AIDA: Algebraic Invariants and their Differential Algebras

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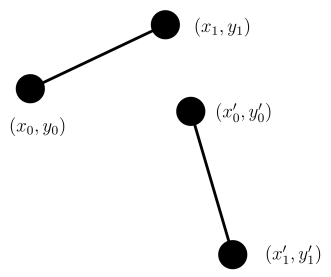
Whether algebraic or differential, one can distinguish two families of computational applications for invariants of group actions: equivalence problems and symmetry reduction. Both applications raise algorithmic issues: can we compute a generating set of invariants and can we determine their syzygies, i.e. the relationships the generating set satisfies.

AIDA is a Maple package to compute generating sets of *differential* invariants and their *differential* syzygies for any given group action. It works on top of *DifferentialGeometry*, *Groebner*, and *diffalg* for non commuting derivations. This poster presents classical examples as introduction, theoretical foundations of the package, and original applications.

Equivalence

Are two objects identical under the action of a group element?

Equivalence of pairs of points

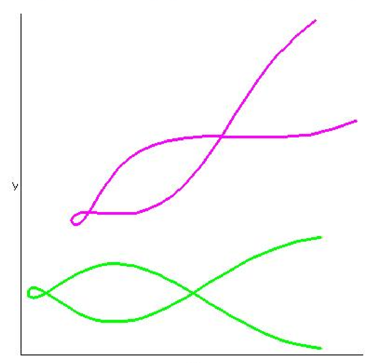


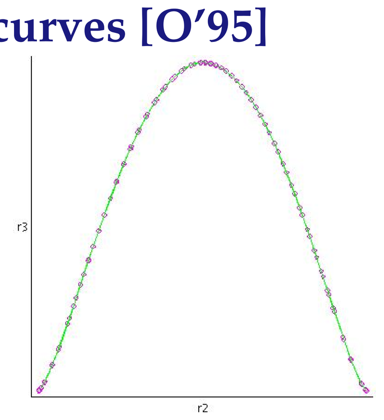
$$\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \epsilon\beta & \epsilon\alpha \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\alpha^2+\beta^2=1,\ \epsilon^2=1$$

$$(x_1-x_0)^2+(y_1-y_0)^2 = (x_1'-x_0')^2+(y_1'-y_0')^2$$

Equivalence of curves $\rightsquigarrow$ signature curves [O'95]





(x, y) $\qquad$ $(\sigma, \frac{d\sigma}{ds})$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \epsilon\beta & \epsilon\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \qquad Y_X = \epsilon\frac{\beta+\alpha y_x}{\alpha-\beta y} \qquad Y_{XX} = \epsilon\frac{y_{xx}}{(\alpha-\beta y)^3}$$

$\alpha^2+\beta^2=1,\ \epsilon^2=1$ $\qquad$ $\rightsquigarrow$ rational action

Differential invariant $\sigma = \sqrt{\frac{y_{xx}^2}{(1+y_x^2)^3}}$ $\qquad$ the curvature

Invariant derivation: $\frac{1}{\sqrt{1+y_x^2}}\frac{d}{dx} = \frac{d}{ds}$ $\qquad$ s the arc length

$\rightsquigarrow$ algebraic functions

Symmetry reduction

- *group of symmetry*
- *generating invariants*
- *relations among those*
- *rewriting algorithm*
- *reduced problem solving*
- *solution lifting*

Biological models redimensioning with A.Sedoglavic

Prey-predator model [Murray, 2002]

$$\begin{cases} \dot n = \left(\left(1-\frac{n}{k_1}\right) r - k_2\frac{p}{n+e}\right) n, \\ \dot p = s\left(1-h\frac{p}{n}\right) p. \end{cases} \rightsquigarrow \begin{cases} \dot{\mathfrak{n}} = \left(1-\mathfrak{n}-\frac{\mathfrak{p}}{\mathfrak{n}+\mathfrak{e}}\right)\mathfrak{n}, \\ \dot{\mathfrak{p}} = \mathfrak{s}\left(1-\mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}}\right)\mathfrak{p}. \end{cases}$$

r, s, e, h, k_1, k_2 parameters. $\qquad$ $\mathfrak{s}, \mathfrak{e}, \mathfrak{h}$ parameters

where

$$\mathfrak{t} = r\,t,\ \mathfrak{n} = \frac{n}{k_1},\ \mathfrak{p} = \frac{k_2 p}{k_1 r},\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{e} = \frac{e}{k_1},\ \mathfrak{h} = \frac{r\,h}{k_2}.$$

are the invariants of the symmetry group

$$\begin{array}{lll} t \to \lambda^{-1} t, & r \to \lambda r, & h \to \nu h, \\ n \to \mu n, & s \to \lambda s, & k_1 \to \mu k_1, \\ p \to \mu\nu^{-1} p, & e \to \mu e, & k_2 \to \lambda\nu k_2 \end{array}$$

