0,\ldots,x_4))_1=I_1=\{0\}$ since $h_{R/(T^{\perp})}(1)=5$. But an explicit computation of $((T^{\perp})_2\colon (x_0,\ldots,x_4))$ gives $\langle x_2,x_3,x_4\rangle$. We obtain a contradiction, so that there is no saturated ideal of degree ≤ 5 such that $I\subset(T^{\perp})$. We deduce that $r_g(T)\geqslant 6$.

The Hilbert Scheme $\mathrm{Hilb}_r(\mathbb{P}^n)$ contains the open set $\mathrm{Hilb}_r^{\mathrm{red}}(\mathbb{P}^n)$ of ideals I defining r simple points. Its closure, denoted $\mathrm{Hilb}_r^{\mathrm{smooth}}(\mathbb{P}^n)$, is the set of smoothable ideals of $\mathrm{Hilb}_r(\mathbb{P}^n)$, that is the schemes which are the limit of simple points.

The set of tensors of rank $\leqslant r$ is the projection $\mathcal{R}_r=\pi_1(\mathcal{W}_r\cap\mathbb{P}(S_d)\times\mathrm{Hilb}_r^{\mathrm{red}}(\mathbb{P}^n))$. Its closure $\Sigma_r=\overline{\mathcal{R}_r}$ is the set of tensors of *border rank* $\leqslant r$.

The projection $\mathcal{S}_r=\pi_1(\mathcal{W}_r\cap\mathbb{P}(S_d)\times\mathrm{Hilb}_r^{\mathrm{smooth}}(\mathbb{P}^n))$ is the set of tensors of *smoothable rank* $\leqslant r$.

We the following inclusions

$$\mathcal{R}_r\subset\Sigma_r\subset\mathcal{S}_r\subset\mathcal{G}_r\subset\mathcal{K}_r.$$

These inclusions can be strict. However for r small compared to d, these varieties behave nicely:

Theorem 75. *For integers r, d, m, m' such that $d\geqslant r$, we have*

$$\Sigma_r=\mathcal{S}_r,\ \ \mathcal{G}_r=\mathcal{K}_r.$$

Proof. For $d\geqslant r$, $\mathrm{Hilb}_r(\mathbb{P}^n)$ can be defined by the intersection of quadrics with the Grassmannian $G_r(S_d^*)$ in $\mathbb{P}(\wedge^r S_d^*)$. The corresponding elements in $G_r(S_d)$ are the linear space $I_d^{\perp}$. The condition $\langle T, I_d\rangle_d=0$ is equivalent to $\bar{T}^*\in I_d^{\perp}$, or to the equation $\bar{T}^*\wedge I_d^{\perp}=0$. This shows that $\mathcal{W}_r$ is a projective variety and its projection $\mathcal{G}_r=\pi_1(\mathcal{W}_r)$ is closed. We deduce that $\mathcal{K}_r=\overline{\mathcal{G}_r}=\mathcal{G}_r$.

As $\mathcal{W}_r\cap\mathbb{P}(S_d)\times\mathrm{Hilb}_r^{\mathrm{smooth}}(\mathbb{P}^n)$ is also a projective variety, we have

$$\mathcal{S}_r=\pi_1(\mathcal{W}_r\cap\mathbb{P}(S_d)\times\mathrm{Hilb}_r^{\mathrm{smooth}}(\mathbb{P}^n))=\overline{\pi_1(\mathcal{W}_r\cap\mathbb{P}(S_d)\times\mathrm{Hilb}_r^{\mathrm{red}}(\mathbb{P}^n))}=\overline{\mathcal{R}_r}=\Sigma_r.\qquad\square$$

Theorem 76. *For integers r, d, m, m' such that $d\geq 2\,r$, $m=\lceil\frac{d}{2}\rceil, m'=\lfloor\frac{d}{2}\rfloor$, we have*

$$\mathcal{G}_r=\mathcal{K}_r=\mathcal{C}_r$$

where $\mathcal{C}_r=\left\{T\in S^d;\mathrm{rank}\,H_T^{m,m'}\leqslant r\right\}$.

Proof. When $d\geqslant 2\,r$, $m=\max\left\{r,\lceil\frac{d}{2}\rceil\right\}=\lceil\frac{d}{2}\rceil, m'=\max\left\{r-1,\lfloor\frac{d}{2}\rfloor\right\}=\lfloor\frac{d}{2}\rfloor$, $m+m'=d$ and $T\in S_d$ has a flat extension of size $\leqslant r$ iff $\mathrm{rank}\,H_T^{m,m'}\leqslant r$. By Proposition 71, this implies $\mathcal{G}_r=\mathcal{C}_r$, which is closed. Thus $\mathcal{G}_r=\mathcal{K}_r=\mathcal{C}_r$. $\square$

Bibliography

[1] Fredrik Andersson, Marcus Carlsson and Maarten V. de Hoop. Nonlinear approximation of functions in two dimensions by sums of wave packets. *Appl. Comput. Harmon. Anal.*, 29(2):198–213, 2010.

