# Symbolic Analysis

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## C6 workshop at FoCM 2011

http://www.damtp.cam.ac.uk/user/na/FoCM11

E. Hubert, G. Labahn and M. Singer organizers

July 12-14th, 2011

| | **July 12th** | **July 13th** | **July 14th** |
|---|---|---|---|
| **2:00-2:45** | F. Winkler, J. Roques | A. Bostan, E. Musso | J.-A. Weil, D. Blazquez-Sanz |
| **2:50-3:30** | Z. Li, T. CLuzeau | P. van der Kamp, G. Mari-Beffa | M. Kauers |
| **3:40-4:25** | C. Schneider, S. Rueda | C.-M. Yuan, F. Chyzak | T. Goncalves, R. Thomson |
| **5:00-5:45** | J. Pohjanpelto, H. Nakayama | G. Regensburger, E. Mansfield | M. Singer, G. Labahn |
| **5:50-6:35** | M. Boutin | W. Hereman | |

## Integrability of linear non autonomous hamiltonians through differential Galois theory

David Blazquez-Sanz

U. Sergio Arboleda

In this talk we introduce a notion of integrability for Hamiltonian systems in the non autonomous sense. For the cases of 1+1/2 degrees of freedom and quadratic homogeneous Hamiltonians of 2+1/2 degrees of freedom we prove that this notion is equivalent to the classical complete integrability of the system in the extended phase space. For the case of quadratic homogeneous Hamiltonians of 2+1/2 degrees of freedom we also give a reciprocal of the Morales-Ramis result. We classify those systems by terms of symplectic change of frames involving algebraic functions of time, and give their canonical forms.

(Joint work with S. Carrilo, U. Nacional de Colombia)

# Symbolic Analysis for Lattice Path Combinatorics

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We give an overview of a recent line of research showing how several problems of enumerative combinatorics can be systematically solved using an experimental-mathematics approach combined with modern computer algebra algorithms. We describe the computer-driven discovery and proof of structural properties and closed forms for generating functions coming from enumeration of lattice walks with small steps in the quarter plane. The results are taken from several joint works with Frédéric Chyzak, Philippe Flajolet, Mark van Hoeij, Manuel Kauers, Lucien Pech and Karol Penson.

## Challenges and pitfalls of automatic learning in the Big Data Age

Mireille Boutin

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Being able to automatically learn from large data sets is viewed by many as a key to solving an ever increasing number of unsolved engineering problems such as automatic Improvised Explosive Device (IED) detection, and baggage screening. The widespread availability of different sensing modalities, which allows us to obtain a large amount of redundant data, combined with the computational power of today's computers, offers the promise of near perfect accuracy at real-time speeds. In this talk, I will summarize some of the computational and conceptual challenges that must be surmounted before this promise can be fulfilled.

## DDMF: A Generated, Online Dictionary of Special Functions

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Special functions are used in many areas of applied mathematics and the continuous need of scientists for lists of their mathematical properties has led to a great deal of reference books on special functions. Formulas in such books are typically collected from the litterature by mathematical experts. Furthermore, more and more powerful algorithms have been developed over the last decades by the computer-algebra community to compute properties of special functions.

Thus, it has become just natural to automate the writing of a mathematical handbook on special functions, insofar as a sufficiently large and well identified class of functions share common algorithmic properties. Our encyclopedia DDMF (for "Dynamic Dictionary of Mathematical Functions") focuses on so-called "differentiably finite functions," that is, functions that are described as solutions of a linear different equation with polynomial coefficients and finitely many initial conditions. These functions enjoy a great deal of common algorithmic properties that have been studied intensively.

For each mathematical function, the current version (v1.6) *algorithmically* computes, then displays: its potential symmetries; Taylor and Chebyshev series expansions; more generally, asymptotic expansions given in closed form or through definitions by recurrence; calculations of guaranteed, arbitrary-precision numerical approximations; real plots; its Laplace transform; expressions in terms of hypergeometric functions. Upon request by the user, more terms in series expansions or more digits in numerical approximations can be computed incrementally. For some of the properties, human-readable proofs are also automatically *generated* and displayed. In addition, our encyclopedia can in principle be augmented with any new function of the class.

