Stable Border Bases for Ideals of Points

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SAGA Workshop, Auron, March 2010

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Aim: Look for a common characterization of $\mathcal{I}(\mathbb{X})$ and $\mathcal{I}(\widetilde{\mathbb{X}})$

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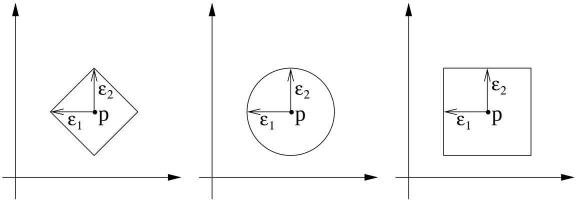
In particular, in the exact case $\mathcal{I}(\mathbb{X})$ is usually computed using the Buchberger-Möller (BM) Algorithm which returns a Gröbner basis of $\mathcal{I}(\mathbb{X})$.
How can we generalize BM in the presence of empirical data?

Empirical point

- **Definition:** An empirical point is a pair (p, ε) where $p \in \mathbb{R}^n$ is the specified value and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ (with each $\varepsilon_i \in \mathbb{R}^+$) is the tolerance.



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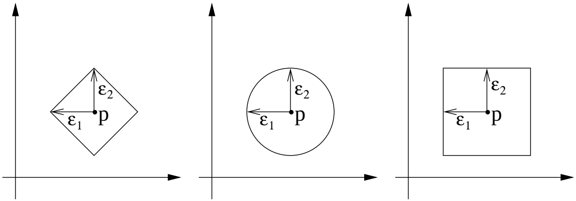

- **Definition:** Any point $\widetilde{p} \in \mathbb{R}^n$ which lies in the ε-neighbourhood of p is called an admissible perturbation of (p, ε).

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- If $\mathcal{O}$ is order ideal, the border $\partial\mathcal{O}$ of $\mathcal{O}$ is defined by

$$\partial\mathcal{O} = (x_1\mathcal{O} \cup \ldots \cup x_n\mathcal{O}) \setminus \mathcal{O}$$

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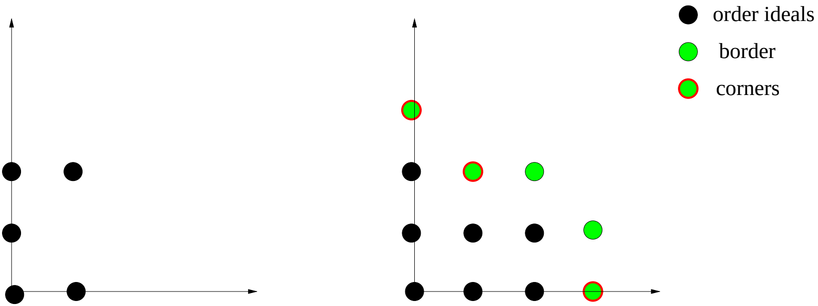
- If $\mathcal{O}$ is order ideal, the elements of the minimal set of generators of the monomial ideal $\mathbb{T}^n \setminus \mathcal{O}$ are called the corners of $\mathcal{O}$.


order ideals
border
corners

Idea of border bases:
describe the quotient ring P/I by an order ideal $\mathcal{O} \subseteq \mathbb{T}^n$ whose residue classes form a K-basis of P/I

Border bases II

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describe the quotient ring P/I by an order ideal $\mathcal{O} \subseteq \mathbb{T}^n$ whose residue classes form a K-basis of P/I

Definition:
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal and $\partial O = \{b_1, \ldots, b_\nu\}$ be its border. Let $\mathcal{B} = \{g_1, \ldots, g_\nu\}$ be a set of polynomials such that

$$g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i \qquad \alpha_{ij} \in K$$

$\mathcal{B}$ is called $\mathcal{O}$-border prebasis of I.
If $\mathcal{B} \subseteq I$ and the residue classes $\overline{\mathcal{O}} = \{\bar{t}_1, \ldots, \bar{t}_\mu\}$ form a K-vector space basis of P/I, then $\mathcal{B}$ is called $\mathcal{O}$-border basis of I.

Border bases III

Proposition (Existence and Uniqueness of Border Bases)
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be a basis of P/I.

- There exists a unique $\mathcal{O}$-border basis $\mathcal{B}$ of I.
- Let $\mathcal{B}$ be an $\mathcal{O}$-border prebasis whose elements are in I. Then $\mathcal{B}$ is the $\mathcal{O}$-border basis of I.

