BOREL IDEALS IN THREE AND (FOUR) VARIABLES

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Fixed any term-ordering $<$ on the set $\mathbf{T}(n)$ of terms in n variables $x_1, \ldots, x_n$, we study homogeneous ideal $\mathfrak{a} \subseteq \mathbf{P}(n)$ of the ring of polynomials in n variables over a field $\mathbf{k}$, via the associated order-ideal $\mathcal{N}(\mathfrak{a})$, consisting of all the terms which are not maximal terms of elements of $\mathfrak{a}$ and called *sous-éscalier* of $\mathfrak{a}$ [7, 12, 14].

In particular we will focus our attention on the following combinatorial properties of subsets $B \subset \mathbf{T}(n)$ (considered by R. Hartshorne [11] and first by N. Gunther [10]):

(1) for every $1 \le j < k \le n$ such that $a_k > 0$

$$x_1^{a_1} \ldots x_j^{a_j} \ldots x_k^{a_k} \ldots x_n^{a_n} \in B \Rightarrow x_1^{a_1} \ldots x_j^{a_j+1} \ldots x_k^{a_k-1} \ldots x_n^{a_n} \in B;$$

(2) for every $1 \le j < k \le n$ such that $a_j > 0$

$$x_1^{a_1} \ldots x_j^{a_j} \ldots x_k^{a_k} \ldots x_n^{a_n} \in B \Rightarrow x_1^{a_1} \ldots x_j^{a_j-1} \ldots x_k^{a_k+1} \ldots x_n^{a_n} \in B.$$

Recalling that, if $char \mathbf{k} = 0$, then for a monomial ideal $\mathfrak{a} \subset \mathbf{P}(n)$ TFAE:

(I) choosing $x_1 > \ldots > x_n$,
$\mathfrak{a}$ satisfies (1);
$\mathfrak{a}$ is *Borel fixed* (i.e. $g(\mathfrak{a}) = \mathfrak{a}$, $\forall\, g \in \mathcal{B}$, the *Borel group* of upper-triangular invertible matrices);
$\mathcal{N}(\mathfrak{a})$ satisfies (2).

(II) choosing $x_1 < \ldots < x_n$,
$\mathfrak{a}$ satisfies (2);
$\mathfrak{a}$ is fixed by the subgroup $\mathcal{B}'$ of lower-triangular invertible matrices;
$\mathcal{N}(\mathfrak{a})$ satisfies (1).

It seems natural to call *Borel subset* of $\mathbf{T}(n)$ any B satisfying (1), thus, as we are dealing with $\mathcal{N}(\mathfrak{a})$, we will consider term-orderings with $x_1 < \ldots < x_n$ and call *Borel ideals* the monomial ideals $\mathfrak{b} \subset \mathbf{P}(n)$ whose $\mathcal{N}(\mathfrak{b})$ is a Borel subset of $\mathbf{T}(n)$.

Borel ideals are the special monomial ideals occurring (given a term-ordering $<$) as initial ideals $in_<(\mathfrak{a})$ of homogeneous ideals $\mathfrak{a} \subseteq \mathbf{P}(n)$, in generic coordinates by the fundamental Galligo's and Bayer-Stillman's results [7, 1]. This initial ideal, denoted $gin_<(\mathfrak{a})$ and called *generic initial ideal* with respect to $<$, has been widely studied together with the algebraic and geometric information it brings on, in particular when the fixed term-ordering is the lexicographical (*lex*) one or the degree reverse lexicographical (*drl*) one.

1. Notation and preliminary results

For a positive integer n, $\mathbf{P}(n) := \mathbf{k}[x_1, \ldots, x_n]$ is the *polynomial ring* over a field $\mathbf{k}$ of characteristic 0 in the variables $x_1, \ldots, x_n$ endowed with a term-ordering such that $x_1 < \ldots < x_n$ and the standard grading $\deg x_i = 1$.

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The multiplicative *semigroup of terms* in n variables $\mathbf{T}(n)$ is the set of monic monomials $x^{\mathbf{a}} := x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$, with $a_i \in \mathbb{N}$; for each $1 \leq i \leq n$.

If $n \leq 4$, $x_1, ..., x_4$ will be replaced respectively by x, y, z, t, in general we will also write $\mathbf{P}$ and $\mathbf{T}$ respectively for $\mathbf{P}(n)$ and $\mathbf{T}(n)$.