In this talk, I will demonstrate the mathematical web site (`http://ddmf.msr-inria.inria.fr/`), present the algorithms used, and briefly touch on the underlying system that generates the web site.

(Joint work in continuous progress with Alexandre Benoit, Alexis Darrasse, Stefan Gerhold, Marc Mezzarobba, and Bruno Salvy.)

## A constructive version of Fitting's theorem on isomorphisms and equivalences of linear systems

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Within the algebraic analysis approach to linear systems theory, a linear functional system can be studied by means of its associated finitely presented left module. Testing whether two linear systems/modules are isomorphic (the so-called equivalence problem) is an important issue in systems/module theory. In this talk, we explicitly characterize the conditions for a homomorphism between two finitely presented left modules to define an isomorphism, and we give an explicit formula for the inverse of an isomorphism. Then, we constructively study Fitting's major theorem, which shows how to enlarge matrices presenting isomorphic modules by blocks of 0 and $I$ to get equivalent matrices. The consequences of this result on the Auslander transposes and adjoints of the finitely presented left modules are given. Finally, we show how to deduce simple proofs of Schanuel's lemma for finitely presented modules and of the fact that Fitting ideals associated with a finitely presented $D$-module $M$ do not depend on any presentation of $M$, when $D$ is a commutative ring. The different results developed are implemented in the OreMorphisms package.

## Symbolic methods for symmetric variational problems: the $SE(3)$ case

Tânia M. N. Goncalves and Elizabeth L. Mansfield

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In the seminal paper "Invariante Variationsprobleme" by Emmy Noether, she showed that for systems derived from a variational principle, the associated conservation laws could be obtained from Lie group actions that left the variational problem unchanged. Recently, we proved that these conservation laws could be rewritten as the divergence of the product of a moving frame and a vector of invariants. The aim of this talk is to illustrate how the knowledge of the conservation laws structure helps reduce the extremising problem, in particular for variational problems that are invariant under the special Euclidean group $SE(3)$.

## Symbolic Computation of Lax Pairs of Integrable Nonlinear Partial Difference Equations on Quad-Graphs

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The presentation deals with two-dimensional nonlinear partial difference equations (P$\Delta$Es) which are completely integrable, i.e., they admit a Lax representation.

Based on work by Nijhoff, Bobenko and Suris, a method to compute Lax pairs will be presented. The method is algorithmic and can be implemented in the syntax of computer algebra systems, such as MATHEMATICA and MAPLE.

A MATHEMATICA program will be demonstrated that automatically computes Lax pairs for *scalar* P$\Delta$Es on quad-graphs, including lattice versions of the potential Korteweg-de Vries (KdV) equations, the modified KdV and sine-Gordon equations, as well as lattices derived by Adler, Bobenko, and Suris.

The generalization of the symbolic code to nonlinear *systems* of integrable P$\Delta$Es on quad-graphs is work in progress. Examples of Lax pairs of systems of P$\Delta$Es will be shown, including the potential KdV and nonlinear Schrödinger lattices, and various Boussinesq-type lattices.

## Complete Integrability of Reductions of Lattice Equations

Peter H van der Kamp, Dinh Tran, Reinout Quispel

La Trobe University, Melbourne

For the complete integrability (in the sense of Arnold-Liouville) of a mapping one needs:

- ⋆ the map to be symplectic,
- ⋆ sufficiently many integrals (half its dimension),
- ⋆ their involutivity,
- ⋆ and their functional independence.

An overview will be given of recent results on the complete integrability of reductions of various integrable lattice equations.

# How a Hard Conjecture in Number Theory was Knocked out with Symbolic Analysis

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We report on a proof of the famous qTSPP conjecture in partition theory, recently obtained in a collaboration with Christoph Koutschan (RISC) and Doron Zeilberger (Rutgers).

