Discriminant method for the homological monodromy

Mario Escario Gil

University of San Jorge, Spain

math.AG/0602297

Contents:

The block decomposition

- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function.
- There exists a finite minimal set $B(f)$ such that the restriction map $f| : f^{-1}(\mathbb{C} \setminus B(f)) \longrightarrow \mathbb{C} \setminus B(f)$ is a locally trivial fibration.
- Given a geometric basis (γ_i) of $\pi_1(T \setminus B(f); \star)$ G. B. one has a direct sum decomposition of $\tilde{H}_q(f^{-1}(\star))$ (reduced homology over $\mathbb{Z}$) which depends essentially on the choice of (γ_i) (see [4, 6, 7, 14, 15, 16] for various degrees of generality).
- With this sum decomposition and if f has only isolated singularities, local monodromy

$$(h_{\gamma_i})_* : \tilde{H}_q(f^{-1}(\star)) \longrightarrow \tilde{H}_q(f^{-1}(\star))$$

has a block decomposition.

The block decomposition

- $(h_{\gamma_i})_*$ has two kinds of blocks:
  - *local blocks* which only depend on the local Milnor fibers.
  - *global blocks* which depend on the embeddings of the local Milnor fibers into the fixed regular fiber $f^{-1}(\star)$.
- These two invariants allow us to compute the intersection matrix of $f^{-1}(\star)$. Example

The block decomposition

- $(h_{\gamma_i})_*$ has two kinds of blocks:
  - *local blocks* which only depend on the local Milnor fibers.
  - *global blocks* which depend on the embeddings of the local Milnor fibers into the fixed regular fiber $f^{-1}(\star)$.
- These two invariants allow us to compute the intersection matrix of $f^{-1}(\star)$. Example
- Several papers dealing with the local blocks and how to compute them:
  - Brieskorn singularities by A. Hefez and F. Lazzeri [13].
  - Certain singularities and unimodal singularities by A. M. Gabriélov [8, 9].
  - General methods: using real morsifications (N. A'Campo [1, 2] and S. M. Gusein-Zade [11, 12]) and using an inductive argument (A. M. Gabriélov [10]).

The block decomposition

- This is not the situation for the global blocks.
  - There are some relations between local and global blocks (A. Dimca and A. Némethi [6], W. Neumann and P. Norbury [14]) which can give useful constraints.
  - Usually these data are computed depending on the particular polynomial f.

The block decomposition

- This is not the situation for the global blocks.
  - There are some relations between local and global blocks (A. Dimca and A. Némethi [6], W. Neumann and P. Norbury [14]) which can give useful constraints.
  - Usually these data are computed depending on the particular polynomial f.
- A practical complete algorithmic method does not exist in the literature.

Case of conjugated polynomials

- Specially interesting is the case of polynomials with coefficients in a number field conjugated by a Galois isomorphism of the field.
- Example: $\left(y^2x-(y+1)^3\right)\left(s^2\left(2\,s-3\right)y+x-3\,s^2\right)$ with $s\in\{3+2\sqrt{3},3-2\sqrt{3}\}$ are conjugated by the Galois isomorphism $a+b\sqrt{3}\mapsto a-b\sqrt{3}$, $a,b\in\mathbb{Q}$.

Case of conjugated polynomials

- Specially interesting is the case of polynomials with coefficients in a number field conjugated by a Galois isomorphism of the field.
- Example: $\left(y^2x-(y+1)^3\right)\left(s^2\left(2\,s-3\right)y+x-3\,s^2\right)$ with $s\in\{3+2\sqrt{3},3-2\sqrt{3}\}$ are conjugated by the Galois isomorphism $a+b\sqrt{3}\mapsto a-b\sqrt{3}$, $a,b\in\mathbb{Q}$.
- Due to the Galois isomorphism both have the same algebraic properties (degree, number of components, global Milnor number, type and position of singularities, ...).

Case of conjugated polynomials

- Specially interesting is the case of polynomials with coefficients in a number field conjugated by a Galois isomorphism of the field.
- Example: $\left(y^2x-(y+1)^3\right)\left(s^2\left(2\,s-3\right)y+x-3\,s^2\right)$ with $s\in\{3+2\sqrt{3},3-2\sqrt{3}\}$ are conjugated by the Galois isomorphism $a+b\sqrt{3}\mapsto a-b\sqrt{3}$, $a,b\in\mathbb{Q}$.
- Due to the Galois isomorphism both have the same algebraic properties (degree, number of components, global Milnor number, type and position of singularities, ...).
- The global blocks reflect how the Milnor fibers sit in the fixed regular fiber and this need not be invariant under Galois isomorphims.