For any non-negative integer j, $\mathbf{P}_j$ is the set of all homogeneous polynomials of degree j and, for any $M \subset \mathbf{P}$, we set $M_j := M \cap \mathbf{P}_j$.

For each $1 \leq i \leq n$, we set $\mathbf{P}(i) := \mathbf{k}[x_1, \dots, x_i]$ and $\mathbf{P}'(i) := \mathbf{k}[x_{n-i+1}, \dots, x_n]$, thought as subrings of $\mathbf{P}$.

If $t = x^{\mathbf{a}} := x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$, we set

$$\mathsf{m}(t); = \min\{i : a_i \neq 0\} \quad \text{and} \quad \mathsf{M}(t); = \max\{i : a_i \neq 0\},$$

moreover, for any $N \subseteq \mathbf{T}$, we let

$$N(i) := \{t \in N : \mathsf{M}(t) \leq i\} \quad \text{and} \quad N'(i) := \{t \in N : \mathsf{m}(t) \geq n - i + 1\}.$$

Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ exponent vectors. Then

(1) $x^{\mathbf{a}}$ is higher than $x^{\mathbf{b}}$ w.r.t. the *degree-lexicographic* term-ordering ($x^{\mathbf{a}} >_{dl} x^{\mathbf{b}}$) if $\deg(x^{\mathbf{a}}) > \deg(x^{\mathbf{b}})$ or if $\deg(x^{\mathbf{a}}) = \deg(x^{\mathbf{b}})$ and the last non-zero entry of $\mathbf{a} - \mathbf{b}$ is positive.

(2) $x^{\mathbf{a}}$ is higher than $x^{\mathbf{a}}$ w.r.t. the *drl* term-ordering ($x^{\mathbf{a}} >_{drl} x^{\mathbf{b}}$) if $\deg(x^{\mathbf{a}}) > \deg(x^{\mathbf{b}})$ or if $\deg(x^{\mathbf{a}}) = \deg(x^{\mathbf{b}})$ and the first non-zero entry of $\mathbf{a} - \mathbf{b}$ is negative.

For each $j \in \mathbb{N}^*$, $1 \leq i \leq n$ and $1 \leq \omega \leq \begin{pmatrix} i-1+j \\ j \end{pmatrix}$, the sets of the ω smallest terms of $\mathbf{T}(i)$ w.r.t. *dl* and *drl*, respectively denoted $\mathbf{L}_{i,\omega,j}$ and $\Lambda_{i,\omega,j}$, are Borel subsets, called *ω-(initial)-l-segment* and *ω-(initial)-rl-segment* of $\mathbf{T}(i)_j$.

The *potential expansion in $\mathbf{T}_{j+\ell}$ of* a $N \subseteq \mathbf{T}_j$ for some $j \in \mathbb{N}^*$, is defined setting $N_{(0)} := N$ and, recursively, for all $\ell \in \mathbb{N}^*$,

$$N_{(\ell)} \subset \mathbf{T}_{j+l} := \{x_i \tau : \tau \in N_{(\ell-1)}, 1 \leq i \leq n\}.$$

If $\mathfrak{a} \subset \mathbf{P}$, the minimal degree α of the generators of $\mathfrak{a}$ is called *initial degree* of $\mathfrak{a}$ and the *minimal system of generators* $G(\mathfrak{a})$ of a monomial ideal $\mathfrak{a} \subseteq \mathbf{P}$, with initial degree $\alpha \in \mathbb{N}^*$ and generated in degrees $\leq \rho$, satisfies

$$G(\mathfrak{a})_j = (\mathcal{N}(\mathfrak{a})_{j-1})_{(1)} \setminus \mathcal{N}(\mathfrak{a})_j \quad \text{for every} \quad \alpha \leq j \leq \rho,$$

note that in the 0-dimensional case one has $G(\mathfrak{a})_\rho = \mathcal{N}(\mathfrak{a})_{\rho-1})_{(1)}$.

For each subset N of $\mathbf{T}$ and $i, j \in \mathbb{N}$ with $1 \leq i < n$, we denote

$$\lambda_{i,j}(N) := \# N'(n-i)_j$$

the number of degree j terms in N not divided by $x_1, \dots, x_i$, we also set $\lambda_{0,j}(N) := \# N'(n-0)_j = \# N_j$.

Note that if $N \subset \mathbf{T}_{\bar{j}}$ for some $\bar{j}$, then $\lambda_{i,j}(N) = 0 \, \forall \, j \neq \bar{j}, 1 \leq i \leq n$.

