Differential implicitization of linear DPPEs:
Linear reparametrizations and differential resultants

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The implicitization and parametrization problems of unirational algebraic varieties have been widely studied, and the results on the computation of the implicit equation of a system of algebraic rational parametric equations by algebraic resultants are well known. The generalization of these results to the differential case is a wide open field of research where many interesting problems arise.

In [3], characteristic set methods were used to solve the differential implicitization problem for differential rational parametric equations. Alternative methods are emerging to treat special cases. Let $\mathbb{K}$ an ordinary differential field with derivation ∂ and $\mathcal{D} = \mathbb{K}[\partial]$ the ring of differential operators with coefficients in $\mathbb{K}$. We focus on the study of systems of linear differential parametric polynomial equations (linear DPPEs),

$$\mathcal{P}(X,U) = \begin{cases} x_1 - a_1 &= \mathcal{L}_{1,1}(u_1) + \cdots + \mathcal{L}_{1,n-1}(u_{n-1}), \\ & \vdots \\ x_n - a_n &= \mathcal{L}_{n,1}(u_1) + \cdots + \mathcal{L}_{n,n-1}(u_{n-1}), \end{cases} \tag{1}$$

where $\mathcal{L}_{i,j} \in \mathcal{D}$, not all $\mathcal{L}_{i,j} \in \mathbb{K}$ and $a_i \in \mathbb{K}$.

In [5], we defined linear complete differential resultants as a generalization of the differential resultant defined by G. Carrà-Ferro in [1] (in the linear case). We proved that when nonzero the differential resultant gives the implicit equation of ID, the implicit ideal of $\mathcal{P}$. As in the algebraic case differential resultants often vanish under specialization. Motivated by Canny's method and its generalizations (see references in [2]) in [4] a linear perturbation of $\mathcal{P}$ is considered. By means of this perturbation an implicitization of $\mathcal{P}$ is computed if ID has dimension $n-1$.

Let o_i be the order of the ith equation of $\mathcal{P}$. The next positive integers were used to define the linear complete differential resultant,

$$\gamma_j(\mathcal{P}) := \min\{o_i - \mathcal{O}(\mathcal{L}_{i,j}) \mid i \in \{1,\ldots,n\}\},$$

where $\mathcal{O}(\mathcal{L}_{i,j}) = \deg(\mathcal{L}_{i,j})$, if $\deg(\mathcal{L}_{i,j}) \geq 0$ and $\mathcal{O}(\mathcal{L}_{i,j}) = 0$, if $\deg(\mathcal{L}_{i,j}) = -\infty$. Given a differential operator $\mathcal{L} = \sum_{\beta\in\mathbb{N}_0} c_\beta \partial^\beta \in \mathcal{D}$, we define $\sigma_\beta(\mathcal{L}) = c_\beta$. The $n \times (n-1)$ matrix $\sigma(\mathcal{P})$ whose ith row contains $(\sigma_{o_i-\gamma_{n-1}}(\mathcal{L}_{i,n-1}), \ldots, \sigma_{o_i-\gamma_1}(\mathcal{L}_{i,1}))$ plays a relevant role in this study.

The implicitization algorithm provided in [4] works for systems $\mathcal{P}$ whose matrix $\sigma(\mathcal{P})$ has maximal rank. It will be shown how in the $(n-1)$-dimensional case a reparametrization $\mathcal{P}'$ of $\mathcal{P}$ (with implicit ideal $\mathrm{ID}' = \mathrm{ID}$) can be constructed so that $\sigma(\mathcal{P}')$ has maximal rank.

References

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