Classifying smooth lattice polytopes via toric fibrations

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Lattice polytopes

Let $P \subset \mathbb{R}^n$ be a convex lattice polytope, i.e., P is the convex hull of a finite set of lattice points.

The *codegree* of P is defined (Batyrev–Nill) by

$$\operatorname{codeg}(P) = \min_{\mathbb{N}}\{m \,|\, mP \text{ has interior lattice points}\}.$$

For example, $\operatorname{codeg}(\Delta_n) = n+1$ and $\operatorname{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil$.

The *degree* of P is $\deg(P) = n + 1 - \operatorname{codeg}(P)$.

Ehrhart series

Let $f_P(m)$ denote the number of lattice points in mP. The *Ehrhart series* is the generating function

$$F_P(t) = \sum_m f_P(m)t^m = \frac{h_P^*(t)}{(1-t)^{n+1}},$$

where $h_P^*(t)$ is a polynomial of degree d equal to the degree of P.

The normalized volume of P is equal to the sum of the coefficients of h_P^*, and the leading coefficient is the number of lattice points in kP, where k is the codegree.This gives an indication of the combinatorial and computational interest in these notions.

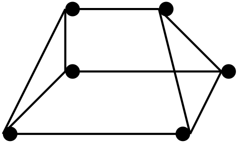
Cayley polytopes

A *Cayley polytope* is a polytope of the form

$$P = \mathrm{Cayley}(P_0, \ldots, P_k),$$

where $k \geq 1$, $P_i = \mathrm{Conv}(\{p_j^i\}) \subset \mathbb{R}^m$, $e_0, \ldots, e_k$ are the vertices of $\Delta_k \subset \mathbb{R}^k$, and $P = \mathrm{Conv}(\{(p_j^i, e_i)\}) \subset \mathbb{R}^{k+m}$.

A *generalized* Cayley polytope is a polytope P^s, where the e_i are replaced by se_i in the definition above.



$Cayley^2(6\Delta_1, 5\Delta_1, 3\Delta_1)$

Batyrev–Nill asked: Given d, does there exist an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Recently, Haase, Nill, and Payne showed that for general polytopes, this holds, with

$$N(d) \leq (d^2 + 19d - 4)/2.$$

We believe that if P is *regular*, then $N(d) = 2d + 1$. We can prove it for regular polytopes satisfying an additional assumption.

(Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.)

Codegree of polytopes

Write

$$P = \cap H^+_{\rho_i,-a_i},$$

where the $H^+_{\rho_i,-a_i}$ are halfplanes defined by the normal vectors ρ_i and the a_i's are integers.

Assume from now on P is regular (aka Delzant, aka smooth), i.e., at any vertex there are precisely n edges, and the first lattice points on these edges form a basis for the lattice.

Set $P^{(r)} = \cap H^+_{\rho_i,-a_i+r}$. Observe that the lattice points of $P^{(1)}$ are the same as the interior lattice points of P. (Note that $P^{(r)}$ does not need to be a lattice polytope, and even if it is, it does not need to be regular nor of the same dimension as P.)

The $\mathbb{Q}$-codegree of P is defined as

$$\operatorname{codeg}_{\mathbb{Q}}(P) = \inf_{\mathbb{Q}}\{\frac{a}{b} \mid (aP)^{(b)} \cap \mathbb{Z}^n \neq \emptyset\}.$$

Note that we have

$$\operatorname{codeg}(P) \geq \operatorname{codeg}_{\mathbb{Q}}(P) \geq \operatorname{codeg}(P) - 1.$$

Example

$\operatorname{codeg}_{\mathbb{Q}}(2\Delta_n) = \frac{n+1}{2}$ and $\operatorname{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil$.

Nef value for polytopes

We say that $P^{(r)}$ is *spanned* if $P^{(r)} \cap \mathbb{Z}^n \neq \emptyset$ and for any vertex $m = (-a_1, \ldots, -a_n)$ (in the lattice basis $\{\rho_1, \ldots, \rho_n\}$) the lattice point $(-a_1 + r, \ldots, -a_n + r)$ is in $P^{(r)}$.

Example

Let P be the polytope obtained from the simplex

$$m\Delta_3 = \mathrm{Conv}\{(0,0,0),(m,0,0),(0,m,0),(0,0,m)\}$$

by removing the simplex

$$\Delta_3 = \mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$$

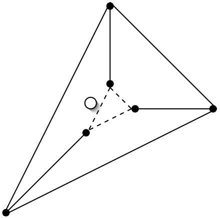
Assume $m \geq 4$. Then $P^{(1)} \cap \mathbb{Z}^n \neq \emptyset$, but $P^{(1)}$ is not spanned.

In fact, the vertex $(1,0,0) = (y = 0) \cap (z = 0) \cap (x + y + z = 1)$ of P "goes to" the lattice point

$$(0,1,1) = (y = 1) \cap (z = 1) \cap (x + y + z = 2)$$

which is not a point in $P^{(1)}$. Similarly for the vertices $(0,1,0)$ and $(0,0,1)$.

Note that if we instead remove $2\Delta_3$, then the polytope is spanned: the vertices $(2,0,0)$, $(0,2,0)$, and $(0,0,2)$ all go to the same lattice point $(1,1,1)$, which is an interior point of the original polytope.