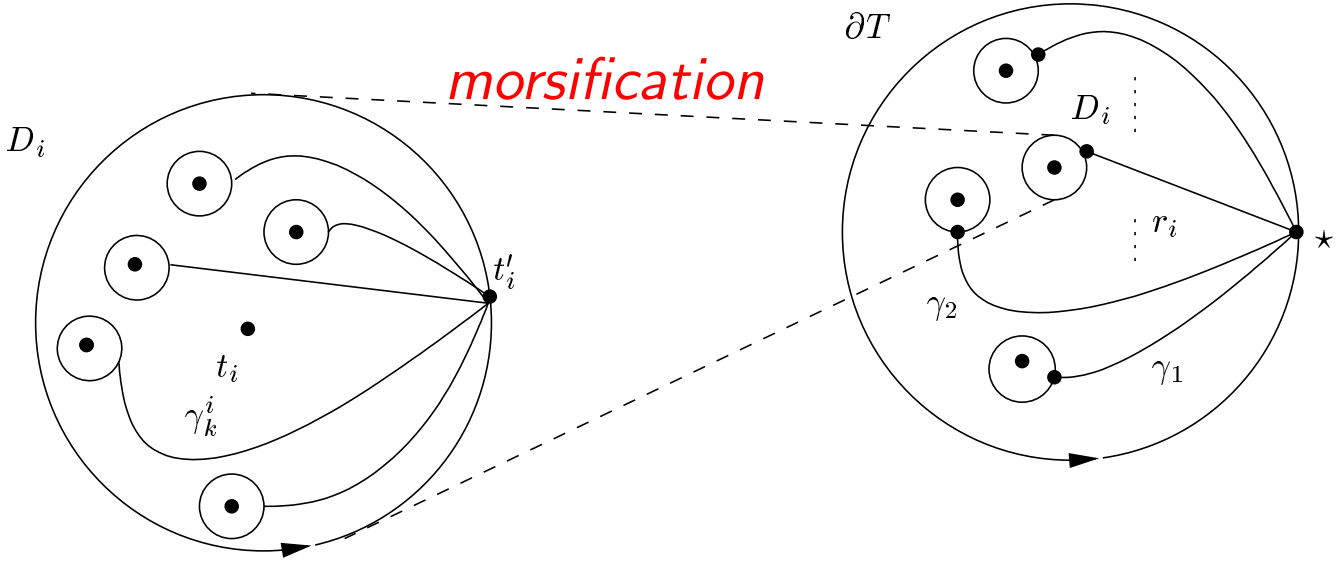
Tame polynomials: reduction to the Morse case

- The discriminant method is a practical complete algorithmic method to compute local monodromies for a tame polynomial f with $n = 2$.
- Let f be a tame polynomial (S. A. Broughton [4])
  - $\Rightarrow$ f is good at infinity $\Rightarrow$ $B(f) = \{t_i\}$ contains only critical values coming from affine singularities.
  - $\Leftrightarrow$ $\mu(f) < \infty$ and $\mu(f)$ is invariant by morsifications $f(x, y) + ag(x, y), g$ generic lineal form $\Rightarrow$ regular fibers are diffeomorphic.

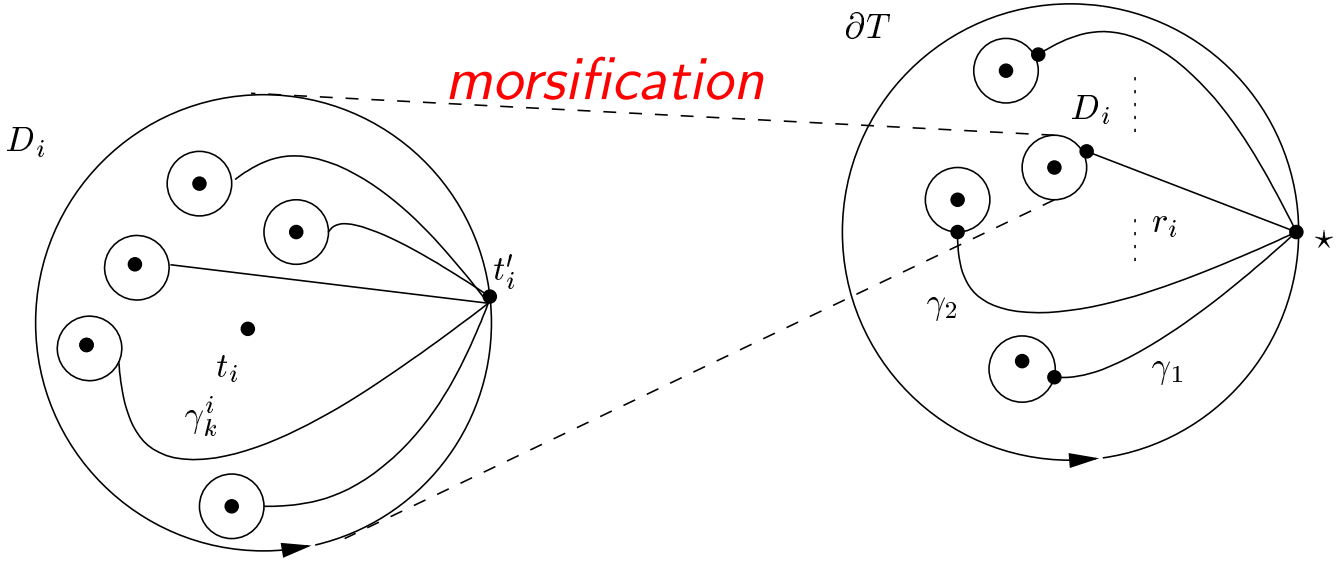
Tame polynomials: reduction to the Morse case

- The discriminant method is a practical complete algorithmic method to compute local monodromies for a tame polynomial f with $n = 2$.
- Let f be a tame polynomial (S. A. Broughton [4])
  - $\Rightarrow$ f is good at infinity $\Rightarrow$ $B(f) = \{t_i\}$ contains only critical values coming from affine singularities.
  - $\Leftrightarrow$ $\mu(f) < \infty$ and $\mu(f)$ is invariant by morsifications $f(x, y) + ag(x, y), g$ generic lineal form $\Rightarrow$ regular fibers are diffeomorphic.
- To obtain the block decomposition of $(h_{\gamma_i})_*$ we need to consider *special geometric bases* of $\pi_1(T \setminus B(f + ag); \star)$.