Note also that for any Borel subset B of $\mathbf{T}(i)_j$ having ω elements it is

$$\lambda_{i,j}(\mathbf{L}_{i,\omega,j}) \geq \lambda_{i,j}(B) \geq \lambda_{i,j}(\Lambda_{i,\omega,j}).$$

THEOREM **1.1.** [15] *Let $B \subseteq \mathbf{T}_j$ be any Borel subset. Then, for every $\ell \in \mathbb{N}^*$, the potential expansion $B_{(\ell)} \subseteq \mathbf{T}_{j+\ell}$ is a Borel set and*

$$B_{(\ell)} = \bigsqcup_{1 \leq i_1 \leq \dots \leq i_\ell \leq n} x_{i_1} \cdots x_{i_\ell} \cdot B'(n - i_\ell + 1). \tag{1}$$

Hence $\#B_{(\ell)} = \sum_{i=1}^{n} \left(\begin{array}{c} i+\ell-2 \\ \ell-1 \end{array} \right) \lambda_{i-1,j}(B)$.

In particular $B_{(1)} = \sqcup_{i=1}^{n} x_i B'(n-i+1) = x_1 B \sqcup x_2 B'(n-1) \sqcup \ldots \sqcup x_n B'(1)$, here we point out that if $B \subset \mathbf{T}_j$ that is at least $x-n^j \notin B$, then $B'(1)=\emptyset$.

Note that for a homogeneous ideal $\mathfrak{a} \subseteq \mathbf{P}$, we have

$$\mathcal{N}(\mathfrak{a})_{j+1} \subseteq (\mathcal{N}(\mathfrak{a})_j)_{(1)}. \tag{2}$$

For all $n \geq 2$, an O-sequence $\mathbf{H} := (1, n, \ldots, h_{s-1}, \ldots)$ is the Hilbert function $\mathbf{H}_{\mathbf{P}/\mathfrak{a}}$ of $\mathbf{P}/\mathfrak{a}$, for some homogeneous ideal $\mathfrak{a} \subset \mathbf{P}$, i.e. $H(j) := h_j = \#(N(\mathfrak{a})_j)$.

For each O-sequence $\mathbf{H}$ there exists a polynomial $p(t) \in K[t]$ such that $h_j = p(j)$ for $j >> 0$, called *Hilbert polynomial of* $\mathbf{P}/\mathfrak{a}$ for each homogeneous ideal $\mathfrak{a} \subset \mathbf{P}$ associated to $\mathbf{H}$..

The *regularity* $reg(\mathbf{H})$ of $\mathbf{H}$ is $s := min\{\bar{t} \ : \ h_j = p(j), \forall j \geq \bar{t}\}$.

$in.deg.\mathbf{H} = \alpha := min\Big\{ j \in \mathbb{N} \mid h_j < \left(\begin{array}{c} j+n-1 \\ j \end{array} \right) \Big\}$.

An O-sequence $\mathbf{H}$ is *not increasing* if $(\Delta \mathbf{H})_j := h_j - h_{j-1} \leq 0$, for all $j > in.deg.\mathbf{H}$.

Letting, for any homogeneous ideal $\mathfrak{a} \subseteq \mathbf{P}$ and an integer i in the range between 1 and n, $\mathfrak{a}[i] := (\mathfrak{a}, x_1, \ldots, x_i)/(x_1, \ldots, x_i)$, we have:

$$\mathbf{H}_{\mathbf{P}'(n-i)/\mathfrak{a}[i]}(j) = \#((\mathcal{N}(\mathfrak{a})_j)'(n-i) = \lambda_{i,j}(\mathcal{N}(\mathfrak{a})).$$

We finally recall that for a Borel ideal $\mathfrak{b} \subset \mathbf{P}$, the *Castelnuovo-Mumford* regularity of $\mathfrak{b}$, denoted $reg(\mathfrak{b})$, equals the highest degree of minimal generators of $\mathfrak{b}$ [1].

For an O-sequence $\mathbf{H} = (1, n, \ldots, h_j, \ldots)$ with Hilbert polynomial $p(t)$ of degree k, $\mathcal{B}^n_{\mathbf{H}}$ denotes the set of all $(k+1)$-dimensional Borel ideals associated to $\mathbf{H}$.

(1) for a $\mathfrak{b} \in \mathcal{B}^n_{\mathbf{H}}$ one has:
   - $\mathcal{N}(\mathfrak{b})_{j+1} \subseteq (\mathcal{N}(\mathfrak{b})_j)_{(1)}$;
   - $\#((\mathcal{N}(\mathfrak{b})_j)_{(l)}) \geq h_{j+l}$, for each $l \geq 0$.

