Noether's Two Theorems, Moving frames and Symbolic Computation

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Noether's seminal paper in 1918 has an enduring appeal: variational problems and their conservation laws continue to be central in physical applications. Their Lie symmetries arise naturally from the physics, and it appears her theorems are valid far more broadly, but just as powerfully, in all the new kinds of variational problems that are arising.

Noether's First Theorem guarantees conservation laws for smooth variational problems which are invariant under a Lie group action, and typically these laws are of serious physical interest in applications, for example, conservation of linear momentum, angular momentum, energy, potential voticity and so on. The conservation laws are written as divergence expressions which are zero on solutions of the system. Since the formulae for the laws are eye-glazingly recursive and complicated (see Olver [1] for the formulae and historical notes), Noether's laws are eminently suited to implementation in the symbolic computation domain. However, the formulae on their own are opaque in the sense that the resulting expressions can be lengthy and unstructured.

Since the variational principle is invariant under a Lie group, one can use a moving frame á la Fels and Olver to structure the expressions for the laws. This yields the laws as a product of the moving frame, in the adjoint representation of the group on its Lie algebra, and a vector of invariants [2]. Using this structure, a great deal more information about the solution set can be gleaned; in joint work with Tania Gonçalves, problems invariant under $SL(2)$ and $SE(3)$ actions have been successfully studied [3,4].

Noether's Second Theorem guarantees syzygies between the Euler-Lagrange equations for a variational problem invariant under a Lie pseudogroup depending on a single free function. The prototypical examples are gauge-invariant Lagrangians for which one of the Euler-Lagrange equations is a differential consequence of the others.

This Theorem has been considerably extended in joint work with Peter Hydon [5]. The extensions raise some potentially interesting questions in differential algebra, concerning the syzygy modules of differential ideals.

Both of Noether's theorems have been extended to finite difference variational problems, the first by several authors (the most general statement and perhaps the clearest exposition is by Hickman and Hereman [6]), and the second Theorem in [5].

The most interesting extensions at the moment are to finite element variational problems on simplicial domains, and to problems defined on topologically nontrivial domains, such as networks and graphs in the one-dimensional case, to spheres in the two-dimensional case, and to domains with holes in the three-dimensional case. In these latter cases, Noether's conservation laws, which are local laws only, need to be extended. The algebraic construction similar to that of the Čech-de Rham double complex (see [7]) allows this extension to be made. For difference systems defined on topologially non-trivial lattice-like "manifolds", it is conjectured that solutions can be counted in terms of the analogues of Betti numbers [8], but nothing appears to be known about global conservation laws.

In this talk, I will discuss a range of issues in symbolic computation that arise in the above circle of ideas.

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