Tame polynomials: reduction to the Morse case

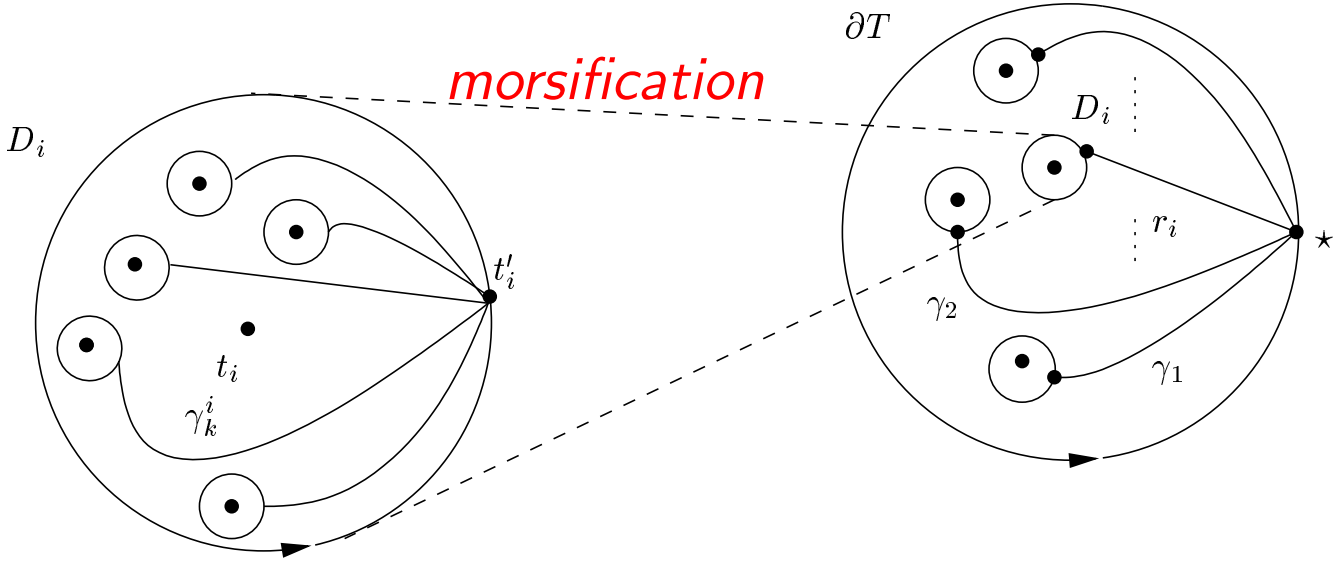


Tame polynomials: reduction to the Morse case



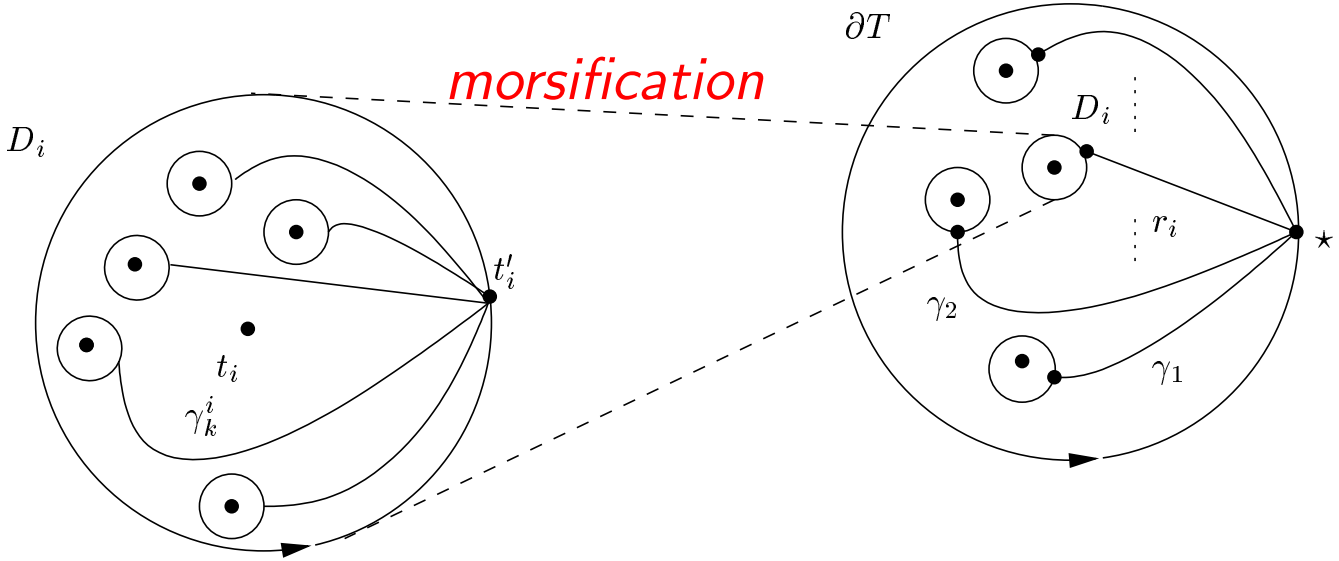
- Order the set $B(f + ag) \cap D_i$ in such a way that the critical values corresponding to the morsification of the same critical point in $f^{-1}(t_i)$ are together.