The qTSPP conjecture, posed by Andrews and Robbins around 1982, is a formula for counting certain integer partitions. It became famous as the last unsolved problem on Stanley's list of conjectures on plane partitions.

Okada had pointed out that in order to prove the qTSPP conjecture, it suffices to prove a certain determinant identity. Using computer algebra, this determinant identity in turn can be reduced to a horrendous summation identity (300Mb in size), and, again making extensive use of computer algebra, an even more horrendous summation certificate (7Gb in size) could finally be constructed for this identity.

Our proof appeared a few months ago in the Proceedings of the National Academy of Science and also attracted the attention of several German-speaking public media.

# Symbolic Computation of Convolution Integrals of Holonomic Functions

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We describe an algorithm for the symbolic computation of convolution integrals of the form

$$\int_0^\infty g(t)f(xt)dt$$

for holonomic functions $g$ and $h$. Such integrals include such well known integral transforms such as Laplace, Fourier and Hankel transforms. The input holonomic functions are represented in terms of the linear differential equations that they solve along with information about their behaviour at 0 and $\infty$. The algorithm produces the linear differential equation solved by the convolution integral along with regions where the solution is valid. Thentechniques make use of both algebra and analysis. The algebraic form makes use of Mellin and inverse Mellin transforms while the analysis determines when the algebra is actually valid. The resulting algorithm generalizes the MeijerG method that is the current standard found in such computer algebra systems such as Maple and Mathematica.

This is joint work with Jason Peasgood (Waterloo, Canada) and Bruno Salvy (INRIA, France).

## Termination Criteria for Zeilberger's Algorithm in Mixed Cases

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We present three criteria on the termination of Zeilberger's algorithm in mixed cases. The first is for the differential and shift case; the second for the differential and $q$-shift case; and the last for shift and $q$-shift case. The criteria describe necessary and sufficient conditions on the existence of telescopers for hyperexponential-hypergeometric solutions in the above mixed case.

We will also review some results on which the criteria are based, including: a structure theorem on compatible rational functions, and various generalizations of Hermite reduction in the mixed cases.

This talk reports joint work with S. Chen, F. Chyzak, R. Feng and G. Fu.

## Noether's Two Theorems, Moving frames and Symbolic Computation

Elizabeth Mansfield

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Noether's seminal paper in 1918 has an enduring appeal: variational problems and their conservation laws continue to be central in physical applications. Their Lie symmetries arise naturally from the physics, and it appears her theorems are valid far more broadly, but just as powerfully, in all the new kinds of variational problems that are arising.

Noether's First Theorem guarantees conservation laws for smooth variational problems which are invariant under a Lie group action, and typically these laws are of serious physical interest in applications, for example, conservation of linear momentum, angular momentum, energy, potential voticity and so on. The conservation laws are written as divergence expressions which are zero on solutions of the system. Since the formulae for the laws are eye-glazingly recursive and complicated (see Olver [1] for the formulae and historical notes), Noether's laws are eminently suited to implementation in the symbolic computation domain. However, the formulae on their own are opaque in the sense that the resulting expressions can be lengthy and unstructured.

Since the variational principle is invariant under a Lie group, one can use a moving frame á la Fels and Olver to structure the expressions for the laws. This yields the laws as a product of the moving frame, in the adjoint representation of the group on its Lie algebra, and a vector of invariants [2]. Using this structure, a great deal more information about the solution set can be gleaned; in joint work with Tania Gonçalves, problems invariant under $SL(2)$ and $SE(3)$ actions have been successfully studied [3,4].

Noether's Second Theorem guarantees syzygies between the Euler-Lagrange equations for a variational problem invariant under a Lie pseudogroup depending on a single free function. The prototypical examples are gauge-invariant Lagrangians for which one of the Euler-Lagrange equations is a differential consequence of the others.

This Theorem has been considerably extended in joint work with Peter Hydon [5]. The extensions raise some potentially interesting questions in differential algebra, concerning the syzygy modules of differential ideals.