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Proposition (Relation with Gröbner bases)
Let σ be a term ordering on $\mathbb{T}^n$ and $\mathcal{O}_\sigma(I) = \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{\mathrm{I}\}$ order ideal. Then

- there exists a unique $\mathcal{O}_\sigma(I)$-border basis $\mathcal{B}$ of I
- the reduced σ-Gröbner basis of I is the subset of $\mathcal{B}$ corresponding to the corners of $\mathcal{O}_\sigma(I)$

Example (Border basis not containing Gröbner basis)
Let $P = \mathbb{Q}[x, y]$ and

$$I = \langle 4xy - 5y^2 - 6x + 9y,\ x^2 - y^2 - 3x + 3y \rangle$$

Let $\mathcal{O} = \{1, x, y, xy\}$; $\mathcal{O}$ is a basis of P/I, so there exists a unique $\mathcal{O}$-border basis $\mathcal{B}$ of I.

But $\mathcal{B}$ does not arise from any term ordering σ:

- if $x <_\sigma y \Rightarrow x^2 <_\sigma xy \Rightarrow \mathrm{LT}_\sigma(I) = \langle y^2, xy, x^3 \rangle \quad \mathcal{O}_\sigma = \{1, y, x, x^2\}$
  $\mathrm{LT}_\sigma(I) = \langle x^4, y \rangle \quad \mathcal{O}_\sigma = \{1, x, x^2, x^3\}$
- if $y <_\sigma x \Rightarrow y^2 <_\sigma xy \Rightarrow \mathrm{LT}_\sigma(I) = \langle x^2, xy, y^3 \rangle \quad \mathcal{O}_\sigma = \{1, y, x, y^2\}$
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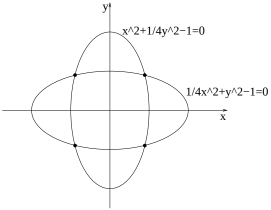
In any case $\mathcal{O}_\sigma(I) \neq \mathcal{O}$

Gröbner bases vs border bases - Example



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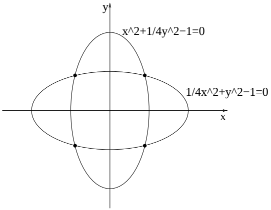

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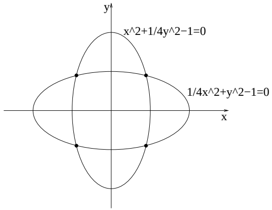

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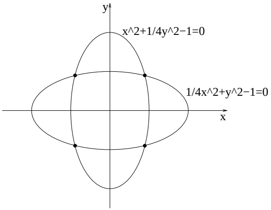

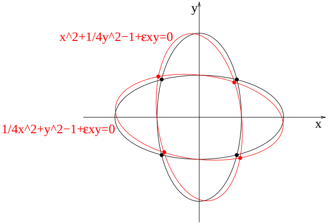

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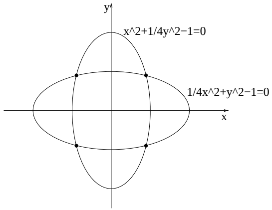

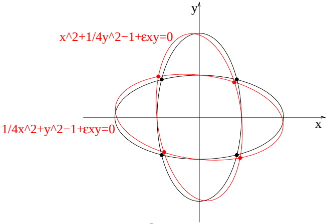

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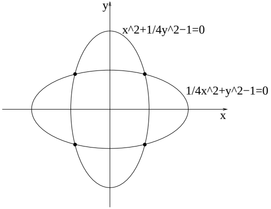

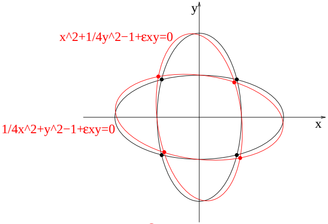

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  σ fixed term ordering
  $g = t - \sum c_i t_i$ added to GB $\Leftrightarrow$ eval. matrix $M_{\mathcal{O}\cup\{t\}}(\mathbb{X})$ rank-deficient
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- Why border bases are MORE STABLE:
  $\mathcal{O}$ basis of $P/\mathcal{I}(\mathbb{X}) \Leftrightarrow$ evaluation matrix $M_{\mathcal{O}}(\mathbb{X})$ non-singular
  $\Leftrightarrow \det(M_{\mathcal{O}}(\mathbb{X})) \neq 0 \Rightarrow$
  $\Rightarrow$ open condition $\Rightarrow$ STABILITY