Tame polynomials: reduction to the Morse case



- $(\gamma_k^i)_{k=1,\dots,k(i)}$ a geometric basis of $\pi_1(D_i \setminus B(f+ag) \cap D_i; t_i')$ which respects this order so that $(r_i \cdot \gamma_k^i \cdot r_i^{-1})_{k=1,\dots,k(i)}^{i=1,\dots,\#B(f)}$ is a geometric basis of $\pi_1(T \setminus B(f+ag); \star)$.

Tame polynomials: reduction to the Morse case



- $\gamma_i = r_i \cdot (\prod_k \gamma_k^i) \cdot r_i^{-1} \Rightarrow$ the ordered product of the associated local monodromies of $f + ag$ gives the block decomposition of $(h_{\gamma_i})_*$.

Hyphotesis

- We can assume $f(x, y)$ to be a tame Morse polynomial function.
- $\ell(x, y)$ generic linear form. One has the polar map

$$\phi_{f,\ell} : \mathbb{C}^2 \to \mathbb{C}^2, (x, y) \mapsto (f(x, y), \ell(x, y)) = (t, \ell)$$

  Let (x, y) be generic coordenates. We take $\ell(x, y) = x$.
- $\mathfrak{D}_f := \{(t, x) \in \mathbb{C}^2 \mid \text{discrim}_y(f(x, y) - t) = 0\}$ the discriminant curve of $\phi_{f,x}$.

Hyphotesis

- The method needs two data which depend on $\mathfrak{D}_f$:
  - The classical monodromy m of the projection

$$\pi| : f^{-1}(\star) \to \mathbb{C}, (x, y) \mapsto x$$

    in a geometric basis associated with the ramification points of $\pi|_{f^{-1}(\star)}$ (the set $\mathbf{x}^\star$ of k points given by $\mathfrak{D}_f \cap \{t = \star\}$).
    First datum
  - The braid monodromy ∇_τ of the discriminant $\mathfrak{D}_f$ in the geometric basis (γ_i). Second datum

The main theorem

Theorem 1 (Discriminant method)

Let $f(x,y) \in \mathbb{C}[x,y]$ be a tame Morse polynomial with (x,y) generic coordinates. Then $(h_{\gamma_i})_*$ is determined by the following data:

$$(m(\mu_1^\tau), \ldots, m(\mu_k^\tau)) \subset \Sigma_N^k$$

$$(\nabla_\tau(\gamma_1), \ldots, \nabla_\tau(\gamma_{\mu(f)})) \subset \mathbb{B}_k^{\mu(f)}$$

Moreover an explicit method to construct $(h_{\gamma_i})_*$ exists.

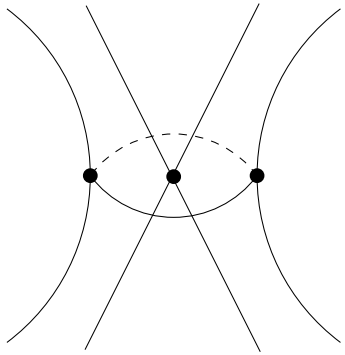
Sketch of the proof.

Sketch of the proof.

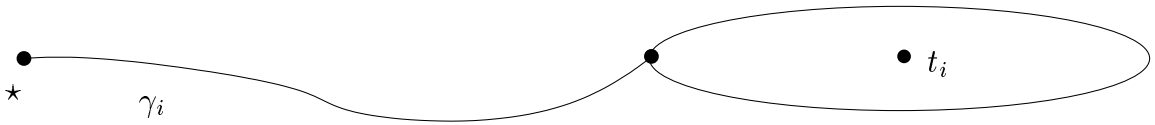
1. Finding the vanishing cycles.

Sketch of the proof.

