Solving structured linear systems with large displacement rank

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Motivation

Problem 1. Recognize that

$$y = 1 + 2\,x - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{3}{20}x^5 + \frac{67}{720}x^6 - \frac{73}{1260}x^7 + \frac{1577}{40320}x^8 + O\left(x^9\right)$$

satisfies $(1+x)y'' + (1-x)y = 0$.

Problem 2. Let $P \in \mathbb{Q}[x,y]$ of total degree ≤ 2, such that

$$P(0,0) = 1,\ P(0,1) = 2,\ P(0,2) = 1,\ P(1,4) = 13,\ P(1,-1) = -2,\ P(2,3) = 36.$$

Find that

$$P = 1 + x + 2y + 3x^2 + 4xy - y^2.$$

These are linear algebra problems, with a lot of structure!

Basic algorithms in linear algebra

Classical approach.

- Most questions of linear algebra in size n (matrix product, inverse, system solving, characteristic polynomial, ...) can be solved in $O(n^3)$ operations.

Faster algorithms.

- Strassen'69: $n \times n$ matrices can be multiplied in $O(n^\omega)$ operations, $\omega < 3$.
- As of now, one can take $\omega \le 2.38$, even though the algorithm is quite impractical (**huge** logarithmic factors and constants hidden in the $O(\ )$).
- Most problems in linear algebra can be solved in time $O(n^\omega)$.

  Upcoming: matrix inversion algorithm using fast matrix multiplication.

However, none of these algorithms takes structure into account.

Toeplitz matrices

A Toeplitz matrix is invariant along its main diagonals:

$$A = \begin{bmatrix} c & d & e \\ b & c & d \\ a & b & c \end{bmatrix}.$$

Then, the Toeplitz displacement operator ϕ:

$$\phi(A) = A - (A \text{ shifted right and down by } 1) = \begin{bmatrix} c & d & e \\ b & 0 & 0 \\ a & 0 & 0 \end{bmatrix}$$

is such that $\phi(A)$ has rank $\alpha = 2$ (in general).

Compact representation

The matrix

$$\phi(A) = \begin{bmatrix} c & d & e \\ b & 0 & 0 \\ a & 0 & 0 \end{bmatrix}$$

can be represented in a compact way as

$$\phi(A) = GH^t, \quad \text{with} \quad G = \begin{bmatrix} c & d \\ b & 0 \\ a & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & e/d \end{bmatrix}.$$

$\longrightarrow$ This feature can be used to obtain algorithms of complexity $O\tilde{}(n)$ for solving the system $Ax = b$ ($O\tilde{}$ means that log. factors are hidden).

- The rank α of $\phi(A)$ is called the displacement rank of A;
- $G, H \in \mathbb{K}^{n\times\alpha}$ are called generators of A, of length α.

More structure . . .

Toeplitz structure: $\begin{bmatrix} c & d & e \\ b & c & d \\ a & b & c \end{bmatrix}$

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Toeplitz structure:

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Hankel structure: $\begin{bmatrix} e & d & c \\ d & c & b \\ c & b & a \end{bmatrix}$

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Hankel structure:

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Vandermonde structure: $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

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Hankel structure:

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Vandermonde structure:

$$\phi(A) = A - (\text{diagonal matrix}) \times (A \text{ shifted right by } 1)$$

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Cauchy structure: $\begin{bmatrix} 1/(a-x) & 1/(a-y) & 1/(a-z) \\ 1/(b-x) & 1/(b-y) & 1/(b-z) \\ 1/(c-x) & 1/(c-y) & 1/(c-z) \end{bmatrix}$

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In all these cases, the **displacement rank** α of A is the rank of $\phi(A)$.

If $\alpha \ll n$, the matrix A is called Toeplitz-like, Hankel-like,. . .

Previous results

Morf, Bitmead & Anderson, Pan, Kaltofen, Gohberg & Olshevsky, ...

Theorem. Let ϕ be one of the Toeplitz, Hankel, Vandermonde, Cauchy operators.

Let A be in $\mathbb{K}^{n\times n}$, given by generators of length α, and let b be in $\mathbb{K}^n$.

