Classifying regular lattice polytopes via toric fibrations

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Let $P \subset \mathbb{R}^n$ be a convex n-dimensional lattice polytope. The *codegree* $\operatorname{codeg}(P)$ of P is the smallest integer m such that the polytope mP contains interior lattice points. The *degree* d of P is the integer $d = n + 1 - \operatorname{codeg}(P)$. Batyrev and Nill asked whether, given d, there exists an integer $N(d)$ such that every polytope P of degree d and dimension $n \geq N(d)$ is a Cayley polytope. Recently, Haase, Nill, and Payne gave a positive answer to this question by showing that $N(d)$ exists and is bounded by a quadratic expression in d. We propose the following answer in the case where the polytopes are assumed to be regular: $N(d) = 2d + 1$.

We deduce our result from a theorem that characterizes regular lattice polytopes with codegree $\geq \frac{n+3}{2}$: we prove, under an additional assumption, that these are precisely the polytopes affinely equivalent to polytopes of the form $P = \operatorname{Cayley}(P_0, \ldots, P_k)$, where $k = \operatorname{codeg}(P) - 1$, $k > \frac{n}{2}$, and the P_i are polytopes in $\mathbb{R}^{n-k}$.

Our proof relies on the study of the nef value and the nef value map of a nonsingular polarized toric variety (X, L). We define the nef value $\tau(P)$ of a regular polytope and show that it is equal to the nef value of the corresponding toric variety. Moreover, we introduce a refined notion of codegree, $\operatorname{codeg}_{\mathbb{Q}}(P)$, which we relate to the codegree and to the nef value of P.