1. Finding the vanishing cycles.

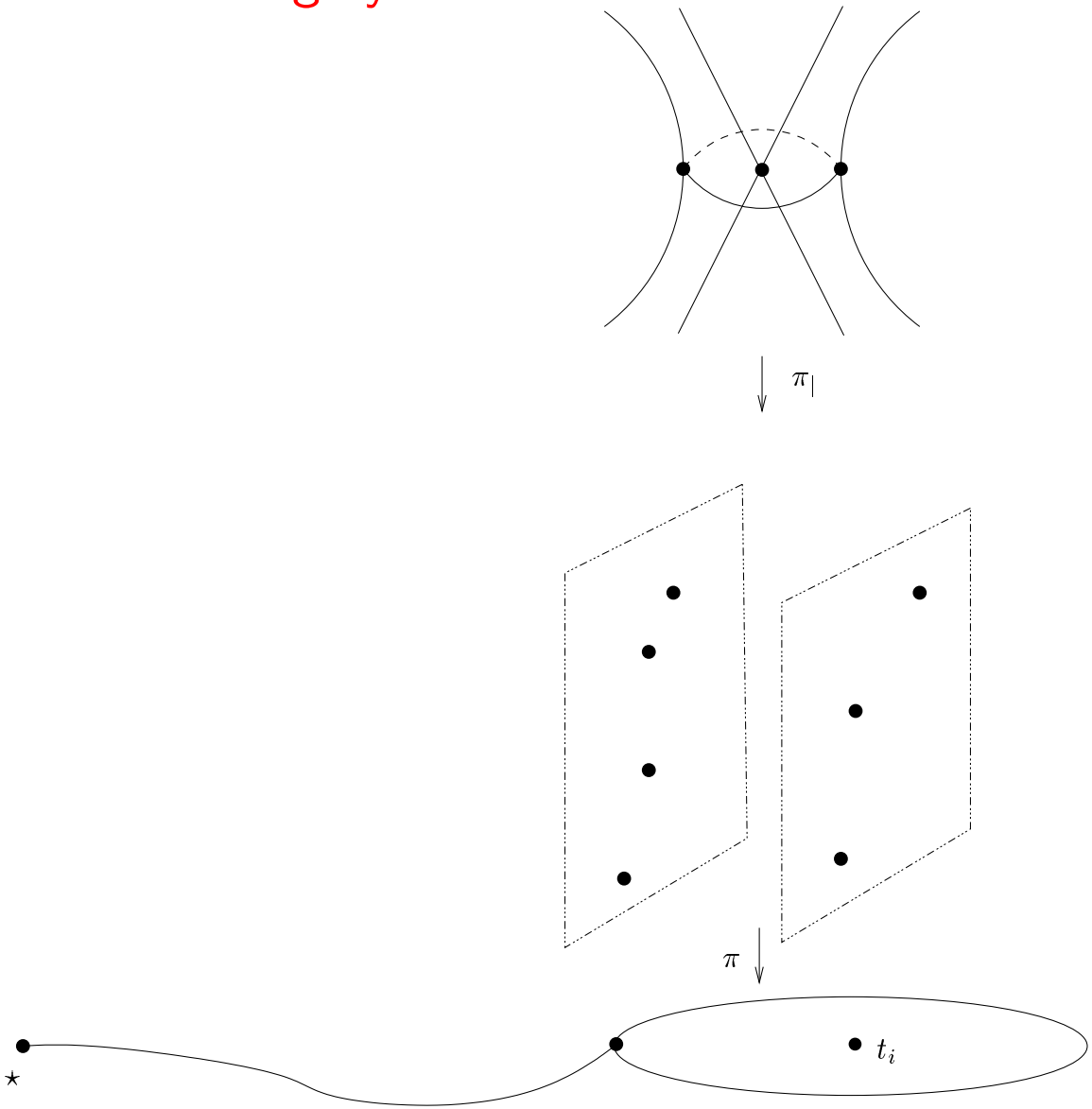


f



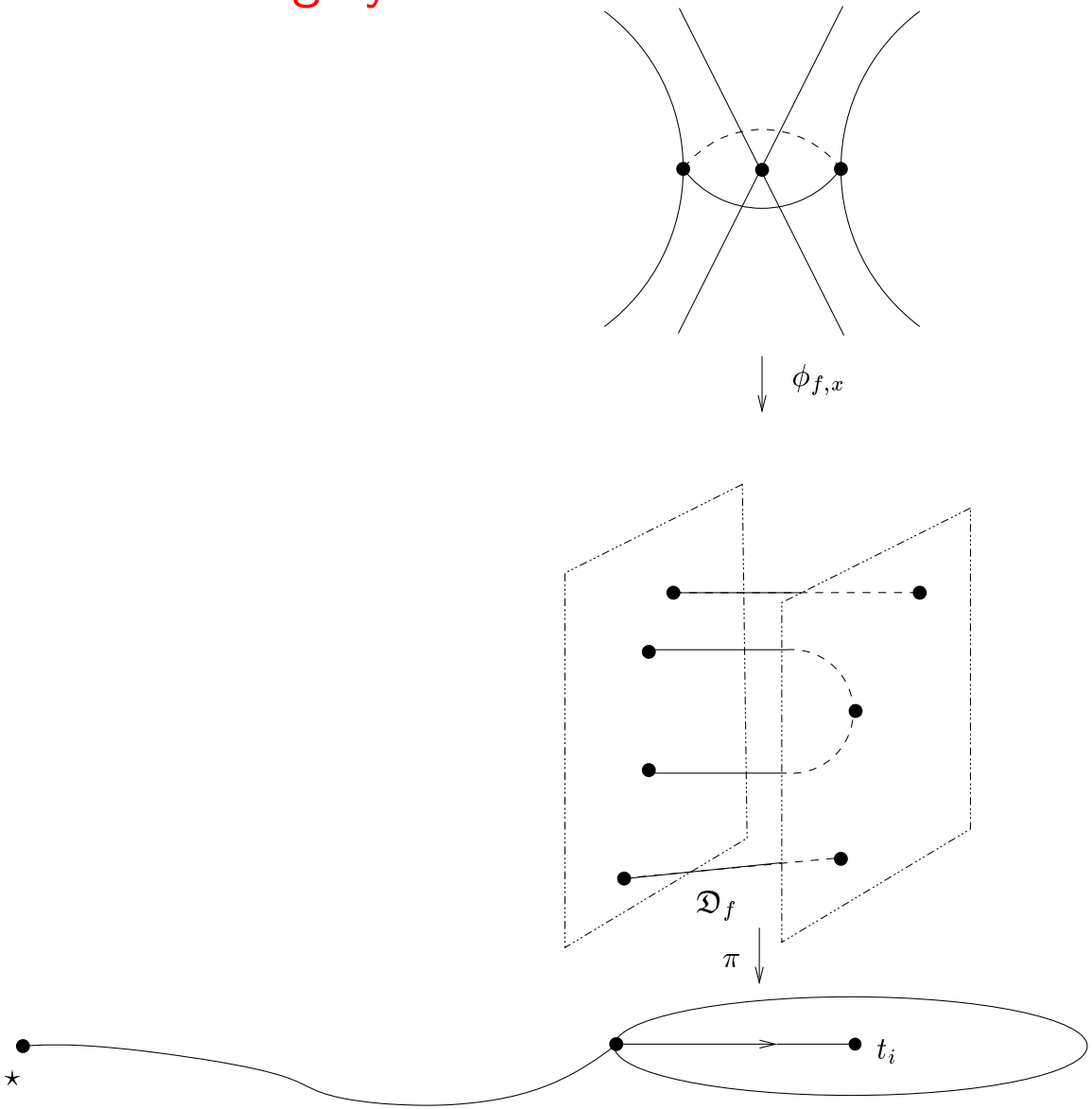