Both of Noether's theorems have been extended to finite difference variational problems, the first by several authors (the most general statement and perhaps the clearest exposition is by Hickman and Hereman [6]), and the second Theorem in [5].

The most interesting extensions at the moment are to finite element variational problems on simplicial domains, and to problems defined on topologically nontrivial domains, such as networks and graphs in the one-dimensional case, to spheres in the two-dimensional case, and to domains with holes in the three-dimensional case. In these latter cases, Noether's conservation laws, which are local laws only, need to be extended. The algebraic construction similar to that of the Čech-de Rham double complex (see [7]) allows this extension to be made. For difference systems defined on topologially non-trivial lattice-like "manifolds", it is conjectured that solutions can be counted in terms of the analogues of Betti numbers [8], but nothing appears to be known about global conservation laws.

In this talk, I will discuss a range of issues in symbolic computation that arise in the above circle of ideas.

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## The Pentagram map and generalizations: discretizations of AGD flows

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In this talk I will discuss very recent work on generalizations of the pentagram map, a map defined on the space of convex polygons in the projective plane. The pentagram map has been recently studied by R. Schwartz, S Tabachnikov and V. Ovsienko who proved that, when defined on twisted polygons, this discrete map is an integrable system. Furthermore, they proved that the pentagram map is a discretization of the Boussinesq equation, a well-known integrable PDE. In this talk we will give a short overview of their results and will discuss possible generalizations to maps defined on twisted k-gons in $\mathbb{RP}^n$, in particular $n = 3, 4$. We will describe conditions that will ensure that the generalizing map is a discretization of higher order AGD flows, and discuss on-going work to establish their integrability.

## Motions of curves in projective plane inducing the Kaup-Kupershmidt hierarchy

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Interrelations between hierarchies of integrable non linear evolution equations and local motions of curves in homogenous spaces have been widely investigated in the last decades. K-S. Chu and C. Qu gave a complete account of integrable hierarchies originated by local motions of plane curves with respect to several transformation groups of $R^2$. In particular, they prove that the fifth order Kaup-Kupershmidt equation is related to a local motion in centro-affine geometry and that modified versions of the Kaup-Kupershmidt equations are related to motions of curves in projective plane. In this talk we will explain how to construct geometrical flows of curves in projective plane having the equations of the Kaup-Kupershmidt hierarchy as their analytical counterparts (for every order). We analyze the congruence curves of the flows and we investigate in more details the congruence curves defined by the cnoidal traveling wave solution of the fifth order Kaup-Kupershmidt equation. More generally, we show that all critical curves of the projective arc-length functional are congruence curves of the flow. We also exhibit new examples of traveling wave solutions in terms of Wieretsrass $\wp$-functions. Our approach to the problem is mostly based on symbolical and numerical computations performed with the software Mathematica 8.

An algorithm of computing inhomogeneous differential equations for definite integrals

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Let us denote by $D = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ the Weyl algebra in $n$ variables, where $K$ is $\mathbb{Q}$ or $\mathbb{C}$ and $\partial_i$ is the differential operator standing for $x_i$. We denote by $D' = K\langle x_1, \ldots, x_m, \partial_1, \ldots, \partial_m\rangle$ the Weyl algebra in $m$ variables, where $m \leq n$ and $D'$ is a subring of $D$.

Let $I$ be a holonomic left $D$-ideal. The integration ideal of $I$ with respect to $x_1, \cdots, x_m$ is defined by the left $D'$-ideal

$$(I + \partial_1 D + \cdots + \partial_m D) \cap D'.$$

Oaku gave an algorithm computing the integration ideal. This algorithm is called the integration algorithm of $D$-module. The Gröbner basis method in $D$ is used in this algorithm.

We give a new algorithm computing not only generators of the integration ideal $J$ but also $P_0 \in I$ and $P_1, \cdots, P_m \in D$ such as

$$P = P_0 + \partial_1 P_1 + \cdots + \partial_m P_m$$

for any generator $P \in J$. Our algorithm is based on Oaku's one. We call these $P_1, \cdots, P_m$ inhomogeneous parts of $P$. As an important application of our algorithm, we can obtain inhomogeneous differential equations for a definite integral with parameters by using generators of the integration ideal and inhomogeneous parts.