One can compute $\det(A)$ and a random solution to the system $Ax = b$, or prove that no such solution exists, in Las Vegas time $O\tilde{}(\alpha^2 n)$.

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Remarks.

- For $\alpha = 2$ (or more generally α constant), this is $O\tilde{~}(n)$, which is optimal, up to logarithmic factors $\longrightarrow$ quasi-optimal gcd, resultant, Padé approximation,. . .
- For large α, not so good: when $\alpha \simeq n$, cost $O\tilde{~}(n^3)$, worse than the cost $O(n^\omega)$ of generic linear algebra algorithms.

Our main result

Theorem. Let ϕ be one of the Toeplitz, Hankel, Vandermonde, Cauchy operators.

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Remarks.

- With $\omega \simeq 2.38$, this is $\tilde{O}(\alpha^{1.38} n)$, compared to an optimal $\tilde{O}(\alpha n)$.
- When α is constant, same cost as before $\tilde{O}(n)$.
- Improvement for large α: for $\alpha \simeq n$, cost $\tilde{O}(n^{\omega})$.

$\longrightarrow$ Our contribution consists in re-introducing fast matrix multiplication in structured matrices algorithms.

Some application examples

Hermite-Padé approximation. Given power series $f_1, \dots, f_m$ known at precision σ, degree bounds d_i, one can find in time $O\tilde{}(m^{\omega-1}\sigma)$ polynomials $p_1, \dots, p_m$ such that

$$\deg(p_i) \le d_i \quad \text{and} \quad \sum p_i f_i = O(x^\sigma) \quad \text{with} \quad \sigma = \sum(d_i + 1) - 1$$

- Beckermann & Labahn (1994) $O\tilde{}(m^{\omega}\sigma)$
- Lecerf, normal cases (2001) $O\tilde{}(m^{\omega-1}\sigma)$
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Generalized simultaneous Hermite-Padé approximation. Given a vector of polynomials $\mathbf{P} \in \mathbb{K}[x]^s$ of degree $\le \sigma/s$ and m vectors $\mathbf{f}_1, \dots, \mathbf{f}_m$ of polynomials in $\mathbb{K}[x]^s$ of degree $< \sigma/s$, one can find in time $O\tilde{~}(m^{\omega-1}\sigma)$ polynomials $p_1, \dots, p_m$ such that

$$\deg(p_i) < \sigma/m \qquad \text{and} \qquad \sum p_i \mathbf{f}_i = 0 \mod \mathbf{P}.$$

Some application examples

Bivariate interpolation. Given the values of a degree-d polynomial $P(x, y)$ at points

$$(a_i, b_j) \quad 0 \leq i + j \leq d,$$

one can recover its coefficients in time $\tilde{O}(d^{\omega+1})$, which is sub-quadratic in the number of terms (generally, interpolation problems whose monomial support indexes the sample points).

Toeplitz-block-Toeplitz systems.

Let $n = pq$ and let A be block-Toeplitz, with p^2 blocks of size q that are Toeplitz. One can solve the system $Ax = b$ in $\tilde{O}(n^{\frac{\omega+1}{2}})$ operations.

Inversion of dense matrices

[Strassen, 1969]

To invert a dense matrix $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in \mathbb{K}^{n\times n}$:

1. Invert $A_{1,1}$ (recursively).
2. Compute the Schur complement $\Delta := A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}$.
3. Invert Δ (recursively).
4. Recover the inverse of A as

$$A^{-1} = \begin{bmatrix} I & -A_{1,1}^{-1}A_{1,2} \\ & I \end{bmatrix} \times \begin{bmatrix} A_{1,1}^{-1} & \\ & \Delta^{-1} \end{bmatrix} \times \begin{bmatrix} I & \\ -A_{2,1}A_{1,1}^{-1} & I \end{bmatrix}$$

Complexity: $C(n) = 2C(\frac{n}{2}) + \mathsf{O}(\mathsf{n}^{\omega})$.

Corollary: $A^{-1}b$ in time $O(n^{\omega})$.

