An algorithm of computing inhomogeneous differential equations for definite integrals

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Let us denote by $D = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ the Weyl algebra in n variables, where K is $\mathbb{Q}$ or $\mathbb{C}$ and ∂_i is the differential operator standing for x_i. We denote by $D' = K\langle x_1, \ldots, x_m, \partial_1, \ldots, \partial_m\rangle$ the Weyl algebra in m variables, where $m \le n$ and D' is a subring of D.

Let I be a holonomic left D-ideal. The integration ideal of I with respect to $x_1, \cdots, x_m$ is defined by the left D'-ideal

$$(I + \partial_1 D + \cdots + \partial_m D) \cap D'.$$

Oaku gave an algorithm computing the integration ideal. This algorithm is called the integration algorithm of D-module. The Gröbner basis method in D is used in this algorithm.

We give a new algorithm computing not only generators of the integration ideal J but also $P_0 \in I$ and $P_1, \cdots, P_m \in D$ such as

$$P = P_0 + \partial_1 P_1 + \cdots + \partial_m P_m$$

for any generator $P \in J$. Our algorithm is based on Oaku's one. We call these $P_1, \cdots, P_m$ inhomogeneous parts of P. As an important application of our algorithm, we can obtain inhomogeneous differential equations for a definite integral with parameters by using generators of the integration ideal and inhomogeneous parts.

For example, we compute an inhomogeneous differential equation for the integral $A(x_2) = \int_a^b e^{-x_1 - x_2 x_1^3} dx_1$. This is the case of $m = 1, n = 2$. The annihilating ideal of the integrand $f(x_1, x_2) = e^{-x_1 - x_2 x_1^3}$ in D is $I = \langle \partial_1 + 1 + 3x_2 x_1^2, \partial_2 + x_1^3\rangle$. The integration ideal of I with respect to x_1 is $J = \langle 27x_2^3\partial_2^2 + 54x_2^2\partial_2 + 6x_2 + 1\rangle = \langle P\rangle$. The operator $P_1 = -(\partial_1^2 + 3\partial_1 + 3)$ is an inhomogeneous part of P.

$$\begin{aligned} P \cdot A(x_2) &= \int_a^b \partial_1 (P_1 \cdot e^{-x_1 - x_2 x_1^3}) dx_1 = \Big[P_1 \cdot e^{-x_1 - x_2 x_1^3}\Big]_{x_1=a}^{x_1=b} \\ &= -\Big[(9x_2^2 x_1^4 - 3x_2 x_1^2 - 6x_2 x_1 + 1)e^{-x_1 - x_2 x_1^3}\Big]_{x_1=a}^{x_1=b}. \end{aligned}$$

In this way, we get an inhomogeneous differential equation for the integral $A(x_2)$. We will give the algorithm to compute inhomogeneous parts of the integration ideal and give some examples.

We implement these algorithms on the computer algebra system Risa/Asir. They are in the program package `nk_restriction.rr`.

References

[1] G. Almkvist, D. Zeilberger, The method of differentiating under the integral sign, Journal of Symbolic Computation **10**, 571-591, 1990.

[2] F. Chyzak, B. Salvy, Non-commutative Elimination in Ore Algebras Proves Multivariate Holonomic Identities, Journal of Symbolic Computation **26**, 187-227, 1998.

[3] H. Nakayama, K. Nisiyama, An Algorithm of Computing Inhomogeneous Differential Equations for Definite Int egrals, Mathematical Software – ICMS2010, Lecture Notes in Computer Science 6327, Spring er, 2010, 221-232

[4] T. Oaku, Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules, *Advances in Applied Mathematics* **19**, 61–105, 1997.

[5] T. Oaku, Y. Shiraki, N. Takayama, Algebraic Algorithm for D-modules and numerical analysis, Computer mathematics (Proceedings of ASCM 2003), 23–39, Lecture Notes Ser. Comput., 10, World Sci. Publ., River Edge, NJ, 2003.

[6] M. Saito, B. Sturmfels, N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer, 2000.

[7] N. Takayama, An Approach to the Zero Recognition Problem by Buchberger Algorithm, Journal of Symbolic Computation **14**, 265–282, 1992.

[8] M. Noro, et al: Risa/Asir, `http://www.math.kobe-u.ac.jp/Asir`

[9] H. Nakayama, K. Nishiyama: Risa/Asir package `nk_restriction.rr`, `http://www.math.kobe-u.ac.jp/~nakayama/nk_restriction.rr`