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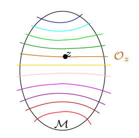
Local invariants of a group action [HK07]

Semi-regular Lie group action

$$g: \mathcal{G}\times\mathcal{M} \to \mathcal{M} \qquad (\lambda\cdot\mu)\star z = \lambda\star(\mu\star z)$$
$$(\lambda, z) \mapsto \lambda\star z$$

$\mathcal{G}$ a r-dimensional Lie group, $\mathcal{M}$ a n-dimensional manifold

Orbit of z: $\mathcal{O}_z = \{\lambda\star z \mid \lambda\in\mathcal{G}\} \subset \mathcal{M}$



Semi-regularity: the orbits have dimension d.

Lie algebra: $v_1, \ldots, v_r$ right invariant vector fields on $\mathcal{G}$

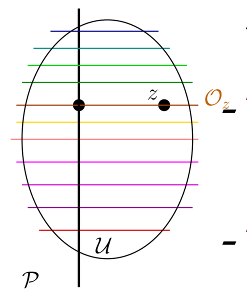
Maurer-Cartan forms: $\omega_1, \ldots, \omega_r$ their duals

Infinitesimal generators: $V_1, \ldots, V_r$ vector fields on $\mathcal{M}$

Local invariant: $f: \mathcal{M}\to\mathbb{R}$ smooth s.t.

$V_1(f)=0, \ldots, V_r(f)=0 \Leftrightarrow f(\lambda\star z) = f(z)$ for $\lambda\in\mathcal{G}_e$

Local cross-section $\mathcal{P}$



- $\mathcal{P}$ an embedded manifold of dimension $n-d$
  $\mathcal{P} = \{z\in\mathcal{M} \mid p_1(z) = \ldots = p_s(z) = 0\}$
- $\mathcal{P}$ is transverse to $\mathcal{O}_z^0$ at $z\in\mathcal{P}$.
  $V(P) = (V_i(p_j))_{ij}$ has rank s at $z_0\in\mathcal{P}$
- $\mathcal{P}$ intersect $\mathcal{O}_z^0$ at a unique point, $\forall z\in\mathcal{M}$.
  $\mathcal{O}_z^0$ is the connected component of $\mathcal{O}_z$.

Local invariants $\cong$ smooth functions on the cross-section.

Invariantization $\bar\iota f$ of a smooth function f

$f: \mathcal{M}\to\mathbb{R}$ smooth $\qquad$ $\bar\iota f(z) = f\left(\mathcal{O}_z^0\cap\mathcal{P}\right)$

$\bar\iota f$ is the unique *local* invariant with $\bar\iota f|_{\mathcal{P}} = f|_{\mathcal{P}}$

Cartan's normalized invariants: $\bar\iota z_1, \ldots, \bar\iota z_n$

Thm: for f local invariant $f(z_1,\ldots,z_n) = f(\bar\iota z_1,\ldots,\bar\iota z_n)$

$\Rightarrow$ Cartan's normalized invariants form a generating set

Thm: a maximally independent set of relations is

$$p_1(\bar\iota z_1,\ldots,\bar\iota z_n) = 0, \ldots, p_r(\bar\iota z_1,\ldots,\bar\iota z_n) = 0$$

Moving frame map [FO'99]

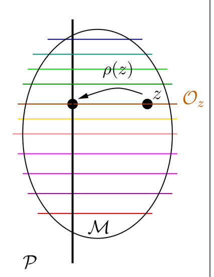
$\rho: \mathcal{U}\to\mathcal{G}$ $\quad$ equivariant $\quad$ $\rho(\lambda\star z) = \rho(z)\cdot\lambda^{-1}$

$\Rightarrow \qquad z\mapsto\rho(z)\star z \quad$ invariant

Locally free action: $\mathcal{G}_z = \{\lambda \mid \lambda\star z = z\}$ finite