(2) The *l-segment ideal* associated to $\mathbf{H}$ is the monomial ideal $\mathcal{L}(H)$ such that $\mathcal{N}(\mathcal{L}(H)) = \cup_{j \in \mathbb{N}} \mathbf{L}_{n,h_j,j}$, namely $\mathbf{L}_{n,\eta,j+1} \subseteq (L_{n,\omega,})_{(1)}$ for all $\eta \leq \#(L_{n,\omega,})_{(1)}$ (see [12]).

(3) Only for a non-increasing O-sequence $\mathbf{H}$, the *rl-segment ideal* associated to it is the monomial ideal $\Lambda(\mathbf{H})$ such that $\mathcal{N}(\Lambda(H)) = \cup_{j \in \mathbb{N}} \Lambda_{n,h_j,j}$, namely $\Lambda_{n,\eta,j+1} \subseteq (L_{n,\omega,})_{(1)}$ only for $\eta \leq \omega$ (see [4, 14]).

EXAMPLE **1.2.** If $\mathbf{H} = (1,3,6,6,\ldots)$, then $\alpha = 3$ and $s = 2$. As $\Lambda(\mathbf{H}) = (z^3, y^2z, yz^2, y^3)$ and $\mathcal{L}(\mathbf{H}) = (z^3, y^2z, yz^2, xz^3, x^2yz, x^4z, y^6)$ it is $reg(\Lambda(\mathbf{H})) = 3$ and $reg(\mathcal{L}(\mathbf{H}) = 6$.

If $\mathfrak{b} \in \mathcal{B}^n_{\mathbf{H}}$ then for every $1 \leq i \leq n$ and $j \geq \alpha - 1$,

$$(\Delta \mathbf{H})_{j+1} = \sum_{i=1}^{n-1} \lambda_{i,j}(\mathcal{N}(\mathfrak{b})) - \#(G(\mathfrak{b})_{j+1}).$$

Since $h_{j+1} = \#(\mathcal{N}(\mathfrak{b})_{j+1})$ and $\lambda_{0,j}(\mathcal{N}(\mathfrak{b})) = h_j$, then

$$h_{j+1} = \#((\mathcal{N}(\mathfrak{b})_j)_{(1)}) - \#(G(\mathfrak{b})_{j+1}) = h_j + \sum_{i=2}^{n} \lambda_{i-1,j}(\mathcal{N}(\mathfrak{b})) - \#(G(\mathfrak{b})_{j+1}).$$

If $\mathfrak{b} \in \mathcal{B}^n_{\mathbf{H}}$ then for every $j \in \mathbb{N}$ and every $1 \leq i \leq n$,

$$\lambda_{i,j}(\mathcal{N}(\mathfrak{b})) \geq \lambda_{i-1,j}(\mathcal{N}(\mathfrak{b})) - \lambda_{i-1,j-1}(\mathcal{N}(\mathfrak{b})).$$

Thus, in particular, $\lambda_{1,j}(\mathcal{N}(\mathfrak{b})) \geq (\Delta\mathbf{H})_j$.

2. The poset and lattice strucrure

The *sous-éscalier (sectional) matrix* of a Borel ideal $\mathfrak{b} \subseteq \mathbf{P}(n)$ (*se.s.m.* for short) is the following $n \times \aleph_0$ matrix with non-negative integral entries:

$$\tilde{\mathcal{M}}(\mathfrak{b}) = (\tilde{m}_{i,j}(\mathfrak{b}))_{1\leq i\leq n,\ j\in\mathbb{N}^*} := (\lambda_{i-1,j-1}(\mathcal{N}(\mathfrak{b})))_{1\leq i\leq n,\ j\in\mathbb{N}^*}.$$

The notion of *se.s.m.* of a Borel ideal has been already introduced (and studied with different aims) in [2] and, for the 0-dimensional case, in [15]. If $\tilde{m}_{i,j}(\mathfrak{b}) = 0$, then all the terms of degree $j-1$ in the variables $x_i, \ldots, x_n$ do not belong to the order ideal $\mathcal{N}(\mathfrak{b})$. Hence, for every $r \geq 0$, it is also $\tilde{m}_{i,j+r} = 0$. In particular, for 0-dimensional ideals $\mathfrak{b} \subset \mathbf{P}$, which are always generated in degrees $\leq s = reg(\mathbf{H}_{\mathbf{P}/\mathfrak{b}})$, it happens that $\tilde{m}_{i,j}(\mathfrak{b}) = 0$ for every i and every $j \geq s$.