Sketch of the proof.

1. Finding the vanishing cycles.



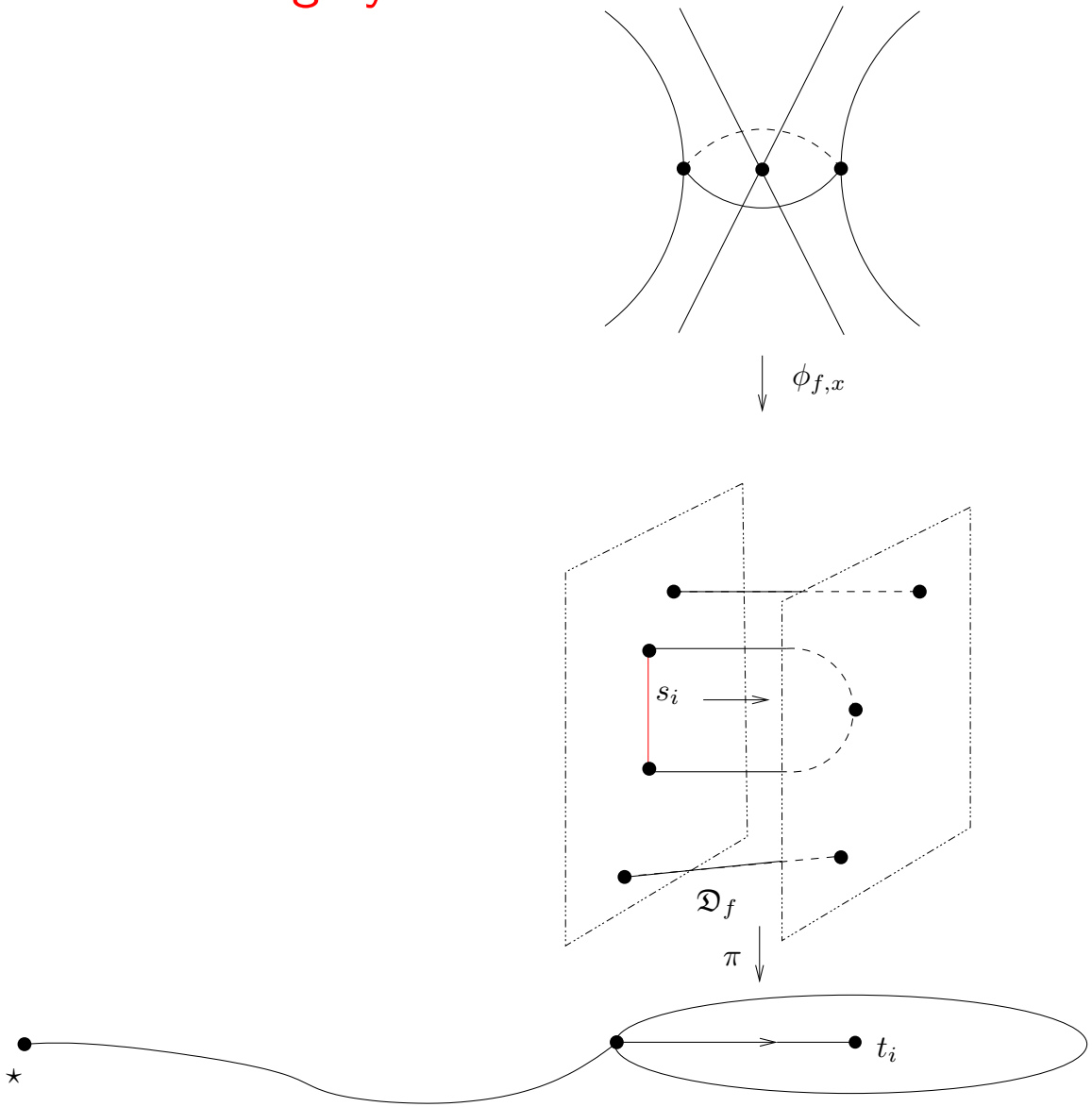
Sketch of the proof.