Inversion of Toeplitz-like matrices

[Morf, 1980], [Bitmead & Anderson, 1980], [Kaltofen 1994], [Pan 2001]

To compute generators of the inverse of a Toeplitz-like $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in \mathbb{K}^{n \times n}$

1. Compute generators of the inverse of $A_{1,1}$ (recursively).
2. Compute generators of Δ.
3. Compute generators of the inverse of Δ (recursively).
4. Compute generators of the inverse of A (by Strassen's formula).

Complexity: If A is given by generators of length α,

$$C(n, \alpha) = 2C\left(\frac{n}{2}, \alpha\right) + O(\mathsf{K}(n, \alpha)) + \tilde{O}(\alpha^{\omega-1} n),$$

where $\mathsf{K}(n, \alpha)$ is the cost of Toeplitz-like matrix multiplication, for $n \times n$ matrices given by generators of size α. Upcoming: $\mathsf{K}(n, \alpha) = \tilde{O}(\alpha^{\omega-1} n)$

$\sum LU$ formula for Toeplitz-like matrices

$$\phi(A) = A - (A \text{ shifted right and down by } 1), \qquad A \in \mathbb{K}^{n\times n}.$$

- The displacement rank of A is the rank α of $\phi(A)$.
- Generators (of length α) are matrices $G, H \in \mathbb{K}^{n\times\alpha}$ such that $\phi(A) = GH^t$.
- $\sum LU$ formula: one can recover A from its generators:

$$A = \sum_{j=1}^{\alpha} L(g_j)U(h_j), \qquad \text{with}$$

$$L(g_j) = \begin{bmatrix} g_{j,1} & & & \\ g_{j,2} & g_{j,1} & & \\ \vdots & \ddots & \ddots & \\ g_{j,n} & g_{j,n-1} & \cdots & g_{j,1} \end{bmatrix} \qquad \text{and} \qquad U(h_j) = \begin{bmatrix} h_{1,j} & h_{2,j} & \cdots & h_{n,j} \\ & h_{1,j} & \ddots & h_{n-1,j} \\ & & \ddots & \vdots \\ & & & h_{1,j} \end{bmatrix}$$

Remark. If $v \in \mathbb{K}^n$, then $L(g)U(h)v \equiv \mathsf{g}(x)\left(\mathsf{h}(x)\mathsf{v}(x) \bmod x^n\right) \operatorname{div} x^{n-1}$.

$\longrightarrow$ the matrix-vector product Av can be computed by FFT in $O\tilde{}(\alpha n)$ operations.

Matrix operations in compact representation

Let A and B be Toeplitz-like, given by generators (T, U) and (G, H) of length α.

- $\big([T \mid G],\ [U \mid H]\big)$ is a generator of length 2α for $A + B$.
- $\big([T \mid W \mid \mathbf{a}],\ [V \mid H \mid -\mathbf{b}]\big)$ is a generator of length $2\alpha + 1$ for $A \times B$, where
  - $V := B^t \times U$
  - $W := (A$ shifted right and down by $1) \times G$
  - $\mathbf{a}$ (resp. $\mathbf{b}$) is the down-shift of the last column of A (resp. B^t).

Thus, in compact representation, one can compute:

- the sum $A + B$ in $O(\alpha n)$ operations.
- the product $A \times B$ in $\mathsf{K}(n, \alpha) = O\tilde{}(\alpha^2 n)$ operations, using the $\sum LU$ formula.

$\longrightarrow$ Our main result is based on improving the cost $\mathsf{K}(n, \alpha)$ of $\times$.

Faster product in compact representation

Through the ΣLU formula, $\mathsf{K}(n, \alpha)$ is seen as the time of computing

$$A_\ell = \sum_{j=1}^{\alpha} G_j \left(H_j V_\ell \bmod x^n\right), \qquad 1 \le \ell \le \alpha$$

with G_j, H_j, V_ℓ in $\mathbb{K}[x]$ of degree $< n$.

Remark: the inner modulo prevents us from factoring out the V_ℓ.