Stable order ideals and stable border bases

Let $\mathbb{X}^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\}$ finite set of distinct empirical points of $\mathbb{R}^n$
$\widetilde{\mathbb{X}} = \{\widetilde{p}_1, \ldots, \widetilde{p}_s\}$ admissible perturbation of $\mathbb{X}^\varepsilon$
$\mathcal{O} = \{t_1, \ldots, t_k\} \subseteq \mathbb{T}^n$ order ideal, $t \in \mathbb{T}^n$

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Proposition If $\mathcal{O}$ is stable contains exactly $\#\mathbb{X}$ terms then

- $\mathcal{O}$ is a basis of the quotient ring $P/\mathcal{I}(\mathbb{X})$
- there is an $\mathcal{O}$-border basis $\widetilde{B}$ for each perturbed ideal $\mathcal{I}(\widetilde{\mathbb{X}})$
- the $\mathcal{O}$-border basis $\mathcal{B}$ of $\mathcal{I}(\mathbb{X})$ exists, and is called stable

How to get stable order ideals?

We generalize the Buchberger-Möller Algorithm

- **Main idea of BM Algorithm:**
  check the linear dependence of the vectors $t(\mathbb{X}), t_1(\mathbb{X}), \ldots, t_k(\mathbb{X})$

How to get stable order ideals?

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- **Main idea of BM Algorithm:**
  check the linear dependence of the vectors $t(\mathbb{X}), t_1(\mathbb{X}), \ldots, t_k(\mathbb{X})$
- **Main idea of new numerical algorithms:**
  check the numerical linear dependence of the above set of vectors, that is check if there exists an admissible perturbation $\widetilde{\mathbb{X}}$ of $\mathbb{X}^\varepsilon$ such that the vectors

  $$t(\widetilde{\mathbb{X}}), t_1(\widetilde{\mathbb{X}}), \ldots, t_k(\widetilde{\mathbb{X}})$$

  are linearly dependent.
  Numerical technique used: analyze the residual $\rho(\widetilde{\mathbb{X}})$, that is the component of $t(\widetilde{\mathbb{X}})$ orthogonal to the vector space spanned by the columns of $M_{\mathcal{O}}(\widetilde{\mathbb{X}})$.

The Stable Order Ideal Algorithm

Let σ be a term ordering on $\mathbb{T}^n$ and let $\mathbb{X}^\varepsilon = \{p_1^\varepsilon, \dots, p_s^\varepsilon\}$ be a finite set of distinct empirical points, with $\mathbb{X} \subset \mathbb{R}^n$ and a common tolerance $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. Let $\mathbf{e} = (e_{11}, \dots, e_{sn})$ be the error variables whose constraints are given by $\|(e_{k1}, \dots, e_{kn})\| \leq 1$ for each k. Consider the following sequence of instructions.

S1 Start with the lists $\mathcal{O} = [1]$, $L = [x_1, \dots, x_n]$, the empty list $C = [\,]$, and the matrices $M_0 \in \mathrm{Mat}_{s,1}(\mathbb{R})$ with all the elements equal to 1, and $M_1 \in \mathrm{Mat}_{s,1}(R)$ with all the elements equal to 0.

S2 If $L = [\,]$ then return the set $\mathcal{O}$ and stop. Otherwise let $t = \min_\sigma(L)$ and delete it from L.

S3 Let v_0 and v_1 be the homogeneous components of degrees 0 and 1 of the evaluation vector $v = t(\widetilde{\mathbb{X}}(\mathbf{e}))$. Solve up to first order the least squares problem $M_{\mathcal{O}}(\widetilde{\mathbb{X}}(\mathbf{e}))\, \alpha(\mathbf{e}) \approx v$, by computing the vectors

$$\begin{aligned} \rho_0 &= v_0 - M_0\alpha_0 \\ \rho_1 &= v_1 - M_0\alpha_1 - M_1\alpha_0 \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= (M_0^t M_0)^{-1} M_0^t v_0 \\ \alpha_1 &= (M_0^t M_0)^{-1}(M_0^t v_1 + M_1^t v_0 - M_0^t M_1 \alpha_0 - M_1^t M_0 \alpha_0). \end{aligned}$$

S4 Let $C_t \in \mathrm{Mat}_{s,sn}(\mathbb{R})$ be such that $\rho_1 = C_t\mathbf{e}$. Compute the minimal 2-norm solution $\hat{\mathbf{e}}$ of the underdetermined system $C_t\mathbf{e} = -\rho_0$.

S5 If $\|\hat{\mathbf{e}}\| > \sqrt{s}\|\varepsilon\|$ then adjoin the vector v_0 as a new column of M_0 and the vector v_1 as a new column of M_1. Append the power product t to $\mathcal{O}$, and add to L those elements of $\{x_1 t, \dots, x_n t\}$ which are not multiples of an element of L or C. Continue with step S2.