1. Finding the vanishing cycles.



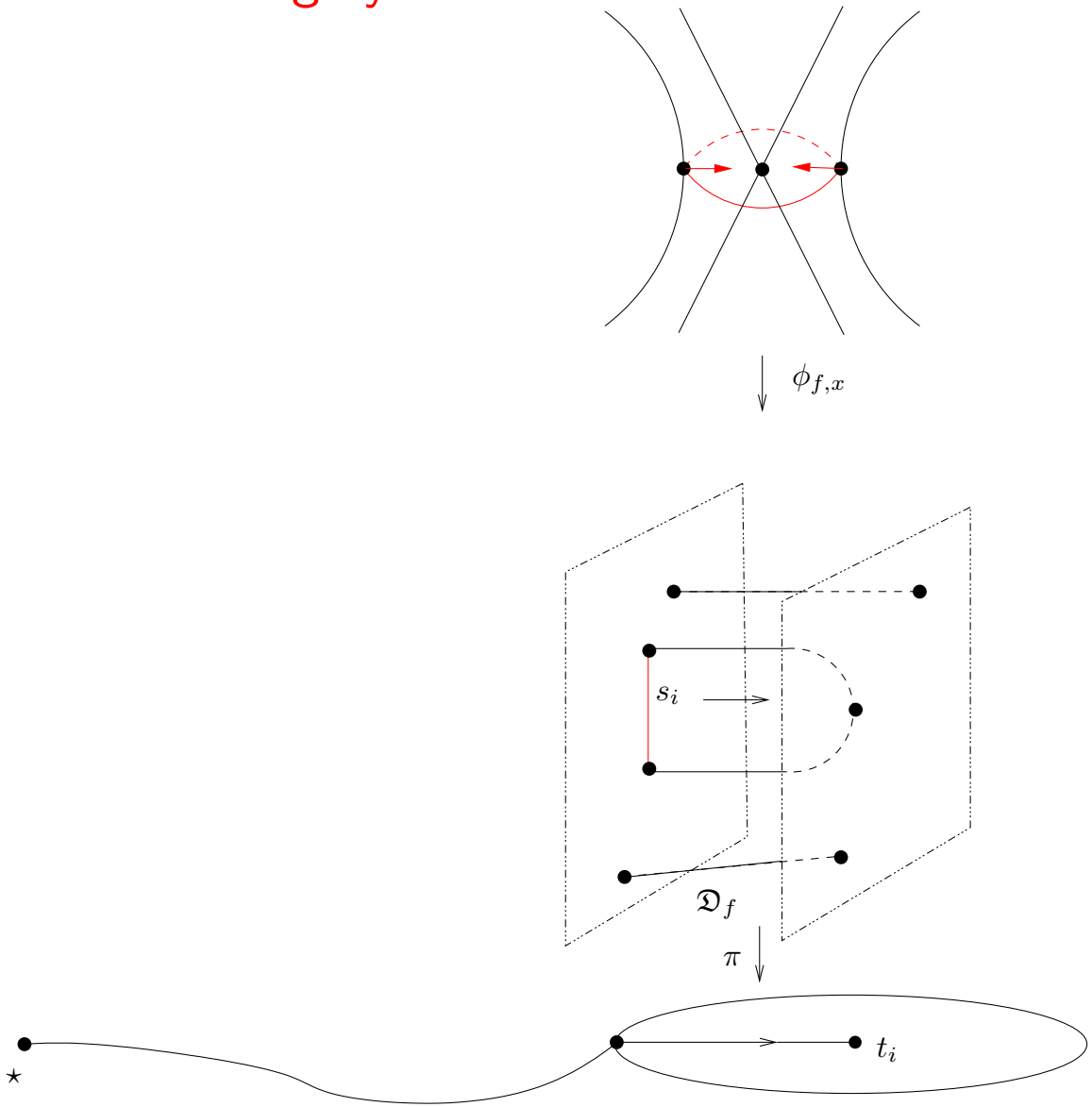
Sketch of the proof.