The equation $\rho(z)\star z\in\mathcal{P}$ defines a m.f. map

Thm: $\bar\iota f(z) = f(\rho(z)\star z)$



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Action on jet spaces

$\mathcal{J}^0 = \mathcal{X}\times\mathcal{U}$

$(x_1,\ldots,x_m)$ coordinates on $\mathcal{X}$; $\qquad$ independent variables

$(u_1,\ldots,u_n)$ coordinate on $\mathcal{U}$; $\qquad$ dependent variables

$(u_\alpha \mid \alpha\in\mathbb{N}^m)$ other coordinates on $\mathcal{J}^\infty$; $\qquad$ derivatives

Total derivations:

$D = (D_1,\ldots,D_m)^t$ s.t. $D_i(x_j) = \delta_{ij}$, $D_i(u_\alpha) = u_{\alpha+\epsilon_i}$

$g: \mathcal{G}\times\mathcal{J}^0\to\mathcal{J}^0$ $\quad$ prolonged to $\quad$ $g: \mathcal{G}\times\mathcal{J}^k\to\mathcal{J}^k$

by $g^*\circ D = \tilde D\circ g^*$ where $\tilde D = A^{-1} D$ with $A = (D_i(g^* x_j))_{ij}$

$g: \mathcal{G}\times\mathcal{J}^0\to\mathcal{J}^0$ effective on subset $\Rightarrow$

$\dim\mathcal{O}^0 \le \dim\mathcal{O}^1 \le \ldots \dim\mathcal{O}^s = \dim\mathcal{O}^{s+1} = r$

Stabilisation order : s. $\qquad$ The action on $\mathcal{J}^{s+k}$ is locally free.

Differential invariant of order k:

$f: \mathcal{J}^k\to\mathbb{R}$ smooth s.t. $V_1^k(f) = 0, \ldots, V_r^k(f) = 0$

$\mathcal{P}$ a cross-section to the orbits on $\mathcal{J}^s$, and therefore on $\mathcal{J}^{s+k}$.

$$\mathcal{P}: p_1(z) = 0, \ldots, p_r(z) = 0$$

$\rho: \mathcal{J}^s\to\mathcal{G}$ the associated moving frame map: $\rho(z)\star z\in\mathcal{P}$

$$\sigma: \mathcal{J}^{s+k}\to\mathcal{G}\times\mathcal{J}^{s+k}$$
$$z\mapsto(\rho(z), z)$$

Invariant derivations

$\mathfrak{D} = (\mathfrak{D}_1,\ldots,\mathfrak{D}_m)^t$ s.t. $V_a\circ\mathfrak{D}_i = \mathfrak{D}_i\circ V_a$

f a diff. inv. of order $k \Rightarrow \mathfrak{D}_i(f)$ a diff. inv. of order $k+1$, $k >> 0$.

Classical construction: $\mathfrak{D} = A^{-1}D$, $A = (D_i(f_j))_{i,j}$, $f_1,\ldots,f_m$ diff. inv.

[F0'99] : $\mathfrak{D} = (\sigma^* A)^{-1} D$ where $A = (D_i(g^* x_j))_{i,j}$

Thm: $\mathfrak{D}(\bar\iota f) = \bar\iota(Df) - K\,\bar\iota(V(f))$ where $K = \bar\iota D(P)\,V(P)^{-1}$

$D(P) = (D_i(p_j))_{ij}$, $V(P) = (V_i(p_j))_{ij}$

Cor: f a diff. inv. $\Rightarrow \mathfrak{D}(f) = \bar\iota(D(f))$

Prop: $\qquad \left[\mathfrak{D}_i, \mathfrak{D}_j\right] = \sum_{k=1}^m \Lambda_{ijk}\mathfrak{D}_k$

where $\Lambda_{ijk} = \sum_{c=1}^d K_{ic}\,\bar\iota(D_j(\xi_{ck})) - K_{jc}\,\bar\iota(D_i(\xi_{ck}))$, $\quad \xi_{ck} = V_c(x_k)$

Cartan Normalized invariants

$\mathfrak{I}^k = \{\bar\iota x_1,\ldots,\bar\iota x_m\}\cup\{\bar\iota u_\alpha \mid |\alpha|\le k\}$

$\mathfrak{I}^{s+k}$ a generating set of differential invariants of order $s+k$.