S6 Otherwise append t to the list C, and remove from L all multiples of t. Continue with step S2.

The Stable Order Ideal (SOI) Algorithm

Input: a finite set $\mathbb{X}^{\varepsilon}$

$\Downarrow$

SOI

$\Downarrow$

Output: stable order ideal $\mathcal{O}$

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SOI

- Parametrizes the empirical points
- At each step it studies $\rho(\widetilde{\mathbb{X}}) =$ component of $t(\widetilde{\mathbb{X}})$ orthogonal to $M_{\mathcal{O}}(\widetilde{\mathbb{X}})$
- Performs a first order error analysis

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Note that:

- once $\mathcal{O}$ stable and $\#\mathcal{O} = s$, then $\mathcal{O}$-border basis $\mathcal{B}$ of $\mathcal{I}(\mathbb{X})$ is simply computed via linear algebra
- as a by-product a set of almost vanishing polynomials (polynomials whose evaluation at the points is minimum) is returned
- algorithm **SOI** is implemented in CoCoA with the name **StableBBasis5**

Example of two conics

Example: The original two conics:

$$\begin{cases} x^2 + \frac{1}{4}y^2 - 1 = 0 \\ \frac{1}{4}x^2 + y^2 - 1 = 0 \end{cases}$$

intersect at the points

$$\mathbb{Y} = \left\{ \left(\sqrt{\frac{4}{5}}, \sqrt{\frac{4}{5}}\right), \left(\sqrt{\frac{4}{5}}, -\sqrt{\frac{4}{5}}\right), \left(-\sqrt{\frac{4}{5}}, \sqrt{\frac{4}{5}}\right), \left(-\sqrt{\frac{4}{5}}, -\sqrt{\frac{4}{5}}\right) \right\}$$

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We consider the new set of points:

$$\mathbb{X} = \left\{ \left(\frac{10}{13}, \frac{10}{13}\right), \left(\frac{10}{9}, -\frac{10}{9}\right), \left(-\frac{10}{9}, \frac{10}{9}\right), \left(-\frac{10}{13}, -\frac{10}{13}\right) \right\}$$

which are the solutions of:

$$\begin{cases} x^2 + \frac{1}{4}y^2 - 1 + \frac{11}{25}xy = 0 \\ \frac{1}{4}x^2 + y^2 - 1 + \frac{11}{25}xy = 0 \end{cases}$$

- We compute Gröbner basis $\mathcal{G}$ of $\mathcal{I}(\mathbb{X})$

$$\mathcal{G} = \begin{cases} xy + \frac{125}{44}y^2 - \frac{25}{11} \\ x^2 - y^2 \\ y^3 + 4400/13689x - 12500/13689y \end{cases}$$

and so

$$\mathrm{LT}(\mathcal{I}(\mathbb{X})) = \{xy, x^2, y^3\} \qquad \mathcal{O}_{\mathcal{G}} = \{1, y, x, y^2\}$$

Note that $\mathcal{O}_{\mathcal{G}}$ is not stable ($\mathcal{O}_{\mathcal{G}}$ is not a basis of $P/\mathcal{I}(\mathbb{Y})$)

Computations with CoCoA II

- Using the function **StableBBasis5**`(Points:LIST, Toler:LIST)` w.r.t. tolerance $\varepsilon = (0.25, 0.25)$ we obtain the stable order ideal

$$\mathcal{O} = \{1, y, x, xy\}$$

whose border is $\partial\mathcal{O} = \{y^2, xy^2, x^2y, x^2\}$, and the $\mathcal{O}$-stable border basis $\mathcal{B}$ of $\mathcal{I}(\mathbb{X})$

$$\mathcal{B} = \begin{cases} y^2 + \frac{44}{125}xy - \frac{4}{5} \\ xy^2 - \alpha x + \beta y \\ x^2y + \beta x - \alpha y \\ x^2 + \frac{44}{125}xy - \frac{4}{5} \end{cases}$$

where

$$\alpha = \frac{913141938782964423990065015706041347067}{1000000000000000000000000000000000000000} \approx 0.91314$$

$$\beta = \frac{40178245306450434655562860691065819271}{125000000000000000000000000000000000000} \approx 0.32142$$

Remarks:

- almost vanishing polynomials minimize the sum of squared evaluations at $\mathbb{X}$
- almost vanishing polynomials do not minimize the squared distances from $\mathbb{X}$

Future work:

- with similar techniques compute varieties lying close to the points $\mathbb{X}$
- use these polynomials to compute a border basis of a perturbed set of points $\widetilde{\mathbb{X}}$

Thank you!

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