Different Borel ideals can have the same *se.s.m.*

Example **2.1.** Let $\mathfrak{b}_1 = (z^2, yz, y^3, xy^2)$ and $\mathfrak{b}_2 = (z^2, yz, y^3, x^2z)$ be two different 1-dimensional Borel ideals of $\mathbf{P}(3) = K[x,y,z]$. They have the same following *se.s.m*:

$$\begin{pmatrix} 1 & 3 & 4 & 3 & 3 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \end{pmatrix}.$$

As the first row of the *se.s.m* consists of the Hilbert function and $G(-)_j = \emptyset$, for every $j < in.deg.(-)$ and $j > \rho = reg(-)$, while for each $\alpha \leq j \leq \rho$,

(i) $G(-)_j = (\mathcal{N}(-)_{j-1})_{(1)} \setminus \mathcal{N}(-)_j$

(ii) $\#G(-)_j = \sum_{i=1}^{n} \lambda_{i-1,j-1}(\mathcal{N}(-))_{(1)} - \lambda_{0,j}(\mathcal{N}(-))$,

we have the following:

Proposition **2.2.** *Two Borel ideals having the same se.s.m also share the same initial degree and Castelnuovo-Mumford regularity.*

Definition **2.3.** [15] *Two Borel ideals* $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}^n_{\mathbf{H}}$ *are* equivalent *(in symbol* $\mathfrak{b} \sim \mathfrak{b}'$*) if they share the same se.s.m.*

In Ex. 2.1: $\mathbf{H} = (1,3,4,3,3,\ldots)$, $\mathfrak{b}_1 = \Lambda(\mathbf{H}) \neq \mathcal{L}(\mathbf{H}) = \mathfrak{b}_2$, but $\Lambda(\mathbf{H}) \sim \mathcal{L}(\mathbf{H})$.

Definition **2.4.** [15] $\prec$ *On* $\mathcal{B}^n_{\mathbf{H}}/\sim$ *is defined a partial order by letting* $\bar{\mathfrak{b}} \prec \bar{\mathfrak{b}}'$ *if* $\bar{\mathfrak{b}} \neq \bar{\mathfrak{b}}'$ *and the se.s.m's entries satisfy* $\tilde{m}_{i,j}(\mathfrak{b}) \leq \tilde{m}_{i,j}(\mathfrak{b}')$.

The above $\prec$ endowes $\mathcal{B}^n_{\mathbf{H}}/\sim$ with a poset structure:

(1) as $\bar{\mathfrak{b}} \prec \overline{\mathcal{L}(\mathbf{H})}$ for all $\mathfrak{b} \notin \overline{\mathcal{L}(\mathbf{H})}$, $\overline{\mathcal{L}(\mathbf{H})}$ is the maximum element of $\mathcal{B}^n_{\mathbf{H}}/\sim$,

(2) if $\mathbf{H}$ is not increasing, then $\overline{\Lambda(\mathbf{H})} \prec \bar{\mathfrak{b}}$ for all $\mathfrak{b} \notin \overline{\Lambda(H)}$ i.e. $\mathcal{B}^n_{\mathbf{H}}$ has a minimum element also if $n \geq 4$.

(3) if $n \geq 4$, for a generic $\mathbf{H}$ (i.e. not necessarily not increasing) $\mathcal{B}^n_{\mathbf{H}}/\sim$ does not have a minimum element but several minimal ones (see [15]).

Letting $n = 3$, $\mathbf{P} := \mathbf{P}(\mathbf{3}) = K[x,y,z]$ and $\mathbf{T} := \mathbf{T}(\mathbf{3})$ endowed with *drl* so that

$$\mathbf{T}_j = x\mathbf{T}_{j-1} \cup y(\mathbf{T}_{j-1})'(2) \cup z(\mathbf{T}_{j-1})'(1);$$

we extend to the positive dimensional case some results of [15]. So, first of all we recall the following notation already introduced in [13, 14, 15].