CN Thm: $\mathfrak{I}^{s+1}$ is a differentially generating set of differential invariants and a complete set of syzygies is given by

$$\mathfrak{D}_i(x_j) = \delta_{ij} + \textstyle\sum_{a=1}^r K_{ia}\,\bar\iota(V(x_j))$$
$$\mathfrak{D}_i(\bar\iota u_\alpha) = \bar\iota u_{\alpha+\epsilon_i} + \textstyle\sum_{a=1}^r K_{ia}\,\bar\iota(V(u_\alpha)),\ |\alpha|\le s$$
$$\mathfrak{D}_i(\bar\iota u_\alpha) - \mathfrak{D}_j(\bar\iota u_\beta) = \textstyle\sum_{a=1}^r K_{ia}\,\bar\iota(V(u_\alpha)) - K_{ja}\,\bar\iota(V(u_\beta))$$
$$\alpha+\epsilon_i = \beta+\epsilon_j,\ |\alpha| = |\beta| = s+1.$$

MO Thm: If $\mathcal{P}$ is a cross-section of *minimal order* then $\mathfrak{I}^0\cup\{\bar\iota(D_i(p_j))\}$ is a differentially generating set of differential invariants.

Maurer-Cartan invariants

$\mathfrak{K}$ the set of entries K_{ia} of the $m\times r$ matrix $K = \bar\iota D(P)V(P)$.

Thm: $\rho^*\omega = -K^T\mathfrak{x} \mod \Theta$

where Θ contact ideal, $\mathfrak{x} = (\mathfrak{x}_1,\ldots,\mathfrak{x}_m)$ duals of $\mathfrak{D}$

Structure equations: $d\omega_c = -\sum_{a<b}\gamma_{abc}\,\omega_a\wedge\omega_b$

$\Rightarrow d\mu_c = -\sum_{a<b}\gamma_{abc}\,\mu_a\wedge\mu_b$, where $\mu = \rho^*\omega$

MC Thm: $\mathfrak{K}$ is a differentially generating set of differential invariants, if the action is transitive on $\mathcal{J}^0$. A complete set of syzygies is given by

$$\mathfrak{D}_j(K_{ic}) - \mathfrak{D}_i(K_{jc}) = \sum_{1\le a<b\le r}\gamma_{abc}\left(K_{ja}K_{ib} - K_{ia}K_{jb}\right) - \sum_{k=1}^m \Lambda_{ijk}K_{kc}$$

Differential invariants for surfaces with P.Olver

With the infinitesimal generators of the group action and a choice of cross-section as data we show that the algebra of differential invariants is generated by a single differential invariants.

The invariant derivations satisfy $[\mathfrak{D}_1, \mathfrak{D}_2] = \phi\,\mathfrak{D}_1 - \psi\,\mathfrak{D}_2$

By differential elimination [H'05] we show that the invariants of theorem MO or MC can be written in terms of the monotone derivatives of ϕ and ψ.

Observe then that $\psi = \frac{1}{\mathfrak{D}_2(\phi)}(\phi\,\mathfrak{D}_1(\phi) - \mathfrak{D}_1\mathfrak{D}_2(\phi) + \mathfrak{D}_2\mathfrak{D}_1(\phi))$.

Conformal geometry

The infinitesimal generators for this action of SO$(4,1)$ are

$$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial u} - u\frac{\partial}{\partial y},\ (x^2-y^2-u^2)\frac{\partial}{\partial x} + 2\,x\,y\frac{\partial}{\partial y} + 2\,x\,u\frac{\partial}{\partial u},$$
$$\frac{\partial}{\partial y},\ x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x},\ 2\,x\,y\frac{\partial}{\partial x} + (x^2-y^2-u^2)\frac{\partial}{\partial y} + 2\,y\,u\frac{\partial}{\partial u},$$
$$\frac{\partial}{\partial u},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\quad 2\,x\,u\frac{\partial}{\partial u} + 2\,y\,u\frac{\partial}{\partial y} + (x^2-y^2-u^2)\frac{\partial}{\partial u}.$$

Cross-section: $x=y=u_{00}=u_{10}=u_{01}=u_{20}-1=u_{11}=u_{02}=u_{21}=u_{12}=0$

Maurer-Cartan matrix: $K = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -\psi & \sigma & \kappa & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \phi & \tau & -\sigma & -\frac12\phi \end{pmatrix}$

where $\phi = \bar\iota u_{03}$, $\psi = \bar\iota u_{30}$, $\tau = \frac12\bar\iota u_{13}$, $\kappa = -\frac12\bar\iota u_{31}$, $\sigma = \frac12\bar\iota u_{22}$

Syzygies: $\tau_{00} - \frac12\phi_{10} = \psi_{00}\phi_{00}$, $\sigma_{10}+\kappa_{01} = 2\,\phi_{00}\kappa_{00} - 2\,\psi_{00}\sigma_{00}$

$\phi_{10}+\psi_{01} = 2\,\tau_{00} - 2\,\kappa_{00}$, $\sigma_{01}-\tau_{10} = \frac12\phi_{00} + 2\,\phi_{00}\sigma_{00} + 2\,\psi_{00}\tau_{00}$, .