Define the *nef value* of the polytope P to be

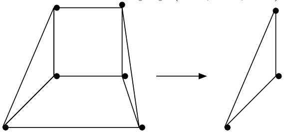
$$\tau(P) = \inf_{\mathbb{Q}}\{\frac{a}{b} \mid (aP)^{(b)} \text{ is spanned }\}.$$

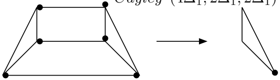
Clearly, $\tau(P) \geq \operatorname{codeg}_{\mathbb{Q}}(P)$, and if $\tau(P)$ is an integer, then $\tau(P) \geq \operatorname{codeg}(P)$.

Example

Take $P = m\Delta_3 \setminus \Delta_3$, $m \geq 4$. Then $\operatorname{codeg}_{\mathbb{Q}}(P) = \operatorname{codeg}(P) = 1$ and $\tau(P) = 2$ (because $P^{(1)}$ is not spanned at the vertices of the small facet).

We believe that when the codegree is big enough, $\tau(P) > \operatorname{codeg}_{\mathbb{Q}}(P)$ cannot happen.





The first polytope P^2 has $n = 3$, $s = 2$, $m = 1$, $k = 2$, and $\mathrm{codeg}_{\mathbb{Q}}(P^2) = \tau(P^2) = \frac{3}{2}$, $\mathrm{codeg}(P^2) = 2$.

The second polytope P^1 has $n = 3$, $s = 1$, $m = 1$, $k = 2$, and $\mathrm{codeg}_{\mathbb{Q}}(P^1) = \mathrm{codeg}(P^1) = \tau(P^1) = 3$.

The theorem

Theorem

Let P be a regular lattice polytope of dimension n, and assume that $\tau(P) = \operatorname{codeg}_{\mathbb{Q}}(P)$. The following are equivalent

(1) $\operatorname{codeg}(P) \geq \frac{n+3}{2}$

(2) $P = \operatorname{Cayley}(P_0, \ldots, P_k)$, *where* $k = \operatorname{codeg}(P) - 1$ *and* $k > \frac{n}{2}$.

Note that a polytope as in (2) is *defective*, with defect $\delta = 2k - n$. (This means that the polarized toric variety (X, L) corresponding to P has defect $\delta = 2k - n$, i.e., its dual variety has codimension $2k - n + 1$.)

Adjunction on toric varieties

Let X be a nonsingular projective variety, L an ample line bundle. Assume the canonical line bundle K_X is not nef. The nef value of (X, L) is

$$\tau_L = \min_{\mathbb{R}}\{t \mid K_X + tL \text{ is nef }\}.$$

It is well known that τ_L is a positive rational number.

Assume X is a toric variety, and let $P = \cap H^+_{\rho_i, -a_i}$ denote the polytope defined by (X, L). Then $L = \sum a_i D_i$, where the D_i are the invariant divisors, and $K_X = -\sum D_i$. On a toric variety, a line bundle is nef if and only if it is spanned (generated by its global sections).

Since $P_{bK_X + aL} = (aP)^{(b)}$, we have $\tau_L = \tau(P)$.

Proof of the theorem

If $\tau = \frac{a}{b}$, the line bundle $bK_X + aL$ defines a morphism from X to a projective space. The Remmert–Stein factorization $\varphi\colon X \to Y$ of this map is called the nef value map.

Lemma

Assume $\tau := \tau(P) = \operatorname{codeg}_{\mathbb{Q}}(P)$*. Then* φ *is not birational.*

It follows from this lemma and $\tau \geq \frac{n+1}{2}$ that there exists a line C (with respect to L) on X which is contracted by φ, i.e., such that $(K_X + \tau L) \cdot C = 0$. It follows that τ is an integer, hence equal to $\operatorname{codeg}(P)$.

By adjunction theory, the inequality $\tau \geq \frac{n+3}{2}$ implies that φ is the contraction of an extremal ray in the nef cone $\operatorname{NE}(X)$.

By a result of Reid, such a contraction is a flat toric fibration, with Y smooth and toric and the general fiber $F = \mathbb{P}^k$, where $k = n - \dim Y$. Hence, $L|_F = \mathcal{O}_{\mathbb{P}}^k(s)$ for some s, and one shows that $s = 1$. This forces all fibers of φ to be $\mathbb{P}^k$, and therefore $X = \mathbb{P}(\varphi_* L)$ is a projective bundle.

To see that $s = 1$, take a line ℓ in $\mathbb{P}^k = F$. We get

$$0 = (K_X + \tau L) \cdot \ell = K_X \cdot \ell + \tau s = -(k+1) + \tau s$$

so that

$$\frac{n+3}{2} \leq \tau = \frac{k+1}{s} \leq \frac{n+1}{s}.$$

So $s = 1$ and $\tau = k + 1$, with $k > \frac{n}{2}$.

Since Y is toric, $\varphi_* L$ splits as a sum of line bundles, and thus φ gives P the structure of a Cayley polytope.

We obtain the following corollary (obtained by Batyrev–Nill in the case of not necessarily regular polytopes).

Corollary

Assume P is a regular n-dimensional lattice polytope.

(1) $\deg(P) = 0$ *if and only if* $P = \Delta_n$,

(2) $\deg(P) = 1$ *if and only if* $P = \mathrm{Cayley}(P_0, \ldots, P_{n-1})$ *is a Lawrence prism (the* P_i *are segments) or* $P = 2\Delta_2$.