For example, we compute an inhomogeneous differential equation for the integral $A(x_2) = \int_a^b e^{-x_1 - x_2 x_1^3} dx_1$. This is the case of $m = 1, n = 2$. The annihilating ideal of the integrand $f(x_1, x_2) = e^{-x_1 - x_2 x_1^3}$ in $D$ is $I = \langle \partial_1 + 1 + 3x_2 x_1^2, \partial_2 + x_1^3\rangle$. The integration ideal of $I$ with respect to $x_1$ is $J = \langle 27x_2^3\partial_2^2 + 54x_2^2\partial_2 + 6x_2 + 1\rangle = \langle P\rangle$. The operator $P_1 = -(\partial_1^2 + 3\partial_1 + 3)$ is an inhomogeneous part of $P$.

$$\begin{aligned} P \cdot A(x_2) &= \int_a^b \partial_1 (P_1 \cdot e^{-x_1 - x_2 x_1^3}) dx_1 = \Big[ P_1 \cdot e^{-x_1 - x_2 x_1^3} \Big]_{x_1 = a}^{x_1 = b} \\ &= - \Big[ (9x_2^2 x_1^4 - 3x_2 x_1^2 - 6x_2 x_1 + 1) e^{-x_1 - x_2 x_1^3} \Big]_{x_1 = a}^{x_1 = b}. \end{aligned}$$

In this way, we get an inhomogeneous differential equation for the integral $A(x_2)$. We will give the algorithm to compute inhomogeneous parts of the integration ideal and give some examples.

We implement these algorithms on the computer algebra system Risa/Asir. They are in the program package `nk_restriction.rr`.

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`http://www.math.kobe-u.ac.jp/~nakayama/nk_restriction.rr`

## Reduction of Exterior Differential Systems under Symmetry Pseudogroups

Juha Pohjanpelto

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We will discuss a symmetry-based method for constructing solutions to systems of differential equations founded on the reduction of exterior differential systems invariant under the action of an infinite dimensional pseudogroup. Any system of differential equations $\Delta = 0$ with a symmetry group $\mathcal{G}$ can be associated with an exterior differential system $\mathcal{I}$ invariant under $\mathcal{G}$ so that solutions of $\Delta = 0$ correspond to integral manifolds of $\mathcal{I}$. With the help of a moving frame, the exterior differential system gives rise to a reduced system on a given cross section to the action of $\mathcal{G}$. All integral manifolds of the original system $\mathcal{I}$ can then be reconstructed from those of the reduced system by the way of an equation of generalized Lie type for the symmetry group parameters. Accordingly, we obtain a two-step algorithm for finding integral manifolds for exterior differential systems akin to Vessiot's method of group foliation. As examples, applications of the reduction process to the construction of analytic solutions to non-linear partial differential equations will be presented.

## Algebraic Properties and Symbolic Aspects of Ordinary Integro-Differential Operators and Applications to Boundary Problems

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An integro-differential algebra combines a differential algebra with a suitable notion of an integral operator. By the fundamental theorem of calculus, the integral should be a right inverse of the derivation. We require additionally a version of integration by parts that allows us to define an "evaluation" in any integro-differential algebra. This is also our vantage point for treating initial and boundary conditions in an algebraic setting.

We discuss the construction of the algebra of ordinary integro-differential operators over an integro-differential algebra. We focus in particular on algebraic properties and algorithmic aspects of the integro-differential operators over the polynomial ring in one indeterminate over a field of characteristic zero with the usual derivation and integration. This algebra has also been studied recently by V. V. Bavula in a series of papers using the fact that it can be constructed as a generalized Weyl algebra.

Integro-differential operators over smooth or analytic functions provide an algebraic structure for computing with boundary problems for linear ODEs as well as their solution operators (Green's operators). Our implementation is based on the fact that every integro-differential operator can be written uniquely as a sum of a differential, an integral, and a boundary operator, and we illustrate it with some sample computations.