Matrix reformulation: Given $\mathbf{H} \in \mathbb{K}[x]^{\alpha \times 1}$, $\mathbf{V} \in \mathbb{K}[x]^{1 \times \alpha}$ and $\mathbf{G} \in \mathbb{K}[x]^{\alpha \times 1}$, all of degree $< n$, compute $\left(\mathbf{HV} \bmod x^n\right) \mathbf{G}$.

$\longrightarrow$ Using short-product techniques (Schönhage'94, Mulders'00), we recast this into a polynomial matrix multiplication in size α and degree n/α.

$\longrightarrow$ We get the bound $\mathsf{K}(n, \alpha) = O\tilde{~}(\alpha^{\omega-1} n)$, which is the basis of our main result.

Short-product techniques

Idea: compute $(\mathbf{HV} \bmod x^n)\mathbf{G}$ by divide-and-conquer, as

$$\left((\mathbf{H_0} + x^{\frac{n}{2}}\mathbf{H_1})\left(\mathbf{V_0} + x^{\frac{n}{2}}\mathbf{V_1}\right) \bmod x^n\right)\left(\mathbf{G_0} + x^{\frac{n}{2}}\mathbf{G_1}\right) \quad = \quad \mathbf{H_0V_0G_0} +$$

$$x^{\frac{n}{2}}\left(\mathbf{H_0V_0G_1} + (\mathbf{H_0V_1} + \mathbf{H_1V_0} \bmod x^{\frac{n}{2}})\mathbf{G_0}\right) + x^n\left((\mathbf{H_0V_1} + \mathbf{H_1V_0} \bmod x^{\frac{n}{2}})\mathbf{G_1}\right)$$

The **desired quantities** for the recursive step read off

$$\left(\begin{bmatrix}\mathbf{H_0} & \mathbf{H_1}\end{bmatrix}\begin{bmatrix}\mathbf{V_1}\\ \mathbf{V_0}\end{bmatrix} \bmod x^{n/2}\right)\begin{bmatrix}\mathbf{G_0} & \mathbf{G_1}\end{bmatrix}$$

Let $\mathsf{K}(d,\alpha,\ell)$ be the cost of: given $\mathbf{A} \in \mathbb{K}[x]^{\alpha\times\ell}$, $\mathbf{B} \in \mathbb{K}[x]^{\ell\times\alpha}$ and $\mathbf{C} \in \mathbb{K}[x]^{\alpha\times\ell}$, of degree $< d$, compute $\left(\mathbf{AB} \bmod x^{d\ell}\right)\mathbf{C}$. Thus

$$\mathsf{K}(n,\alpha) = \mathsf{K}(n,\alpha,1) \leq \mathsf{K}(n/2,\alpha,2) \leq \mathsf{K}(n/4,\alpha,4) \leq \ldots \leq \mathsf{K}(n/\alpha,\alpha,\alpha) = \tilde{O}(\alpha^{\omega-1}n)$$

Here $\mathsf{K}\left(\frac{n}{\alpha},\alpha,\alpha\right)$ = cost of polynomial matrix multiplication in size α and degree $\frac{n}{\alpha}$.

Vandermonde and Cauchy

[Pan 1990] [Gohberg-Olshevsky 1994]

One can reduce the study of Vandermonde operators

$$\phi(A) = A - (\text{diagonal matrix}) \times (A \text{ shifted right by } 1)$$

and Cauchy operators

$$\phi(A) = A - (\text{diagonal matrix}) \times A \times (\text{diagonal matrix})'$$

to that of Toeplitz operators.

- The reduction involves a question similar to the one before: multiply a Vandermonde-like or Cauchy-like matrix, given by α generators, by α vectors.
- Similar techniques apply.

Conclusion

- Positive aspects: we can speed up the resolution for systems with large displacement rank (at least, theoretically).
- To do: make it automatic (Pan & Wang).
- Loose ends: often, a large displacement rank hides a multi-level structure.
  - Toeplitz-block-Toeplitz;
  - Multivariate interpolation: multilevel Vandermonde structure;
  - Algebraic / differential approximants (Hermite-Padé for powers / derivatives of a single power series).

For these questions, we are **far** from exploiting the structure as much as we would want.