By differential elimination on the syzygies we obtain:

$$\tau = \tfrac12\phi_{10} + \psi_{00}\phi_{00} \qquad \kappa = \psi_{00}\phi_{00} - \tfrac12\psi_{01}$$

$$\sigma = \frac{\tau_{20}+\kappa_{02}}{4(\kappa_{00}-\tau_{00})} + \frac{5(\psi_{00}\tau_{10}-\phi_{00}\kappa_{01}) + 2(\psi_{10}\tau_{00}-\phi_{01}\kappa_{00})}{4(\kappa_{00}-\tau_{00})} + \frac{12(\phi_{00}{}^2\kappa_{00}+\psi_{00}{}^2\tau_{00}) + \psi_{00}\phi_{00} + 2\,\tau_{00}}{8(\kappa_{00}-\tau_{00})}$$

Projective geometry

The infinitesimal generators of the action are:

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial u},\ x\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ u\frac{\partial}{\partial x},\ x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial y},\ u\frac{\partial}{\partial y},\ x\frac{\partial}{\partial u},\ y\frac{\partial}{\partial u},\ u\frac{\partial}{\partial u},$$
$$x^2\frac{\partial}{\partial x} + x\,y\frac{\partial}{\partial y} + x\,u\frac{\partial}{\partial u},\quad x\,y\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + y\,u\frac{\partial}{\partial u},\quad x\,u\frac{\partial}{\partial x} + y\,u\frac{\partial}{\partial y} + u^2\frac{\partial}{\partial u}.$$

Minimal order cross-section:

$x=0,\ y=0,\ u_{00}=0,\ u_{10}=0,\ u_{01}=0,\ u_{20}=0,\ u_{11}=1,\ u_{02}=0,$
$u_{30}=1,\ u_{21}=0,\ u_{12}=0,\ u_{03}=1,\ u_{31}=0,\ u_{22}=0,\ u_{13}=0.$

Then $\quad K = \begin{pmatrix} 1 & 0 & 0 & 2\psi & 0 & \kappa & -\frac12 & \psi & \tau & 0 & 1 & 3\psi & -\tau & \frac14-\kappa & \frac12\sigma+\frac38\psi \\ 0 & 1 & 0 & \phi & -\frac12 & \sigma & 0 & 2\phi & \eta & 1 & 0 & 3\phi & \frac14-\eta & -\sigma & \frac38\phi+\frac12\tau \end{pmatrix}$

where $\phi = -\frac13 u_{04}$, $\psi = -\frac13 u_{40}$, $\eta = -\frac12 u_{14} - \frac14$, $\tau = \frac14 u_{04} - \frac12 u_{23}$,
$\sigma = \frac14 u_{40} + \frac14 u_{40} - \frac12 u_{32}$, $\kappa = -\frac12 u_{41} - \frac14$

Syzygies:

$\phi_{10} - 2\,\psi_{01} = \frac12 - 2\,\eta + \phi\,\psi$, $\sigma_{10} - \kappa_{01} = \frac38\phi + 3\,\phi\,\kappa + 2\,\psi\,\sigma$
$2\,\phi_{10} - \psi_{01} = 2\,\kappa + \phi\,\psi - \frac12$, $\eta_{10} - \tau_{01} = -3\,\psi\,\eta - \frac38\psi - 2\,\phi\,\tau$
$\frac12\tau_{10} - \frac12\sigma_{01} + \frac38\phi_{10} - \frac38\psi_{01} = \frac14\kappa - \frac14\eta - 2\,\phi\,\sigma + 2\,\psi\,\tau$

From which: $\eta = \frac14 - \frac12\phi_{10} + \psi_{01} - \frac12\phi_{00}\psi_{00}$, $\kappa = \frac14 + \phi_{10} - \frac12\psi_{01} - \frac12\phi_{00}\psi_{00}$

A challenging differential elimination with non-commuting derivations produces τ and σ in terms of ψ and ϕ and their monotone derivatives.