Linear Syzygies and Implicitization

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INRIA, November 20th, 2007

Plane curve implicitization

• Suppose given a parametrization ($\mathbb{K}$ is a field)

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{K}} & \xrightarrow{\phi} & \mathbb{P}^2_{\mathbb{K}} \\ (X_1 : X_2) & \mapsto & (f_1 : f_2 : f_3)(X_1 : X_2) \end{array}$$

of a plane curve $\mathcal{C}$ in $\mathbb{P}^2$. Set $d := \deg(f_i) \geq 1$.

• **Implicitization**: find a (homogeneous) polynomial $P \in \mathbb{K}[T_1, T_2, T_3]$ such that $P(f_1, f_2, f_3) \equiv 0$ with the smallest possible degree – it is called an implicit equation of $\mathcal{C}$.

• **Degree formula**: $\deg(\phi)\deg(\mathcal{C}) = d - \deg(\gcd(f_1, f_2, f_3))$.
$\Rightarrow$ For simplicity, assume $\gcd(f_1, f_2, f_3) \in \mathbb{K} \setminus \{0\} \Leftrightarrow$ no base point.

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Linear Syzygies

$$\mathscr{L}(\mathbf{f}) := \left\{ \sum_{i=1}^{3} a_i(X_1, X_2) T_i \in \mathbb{K}[X_1, X_2][T_1, T_2, T_3] \right.$$

$$\left. \text{such that } \sum_{i=1}^{3} a_i(X_1, X_2) f_i(X_1, X_2) \equiv 0 \right\}$$

- It is a **graded** $\mathbb{K}[X_1, X_2]$-module: $\mathscr{L}(\mathbf{f}) = \bigoplus_{\nu \geq 0} \mathscr{L}(\mathbf{f})_\nu$
- $\mathscr{L}(\mathbf{f})_\nu \simeq \mathbb{K}$-vector space. For any $L \in \mathscr{L}(\mathbf{f})_\nu$ set

$$L = \sum_{i=0}^{\nu} L_i(T_1, T_2, T_3) X_1^i X_2^{\nu - i}$$

Notice that $L_i(T_1, T_2, T_3)$ is a **linear** form in $\mathbb{K}[T_1, T_2, T_3]$.

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Matrix of linear syzygies (1)

• Following [Sederberg, Chen], given an integer ν one builds the matrix $\mathtt{M}(\mathbf{f})_\nu$ as follows:

1. Compute a basis $L^{(1)}, \ldots, L^{(n_\nu)}$ of $\mathscr{L}(\mathbf{f})_\nu$ (i.e. solve a linear system)
2. $\mathtt{M}(\mathbf{f})_\nu$ is the matrix of coefficients of this basis, that is

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Matrix of linear syzygies (2)

We have a **family** of matrices indexed by ν : $\mathtt{M}(\mathbf{f})_\nu$.

$$\begin{cases} 0 \leq \nu \leq d-2 & \sharp\text{columns} < \sharp\text{rows} = \nu + 1 \\ \nu = d-1 & \mathtt{M}(\mathbf{f})_{d-1} \text{ is a } \textbf{square} \text{ matrix of size } d = \deg(f_i) \\ \nu \geq d & \sharp\text{columns} > \sharp\text{rows} = \nu + 1 \end{cases}$$

Proposition

- **For all** $\nu \geq d-1$ *we have the two following properties:*
  1. *the GCD of the minors of (maximum) size* $\nu + 1$ *of* $\mathtt{M}(\mathbf{f})_\nu$ *is equal to* $C(T_1, T_2, T_3)^{\deg(\phi)}$
  2. $\mathtt{M}(\mathbf{f})_\nu$ *is generically full rank and its rank drops exactly on* $\mathcal{C}$

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Link with μ-bases (Cox, Sederberg, Chen)

• $\mathscr{L}(\mathbf{f})$ turns out to be a **free** $\mathbb{K}[X_1, X_2]$-module of **rank 2**: a μ-basis is a basis (P, Q) of $\mathscr{L}(\mathbf{f})$. As a property, there exists an integer μ such that

$$P \in \mathscr{L}(\mathbf{f})_\mu \text{ and } Q \in \mathscr{L}(\mathbf{f})_{d-\mu}.$$

• One can reformulate the construction of $\mathtt{M}(\mathbf{f})_\nu$, for all $\nu \geq 0$, as the matrix of the multiplication map:

$$\mathbb{K}[X_1, X_2]_{\nu-\mu} \oplus \mathbb{K}[X_1, X_2]_{\nu-d+\mu} \xrightarrow{(P\ Q)} \mathbb{K}[X_1, X_2]_\nu$$

$\Rightarrow \mathtt{M}(\mathbf{f})_{d-1}$ is the Sylvester Matrix of P, Q:

$$\det(\mathtt{M}(\mathbf{f})_{d-1}) = \mathrm{Res}(P, Q) = C^{\deg(\phi)}.$$

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How to generalize to parametrized surfaces in $\mathbb{P}^3_{\mathbb{K}}$?

• For parametrized curves, the matrix $\mathtt{M}(\mathbf{f})_{d-1}$ is the perfect candidate to represent the curve $\mathcal{C}$: it is a **square** matrix built from **linear syzygies**.