This talk is based in part on joint work with Anja Korporal, Johannes Middeke, Alban Quadrat, and Markus Rosenkranz.

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## Discrete Morales-Ramis theory : an example.

Julien Roques[a]

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We will give an application of a non integrability criterion, of Morales-Ramis type, for discrete dynamical systems. This is part of a work in collaboration with G. Casale.

Differential implicitization of linear DPPEs:
Linear reparametrizations and differential resultants

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The implicitization and parametrization problems of unirational algebraic varieties have been widely studied, and the results on the computation of the implicit equation of a system of algebraic rational parametric equations by algebraic resultants are well known. The generalization of these results to the differential case is a wide open field of research where many interesting problems arise.

In [3], characteristic set methods were used to solve the differential implicitization problem for differential rational parametric equations. Alternative methods are emerging to treat special cases. Let $\mathbb{K}$ an ordinary differential field with derivation $\partial$ and $\mathcal{D} = \mathbb{K}[\partial]$ the ring of differential operators with coefficients in $\mathbb{K}$. We focus on the study of systems of linear differential parametric polynomial equations (linear DPPEs),

$$\mathcal{P}(X,U) = \begin{cases} x_1 - a_1 & = & \mathcal{L}_{1,1}(u_1) + \cdots + \mathcal{L}_{1,n-1}(u_{n-1}), \\ & \vdots & \\ x_n - a_n & = & \mathcal{L}_{n,1}(u_1) + \cdots + \mathcal{L}_{n,n-1}(u_{n-1}), \end{cases} \tag{1}$$

where $\mathcal{L}_{i,j} \in \mathcal{D}$, not all $\mathcal{L}_{i,j} \in \mathbb{K}$ and $a_i \in \mathbb{K}$.

In [5], we defined linear complete differential resultants as a generalization of the differential resultant defined by G. Carrà-Ferro in [1] (in the linear case). We proved that when nonzero the differential resultant gives the implicit equation of ID, the implicit ideal of $\mathcal{P}$. As in the algebraic case differential resultants often vanish under specialization. Motivated by Canny's method and its generalizations (see references in [2]) in [4] a linear perturbation of $\mathcal{P}$ is considered. By means of this perturbation an implicitization of $\mathcal{P}$ is computed if ID has dimension $n-1$.

Let $o_i$ be the order of the $i$th equation of $\mathcal{P}$. The next positive integers were used to define the linear complete differential resultant,

$$\gamma_j(\mathcal{P}) := \min\{o_i - \mathcal{O}(\mathcal{L}_{i,j}) \mid i \in \{1,\ldots,n\}\},$$

where $\mathcal{O}(\mathcal{L}_{i,j}) = \deg(\mathcal{L}_{i,j})$, if $\deg(\mathcal{L}_{i,j}) \geq 0$ and $\mathcal{O}(\mathcal{L}_{i,j}) = 0$, if $\deg(\mathcal{L}_{i,j}) = -\infty$. Given a differential operator $\mathcal{L} = \sum_{\beta\in\mathbb{N}_0} c_\beta \partial^\beta \in \mathcal{D}$, we define $\sigma_\beta(\mathcal{L}) = c_\beta$. The $n \times (n-1)$ matrix $\sigma(\mathcal{P})$ whose $i$th row contains $(\sigma_{o_i-\gamma_{n-1}}(\mathcal{L}_{i,n-1}),\ldots,\sigma_{o_i-\gamma_1}(\mathcal{L}_{i,1}))$ plays a relevant role in this study.

The implicitization algorithm provided in [4] works for systems $\mathcal{P}$ whose matrix $\sigma(\mathcal{P})$ has maximal rank. It will be shown how in the $(n-1)$-dimensional case a reparametrization $\mathcal{P}'$ of $\mathcal{P}$ (with implicit ideal $\mathrm{ID}' = \mathrm{ID}$) can be constructed so that $\sigma(\mathcal{P}')$ has maximal rank.