For $a, i, j \in \mathbb{N}$ with $j \neq 0$, $1 \leq i \leq j+1$,$0 \leq a \leq j+1$:

$$\ell_{i,j} := \mathbf{T}(\mathbf{2})_{j-i+1} \cdot z^{i-1} \subset \mathbf{T}_j \quad \text{and} \quad R_{a,j} := \bigsqcup_{i=1}^{a} \ell_{i,j}.$$

All terms of degree j can be arranged in the following table which takes into account the Borel condition (1), namely each term, via (1), reaches all the terms placed on its left and above it. Moreover, in the case $x < y < z$ (resp. $x > y > z$), on each row from left to right (resp. from right to left), starting from top and moving to bottom (resp. from bottom to top) one reads the degree j terms in increasing order w.r.t. lex, while on each column, starting from the leftmost (resp. rightmost) from top to bottom, (resp. from bottom to top) one reads the degree j terms in increasing order w.r.t. dr-lex :

$$\begin{array}{ccccccc} x^j & x^{j-1}y & x^{j-2}y^2 & \dots & xy^{j-1} & y^j \\ & x^{j-1}z & x^{j-2}yz & \dots & xy^{j-2}z & y^{j-1}z \\ & & x^{j-1}z^2 & \dots & xy^{j-3}z^2 & y^{j-2}z^2 \\ & & & \ddots & \vdots & \vdots \\ & & & & xz^{j-1} & yz^{j-1} \\ & & & & & z^j \end{array}.$$

Note that $\ell_{i,j}$ consists of all terms on the i-th row of the above table and that for a Borel set $B \subseteq \mathbf{T}_j$, if $y^{j-(a-1)}z^{a-1}$ belongs to B for some $a \geq 1$, then $\ell_{a,j} \subset R_{a,j} \subset B$. In particular, if $\lambda_{1,j}(\mathcal{N}(\mathfrak{b})) = a$, then $R_{a,j} \subset \mathcal{N}(\mathfrak{b})$.

Definition **2.5.** *Given* $\mathbf{H} := (1, 3, \dots, h_\alpha, \dots, h_s, \dots)$ *Hilbert function of* $\mathbf{P}/\mathfrak{a}$, *with* $s = reg(\mathbf{H})$ *and* $\mathfrak{a} \subset \mathbf{P}$ *homogeneous ideal of Krull-dim*≤ 2, *in.deg.* $= \alpha$. *The* increasing character *of* $\mathbf{H}$ *in degree* $\alpha + \ell$ *for each* $\ell \in \mathbb{N}$ *is*

$$a_\ell := \max\{0, \max_{i \geq \alpha+\ell}\{(\Delta\mathbf{H})_i\}\}.$$

Example **2.6.** If $\mathbf{H} := (1, 3, 6, 10, 15, 21, 27, 26, 29, 42, 55 \dots)$, $\alpha = 6$, $s = 8$, $p(t) = 2t + 13$, $\Delta\mathbf{H} = (1, 2, 3, 4, 5, 6, 6, -1, 3, 2, 2, 2 \dots)$ and $a_0 = 6, a_1 = a_2 = 3, a_{3+r} = 2\ \forall\, r \in \mathbb{N}$.
If $\mathbf{H} := (1, 3, 6, 10, 10, 6, 3, 1)$, $\alpha = 4$, $s = 8$, $p(t) = 0$, $\Delta\mathbf{H} = (1, 2, 3, 4, 0, -4, -3, -2, -1)$ and $a_{0+r} = 0\ \forall\ r \in \mathbb{N}$.

Theorem **2.7.** *Let* $\rho = reg(\mathcal{L}(\mathbf{H}))$. *Then for each collection of non-negative integers* $\mu_\alpha \geq \mu_{\alpha+1} \geq \dots \geq \mu_\rho \geq \dots$ *such that*

$$a_{j-\alpha} \leq \mu_j \leq \lambda_{1,j}(\mathcal{N}(\mathcal{L}(\mathbf{H}))),\ \forall\, \alpha \leq j,$$

there exists an ideal $\mathfrak{d} \in B^3_{\mathbf{H}}$ *with* $\lambda_{1,j}(\mathcal{N}(\mathfrak{d})) = \mu_j$.