[2] M. Ben-Or and P. Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, STOC '88, pages 301–309. New York, NY, USA, 1988. ACM.

[3] A. Bernardi, J. Brachat, P. Comon and B. Mourrain. General Tensor Decomposition, Moment Matrices and Applications. *Journal of Symbolic Computation*, 52(May):51–71, May 2013.

[4] A. Bernardi, A. Gimigliano and M. Idà. Computing symmetric rank for symmetric tensors. *J. Symb. Comput.*, 46:34–53, January 2011.

[5] Gregory Beylkin and Lucas Monzón. On approximation of functions by exponential sums. *Appl. Comput. Harmon. Anal.*, 19(1):17–48, 2005.

[6] J. Brachat, P. Comon, B. Mourrain and E. Tsigaridas. Symmetric tensor decomposition. *Linear Algebra and Applications*, 433:1851–1872, 2010.

[7] W. Buczyńska and J. Buczyński. Secant varieties to high degree veronese reembeddings, catalecticant matrices and smoothable gorenstein schemes. To appear in Journal of Algebraic Geometry, http://arxiv.org/abs/1012.3563, 12 2010.

[8] E. J. Candès, J. K. Romberg and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. *Communications on Pure and Applied Mathematics*, 59(8):1207–1223, 2006.

[9] Baron Gaspard Riche de Prony. Essai expérimental et analytique: sur les lois de la dilatabilité de fluides élastique et sur celles de la force expansive de la vapeur de l'alcool, à différentes températures. *J. Ecole Polyt.*, 1:24–76, 1795.

[10] D. Eisenbud. *Commutative Algebra with a view toward Algebraic Geometry*, volume 150 of *Graduate Texts in Math.* Berlin, Springer-Verlag, 1994.

[11] Mohamed Elkadi and Bernard Mourrain. *Introduction à la résolution des systèmes polynomiaux*, volume 59 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2007.

[12] Mark Giesbrecht, George Labahn and Wen-shin Lee. Symbolic-numeric sparse interpolation of multivariate polynomials. *J. Symb. Comput.*, 44(8):943–959, aug 2009.

[13] G. Golub and V. Pereyra. Separable nonlinear least squares: the variable projection method and its applications. In *Institute of Physics, Inverse Problems*, pages 1–26. 2002.

[14] G. Gotzmann. Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes. *Math. Z.*, 158:61–70, 1978.

[15] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci*, volume 1721 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1999.

[16] M. Laurent and B. Mourrain. A Sparse Flat Extension Theorem for Moment Matrices. *Archiv der Mathematik*, 93:87–98, 2009.

[17] L.-H. Lim and P. Comon. Blind multilinear identification. *IEEE Transactions on Information Theory*, 60(2):1260–1280, 2014.

[18] B. Mourrain. A new criterion for normal form algorithms. In M. Fossorier, H. Imai, Shu Lin and A. Poli, editors, *Proc. AAECC*, volume 1719 of *LNCS*, pages 430–443. Springer, Berlin, 1999.

[19] B. Mourrain and V.Y. Pan. Multivariate Polynomials, Duality, and Structured Matrices. *Journal of Complexity*, 16(1):110–180, 2000.

[20] B. Mourrain and P. Trébuchet. Generalized normal forms and polynomials system solving. In M. Kauers, editor, *ISSAC: Proceedings of the ACM SIGSAM International Symposium on Symbolic and Algebraic Computation*, pages 253–260. 2005.

[21] Daniel Potts and Manfred Tasche. Parameter estimation for exponential sums by approximate prony method. *Signal Processing*, 90(5):1631–1642, 2010.

[22] Daniel Potts and Manfred Tasche. Parameter estimation for multivariate exponential sums. http://www-user.tu-chemnitz.de/~potts/paper/multiprony.pdf, 2012.

[23] V. Pereyra and G. Scherer. *Exponential Data Fitting and its Applications*. Bentham Science Publishers, Sharjah, 2010.

[24] R. Roy and T. Kailath. Signal processing part ii. Chapter ESPRIT-estimation of signal parameters via rotational invariance techniques, pages 369–411. Springer-Verlag New York, Inc., New York, NY, USA, 1990.

[25] A. Swindlehurst and T. Kailath. A performance analysis of subspace-based methods in the presence of model errors – part i: the music algorithm. *IEEE Trans. on Signal Processing*, 40:1758–1774, 1992.

[26] W.Buczyńska and J.Buczyński. *In Preparation.*

[27] Richard Zippel. Probabilistic algorithms for sparse polynomials. In *Proceedings of the International Symposiumon on Symbolic and Algebraic Computation*, EUROSAM '79, pages 216–226. London, UK, UK, 1979. Springer-Verlag.