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## Symbolic Summation in Perturbative Quantum Field Theory

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We present summation algorithms in the context of difference fields that assist in the task to evaluate 3-loop massive single scale Feynman integrals with operator insertion. Special emphasis is put on new evaluations that are relevant for the the computations at the Large Hadron Collider at CERN.

# Factoring Partial Differential Operators

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It has been known for a long time that an ordinary linear differential operator can be factored into a product of irreducible ordinary linear operators and that this factorization is unique up to a certain equivalence. This result follows from a Jordan-Hoelder type theorem for certain modules over the ring of ordinary differential operators.

A consequence of the above result is that the number of irreducible factors of an operator and their orders are unique. This is no longer true for partial differential operators and we discuss several examples to show in which ways this can fail. Instead of looking for factors one can consider subspaces of the solution space that are again defined by the vanishing of linear operators. I will discuss a result that states that there exists a finite tower of such subspaces where the successive quotients are "simple" in a certain sense and show that these quotients are unique up to a certain kind of equivalence. This is joint work with Phyllis Cassidy.

# Group foliation using moving frames

Rob Thomson

School of Mathematics, University of Minnesota, Minneapolis, USA.

The method of group foliation, first proposed by Lie and later developed by Vessiot, provides a technique for using the symmetry group or pseudo-group of a differential equation to find solutions which possess no symmetry. The method has found frequent use for finding new solutions to important physical equations. We'll describe the use of moving frames to accomplish the group foliation algorithm. This algorithm is also closely related to the EDS reduction method recently discovered by Anderson, Fels and Pohjanpelto.

## From algebra and algebraic geometry to differential equations

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An algebraic ordinary differential equation (AODE) is of the form $F(x, y, y', ..., y^{(n)}) = 0$, where F is an $(n+2)$-variate polynomial over a differential field K. Solving AODEs is an open problem. There are currently no general methods for solving such non-linear differential equations. We describe an algorithm for determining the rational solvability of AODEs of order 1, and, in the positive case, finding a general rational solution. Moreover, we introduce a group of affine transformations preserving rational solvability. The orbits of this group provide a classification of AODEs of order 1, where the classes consist of equations having the same computational complexity with respect to the computation of general rational solutions.

# Reduced Forms of Linear Differential Systems

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A linear differential system $[A]$ : $Y' = AY$, with $A \in \mathrm{Mat}(n, \overline{k})$ is said to be in reduced form if $A \in \mathfrak{g}(\overline{k})$ where $\mathfrak{g}$ is the Lie algebra of the differential Galois group $G$ of $[A]$.
In this talk, we will first explain why this notion is natural and desirable. A classical result of Kolchin and Kovacic shows that any linear differential system admits reduced forms; we will propose a procedure to achieve this reduction constructively (when the Galois group is reductive and unimodular). The key ingredient is the following result : when $G$ is reductive and unimodular, the system $[A]$ is in reduced form if and only if all of its invariants (rational solutions of appropriate symmetric powers) have constant coefficients (instead of rational functions). When $G$ is non-reductive, we give a similar characterization via the semi-invariants of $G$.

## Differential Chow Form and Differential resultant

Chun-Ming Yuan[a]

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In this talk, an intersection theory for generic differential polynomials is presented. Based on the intersection theory, the Chow form for an irreducible differential variety is defined and most of the properties of the Chow form in the algebraic case are established for its differential counterpart. Furthermore, the generalized differential Chow form is defined and its properties are proved. As an application of the generalized differential Chow form, the differential resultant of $n+1$ generic differential polynomials in n variables is defined and properties similar to that of the Macaulay resultant for multivariate polynomials are proved.

The concept of sparse differential resultant for a system of quasi-generic differential polynomials is introduced and its propertied are proved. In particular, a degree bound for the sparse differential resultant is given. Based on the degree bound, an algorithm to compute the sparse differential resultant is proposed, which is single exponential in terms of the order, the number of variables, and the size of the quasi-generic polynomials.