Distance Computation to the Resultant Variety

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The problem

Let f,g two univariate polynomials aznd $R(f,g)$ their resultant. We want to compute two polynomials f^* and g^* such that for all polynomial u,v such that $R(u,v) = 0$ we have :

$$||f - f^*||_2^2 + ||g - g^*||_2^2 \leq ||f - u||_2^2 + ||g - v|_2^2 := F(f,g;u,v).$$

Remark The distance $F(f,g;f^*,g^*)$ is greater than the distance of the Sylvester matrix to the variety of the singular matrix.

Example

Let

$f = x^2 - 6x + 5,\ g = x^2 - 6.3x + 5.72$

then

$f^* = 0.985x^2 - 6.0029x + 4.9994,\ g^* = 1.015x^2 - 6.2971x + 5.7206$

and

$$F(f, g; f^*, g^*) = 0.0305.$$

But

$$dist(S(f, g), \Sigma) = \sigma_{min} = 0.1052 < F(f, g; f^*, g^*).$$

A theoretical computation of the distance from a point to an algebraic hypersurface 1

J.P Dedieu, X. Gourdon, J.-C. Yak. (1995)
Let $x \in \mathbb{C}^n$ and $V = \{z : R(z) = 0\}$ and $degree(R) = d$. Let

$$R^{[N]}(z) := \prod_{j=0}^{2^N-1} R(\omega^j z) = \sum_{k=0}^{d} R_k^{[N]}(z), \quad \omega = e^{2i\pi/N},$$

where the $R_k^{[N]}(z)$'s are homogeneous polynomial of degree $2^N k$. Let ρ_N the first positive root of the polynomial:

$$|R^{[N]}(x)| - \sum_{k=1}^{d} ||R_k^{[N]}(x)||t$$

where $||R_k^{[N]}(z)||$ are the sum of absolute value of the coefficients.

A theoretical computation of the distance from a point to an algebraic hypersurface 2

Then we have

$$\rho_N^{2^{-N}} \leq \mathrm{distance}(x, V) \leq c_N \rho_N^{2^{-N}}$$

where

$$c_n = \left(\frac{1}{2^{1/d} - 1} \sqrt{\binom{2^N + n - 1}{n - 1}} \right)^{2^{-N}} \to 1 \text{ when } N \to \infty$$

A theoretical computation of the distance from a point to an algebraic hypersurface 3

But the number of coefficients of each homogeneous polynomials $F_k^{[N]}(z)$ is

$$\binom{n+2^N k-1}{n-1}$$

. Consequently the complexity of the computation is too huge to perform efficiently the distance computation.

Problem Formulation

Let an hypersurface $V = \{z \, : \, R(z) = 0\}$ with $degree(F) = d$ and a point $x \in \mathbb{C}^n$. We consider the minimization problem

$$\min_{y \in V} ||y - x||^2 \qquad (1)$$

The first order necessary condition for a regular point $x^* \in V$ to be a local minimum of 1 is there exist λ S.t. :

$$x^* - x + \lambda \nabla R(x^*) = 0, \quad (CO1)$$

The second order necessary condition is the matrix

$$I + H_R(x^*)$$

semi-definite on the tangent plane $T_{x^*}V$.
The second order sufficient for a regular point x^* which satisfies $C01$ to be a strict local minimum of 1 is