Proof. (sketch) A constructive proof of this statement produces for each $j \in \mathbb{N}$ a Borel subset $D_j \subset \mathbf{T}_j$ such that $D_j \subset (D_{j-1})_{(1)}, \# D_j = h_j$ and $\lambda_{1,j}(D_j) = \mu_j$, $\forall\, j \geq \alpha$, then

$$\mathcal{N}(\mathfrak{d}) := \sqcup_{j \in \mathbb{N}} D_j.$$

We use the following facts

- $a_{j-\alpha} \leq \mu_j \leq \lambda_{1,j}(\mathcal{N}(\mathcal{L}(\mathbf{H}))) \leq j, \forall\, j \geq \alpha$ as $z^\alpha \in \mathcal{L}(\mathbf{H})$ for any j,
- $\rho = reg(\mathcal{L}(\mathbf{H}))$ implies in particular that $\forall\, j \geq \rho$, $\lambda_{1,j}(\mathcal{N}(\mathcal{L}(\mathbf{H}))) = p_0$,
- $\#(R_{\mu_j,j}) \subset \mathcal{N}(\mathcal{L}(\mathbf{H}))_j \Longrightarrow \#(R_{\mu_j,j}) \leq h_j$,

- any of the the above sequences of $\mu_j's$ is definitely constant (equal to p_0 for some index $\sigma \leq \rho$),
- $h_j - h_{j-1} \leq \mu_j \, \forall j \geq \alpha$,
- $R_{\mu_j,j}$ contains exactly μ_j terms t with $\mathsf{m}(t) = 2$,

We actually set

(1) $D_j := \mathbf{T}_j$, $\forall\, 0 \leq j \leq \alpha - 1$,
(2) $D_\alpha := R_{\mu_\alpha,\alpha} \cup \{t_{\alpha_1}, \ldots, t_{\alpha_{b(\alpha)}}\}$ where $\{t_{\alpha_1} < \ldots < t_{\alpha_{b(\alpha)}}\}$ are the smallest $b(\alpha) := h_\alpha - \# R_{\mu_\alpha,\alpha}$ terms (w.r.t *drl*) in $\mathbf{T}_\alpha \setminus R_{\mu_\alpha,\alpha}$ (all divisible by x).
(3) $D_j := R_{\mu_j,j} \cup \{t_{j_1}, \ldots, t_{j_{b(j)}}\}$, recursively for each $\alpha < j \leq \sigma$, where $\{t_{j_1} < \ldots < t_{j_{b(j)}}\}$ are the smallest $b(j) := h_j - \# R_{\mu_j,j}$ terms (w.r.t. *drl*) in $(D_{j-1})_{(1)} \setminus R_{\mu_j,j}$ (all divisible by x, namely, by construction it is $\#(D_{j-1})_{(1)} = h_{j-1} + \mu_{j-1} \geq h_{j-1} + \mu_j \geq h_j$ and $(D_{j-1})_{(1)}$ contains h_{j-1} terms τ with $\mathsf{m}(\tau) = 1$ and μ_{j-1} terms τ with $\mathsf{m}(\tau) = 2$).
(4) $D_j := D_{j-1})_{(1)}$ for all $j > \sigma$.

$\square$

Note that in this way we get all possible sous-éscalier sectional matrices of ideals in $\mathcal{B}^3_{\mathbf{H}}$. Namely, the second row of the sous-éscalier sectional matrix of $\mathfrak{b}$ must be:

$$(1 \quad 2 \quad 3 \quad \cdots \quad \alpha \quad \mu_0 \quad \cdots \quad \mu_{j-\alpha} \quad \ldots).$$

Definition **2.8.** *The monomial ideal with sous-éscalier the set* $\cup_{j\in\mathbb{N}} L_j$ *constructed in correspondence with* $\mu_j = a_{j-\alpha}$, *is called* generalized-rl-segment-ideal corresponding to $\mathbf{H}$ *and denoted* $\pounds(\mathbf{H}) \in \mathcal{B}^3_{\mathbf{H}}$.

Example **2.9.** *If* $\mathbf{H} = (1,3,6,10,15,21,28,28,\ldots)$, $s = 6 < \alpha = 7 < \beta = 28$ *and* $\pounds(\mathbf{H}) = \Lambda(\mathbf{H}) = (z^7, yz^6, y^2z^5, y^3z^4, y^4z^3, y^5z^2, y^6z, y^7)$, *while* $\mathcal{L}(\mathbf{H}) = (z^7, yz^6, xz^6, y^2z^5, xyz^5, x^2z^5, y^3z^4, xy^2z^4, x^3yz^4, x^4z^4, y^5z^3, xy^4z^3, x^3y^3z^3, x^4y^2z^3, x^5yz^3, x^7z^3, y^8z^2, xy^7z^2, x^3y^6z^2, x^4y^5z^2, x^6y^4z^2, x^7y^3z^2, x^9y^2z^2, x^{10}yz^2, x^{12}z^2, y^{13}z, x^2y^{12}z, x^4y^{11}z, x^6y^{10}z, x^8y^9z, x^{10}y^8z, x^{12}y^7z, x^{14}y^6z, x^{16}y^5z, x^{18}y^4z, x^{20}y^3z, x^{22}y^2z, x^{24}yz, x^{26}z, y^{28})$, *so that* $j_{\mathcal{L}(\mathbf{H})} = 28 = \beta = s + 22$.