• For parametrized surfaces, such a matrix does not exist in general. So we have two options:

Option 1: only look for a square matrix of syzygies. Require to introduce higher order syzygies. There are many results using quadratic syzygies (assuming proper parametrization, local complete intersection base points and some other technical conditions...)

Option 2: just fill a matrix with a basis of the linear syzygies. . .

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Linear Syzygies of a parametrized surface

- Consider a surface $\mathcal{S}$ parametrized by

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with $d := \deg(f_i) \geq 1$. Assume that $\gcd(f_1, \ldots, f_4) \in \mathbb{K} \setminus \{0\}$.

- The **graded** $\mathbb{K}[X_1, X_2, X_3]$-module of linear syzygies is

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Definition

The matrix $\mathtt{M}(\mathbf{f})_\nu$ is a representation of the homogeneous polynomial $P \in \mathbb{K}[T_1, \ldots, T_4]$ if

- i) *$\mathtt{M}(\mathbf{f})_\nu$ is generically full rank*
- ii) *the rank of $\mathtt{M}(\mathbf{f})_\nu$ drops exactly on the surface $P = 0$*
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Proposition (-, Chardin, Jouanolou)

For all $\nu \geq 2(d-1)$ we have:

- *if the base points are local complete intersections then* $\mathtt{M}(\mathbf{f})_\nu$ *represents* $S^{\deg(\phi)}$, S *being an implicit equation of* $\mathcal{S}$
- *if the base points are almost local complete intersections then* $\mathtt{M}(\mathbf{f})_\nu$ *represents*

$$S^{\deg(\phi)} \times \prod_{\mathfrak{p} \in V(f_1,\ldots,f_4) \subset \mathbb{P}^2_{\mathbb{K}}} L_{\mathfrak{p}}(T_1, \ldots, T_4)^{e_{\mathfrak{p}} - d_{\mathfrak{p}}}$$

• One can improve the bound $2(d-1)$ by taking into account the geometry of the base points.
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Surface parametrized by $\mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^1_{\mathbb{K}}$

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with the f_i's of **bi-degree** (d, d) and $\gcd(f_1, \ldots, f_4) \in \mathbb{K} \setminus \{0\}$.

- For all $\nu \geq 0$, one can consider the matrix $\mathtt{M}(\mathbf{f})_\nu$ of the **bi-homogeneous linear syzygies of bi-degree** (ν, ν).

Proposition (-,Dohm)

For all $\nu \geq 2d - 1$ then $\mathtt{M}(\mathbf{f})_\nu$ is a representation of

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- same remarks as in the previous case.

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How to manipulate these representations of surfaces ?

- Given a parametrized surface, it is easy to compute $\mathtt{M}(\mathbf{f})_\nu$ which is a very compact representation in many cases.

- **Questions**: How can we perform basic operations at the level of matrices such as:
  - space curve/surface intersections
  - surface/surface intersections
  - detection of singular locus
  - ...

- **Example**: Given $P \in \mathbb{P}^3$, one can test if $P \in \mathcal{S}$ as follows:

  i) After computing S, evaluate $S(P)$ and check that $|S(P)| < \epsilon$.

  ii) Compute an SVD of $\mathtt{M}(\mathbf{f})_\nu(P)$ and check its ϵ-rank.

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Link with μ-bases (1)

- Given $f_1, \ldots, f_4 \in \mathbb{K}[X_1, X_2, X_3]$ that parametrizes a surface, $\mathscr{L}(\mathbf{f})$ is not free in general. BUT, it becomes free of rank 3 after dehomogenization ($X_3 = 1$).
- Set $\tilde{f}_i(X_1, X_2) = f_i(X_1, X_2, 1)$ for all $i = 1, 2, 3, 4$.

Definition (Chen,Cox,Liu)

A μ-basis is a basis (P, Q, R) of the linear syzygies $\mathbb{K}[X_1, X_2]$-module $\mathscr{L}(\tilde{f}_1, \ldots, \tilde{f}_4)$

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If $\mathrm{Res}(P,Q,R) \neq 0$ *then it is equal to*

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- The fact that $\mathrm{Res}(P,Q,R)$ is equal to zero or not does depend on the choice of the μ-basis.
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Consider of the surface parametrized by (taken from [CCL05])

$$f_1 = 2X_1X_2,\ f_2 = 2X_2X_3,\ f_3 = 2X_1X_3,\ f_4 = X_1^2 + X_2^2 + X_3^3.$$

By dehomogenizing with respect to X_3 we get

$$\tilde{f}_1 = 2X_1X_2, \tilde{f}_2 = 2X_2, \tilde{f}_3 = 2X_1, \tilde{f}_4 = X_1^2 + X_2^2 + 1$$

and a basis of the linear syzygy module of the $\tilde{f}_i$'s is given by the matrix

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- $\text{Res}(P, Q, R) = {T_2}^4 H(T_1, T_2, T_3, T_4)$
- The extraneous factor T_2^4 is associated to the almost local complete intersection base point $X_1 = X_3 = 0$ (it is the unique base point).

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