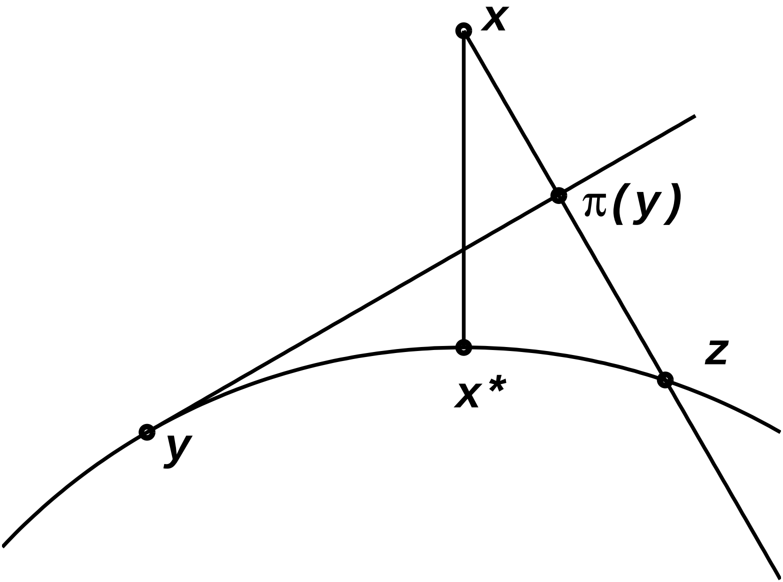
$$I + H_R(x^*)$$

definite positive on the tangent plane $T_{x^*}V$.

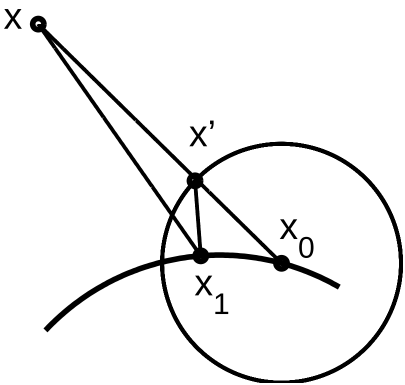
Tir and Projection Algorithm 1

1. Inputs $x \in K^n$, $u_0 \in S^n$, $\epsilon > 0$
2. compute $\lambda_0 \in K$ such that
   $|\lambda_0| = \min\{|\lambda| \,:\, g_0(\lambda) := R(x + \lambda u_0) = 0\}$.
3. Put $x_0 = x + \lambda_0 u_0$
4. Step $k \geq 1$: let $u_k = \frac{\nabla R(x_{k-1})}{||\nabla R(x_{k-1})||}$
5. compute $\lambda_k \in K$ such that
   $|\lambda_k| = \min\{|\lambda| \,:\, g_k(\lambda) := R(x + \lambda u_k) = 0\}$
6. Put $x_k = x + \lambda_k u_k$.
7. If $||x_k - x_{k-1}|| \leq \epsilon$, stop, else $k = k + 1$
8. Output x_k

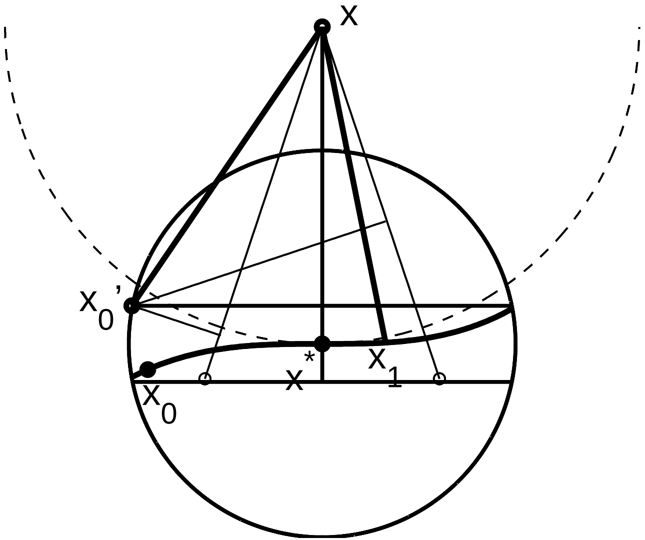
Tir and Projection Local Algorithm 2



Tir and Projection Global Algorithm 2



Convergence of Tir and Projection Local Algorithm :γ-theorem



Convergence of Tir and Projection Local Algorithm : α-theorem