As for each $\mathfrak{b} \in \mathcal{B}^3_{\mathbf{H}}$, $\mathfrak{b} \notin \overline{\pounds(\mathbf{H})}$ and $\mathfrak{b} \notin \overline{\mathcal{L}(\mathbf{H})}$, it is $\overline{\pounds(\mathbf{H})} \preceq \bar{\mathfrak{b}} \preceq \overline{\mathcal{L}(\mathbf{H})}$, we have:

Proposition **2.10.** *$(\mathcal{B}^3_{\mathbf{H}}/\sim, \prec)$ is a poset with universal extremes* $\mathbf{0} = \overline{\pounds(\mathbf{H})}$ *and* $\mathbf{1} = \overline{\mathcal{L}(\mathbf{H})})$.

Proposition **2.11.** *The poset $\mathcal{B}^3_{\mathbf{H}}/\sim$ has a lattice structure.*

Proof. (sketch) For $\bar{\mathfrak{b}}, \bar{\mathfrak{b}'} \in \mathcal{B}^3_{\mathbf{H}}/\sim$, let $\mu_0 \geq \cdots \geq \mu_{j-\alpha} \geq \ldots$ (resp. $\mu'_0 \geq \cdots \geq \mu'_{j-\alpha} \geq \ldots$) be the collection of integers defined by:

$\mu_{j-\alpha} := min\{\lambda(\mathcal{N}(\mathfrak{b})_j), \lambda(\mathcal{N}(\mathfrak{b}')_j)\}$, (resp. $\mu'_{j-\alpha} := max\{\lambda(\mathcal{N}(\mathfrak{b})_j), \lambda(\mathcal{N}(\mathfrak{b}')_j\})$, we set: $\bar{\mathfrak{b}} \wedge \bar{\mathfrak{b}'} := \bar{\mathfrak{d}}$ and $\bar{\mathfrak{b}} \vee \bar{\mathfrak{b}'} := \bar{\mathfrak{d}}'$, with $\mathfrak{d} \in \mathcal{B}^3_H$, $\mathfrak{d}' \in \mathcal{B}^3_{\mathbf{H}}$ the ideal constructed from the above collection of integers. $\square$

Recently, C.A. Francisco in [6], letting $\mathfrak{b} \approx \mathfrak{b}'$ if the graded Betti numbers of two Borel ideals $\mathfrak{b}, \mathfrak{b}' \subset \mathbf{P}(n)$ are ordinately equal (i.e. $\beta_{q,j+q}(\mathfrak{b}) = \beta_{q,j+q}(\mathfrak{b}')$) and $\tilde{\tilde{\mathfrak{b}}} \prec' \tilde{\tilde{\mathfrak{b}}}'$ if $\tilde{\tilde{\mathfrak{b}}} \neq \tilde{\tilde{\mathfrak{b}}}'$ with $\beta_{q,j+q}(\mathfrak{b}) \leq \beta_{q,j+q}(\mathfrak{b}')$ for each $\mathfrak{b} \in \tilde{\tilde{\mathfrak{b}}}, \mathfrak{b}' \in \tilde{\tilde{\mathfrak{b}}}'$, proved the existence of a minimum element in $\mathcal{B}^3_{\mathbf{H}}/\approx$. Using the Eliahou-Kervaire formula of [5] (extending [15] where it is proved for the 0-dimensional case) one can show:

Proposition **2.12.** *Two Borel ideals $\mathfrak{b}, \mathfrak{b}' \subset \mathbf{P}(n)$ have the same se.s.m iff have the same graded Betti numbers.*

Thus, in particular the equivalence relations $\sim$ and $\approx$ in $\mathcal{B}^n_{\mathbf{H}}$ coincide, moreover, as if $n = 3$ the $\beta_{q,j+q}(-)'s$ are all expressed in terms of the $h'_j s$ and $\beta_{0,j}(-)'s$, also the two partial orderings $\prec$ and $\prec'$ coincide, and so one has a lattice structure also w.r.t. the partial ordering arising from the graded Betti numbers. Nevertheless this is false in the case $n \geq 4$ as it is shown in [15]

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