1. Finding the vanishing cycles.



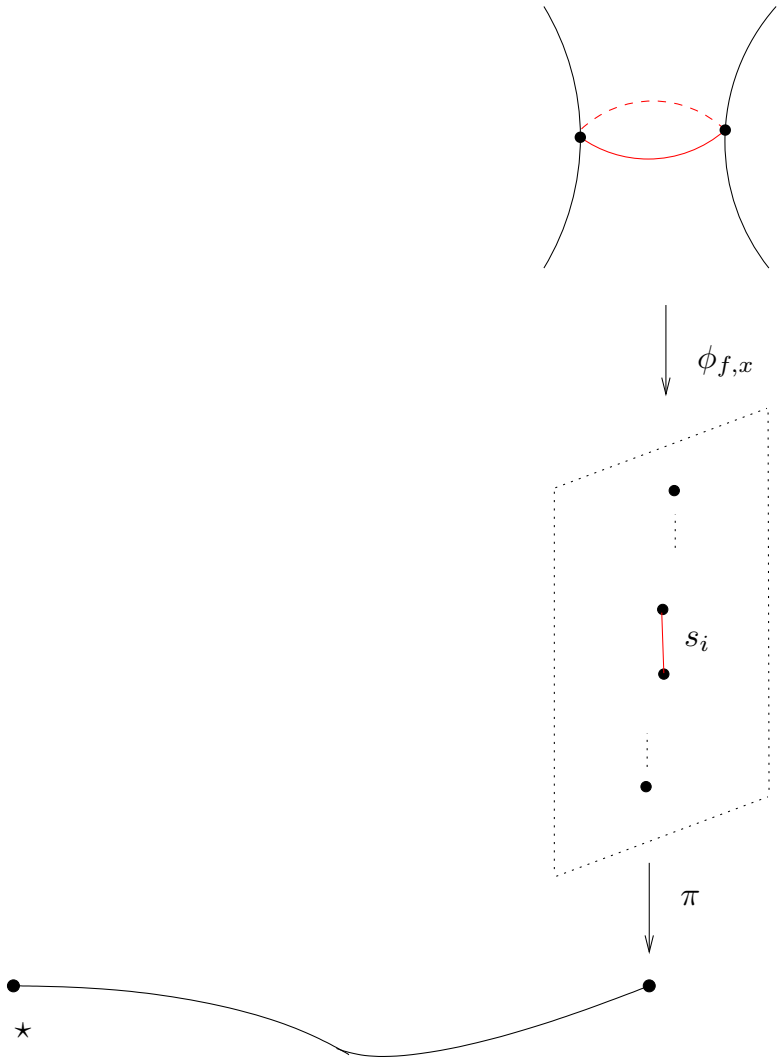
Sketch of the proof.

1. Finding the vanishing cycles.



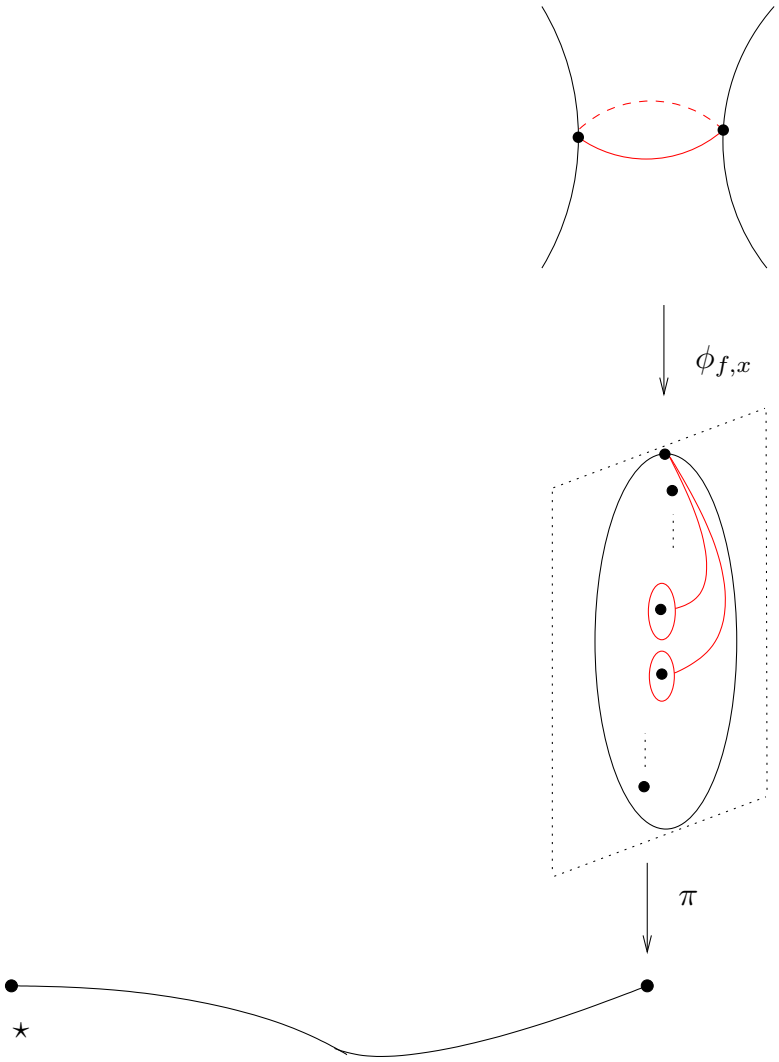
Sketch of the proof.

2. A distinguished basis.



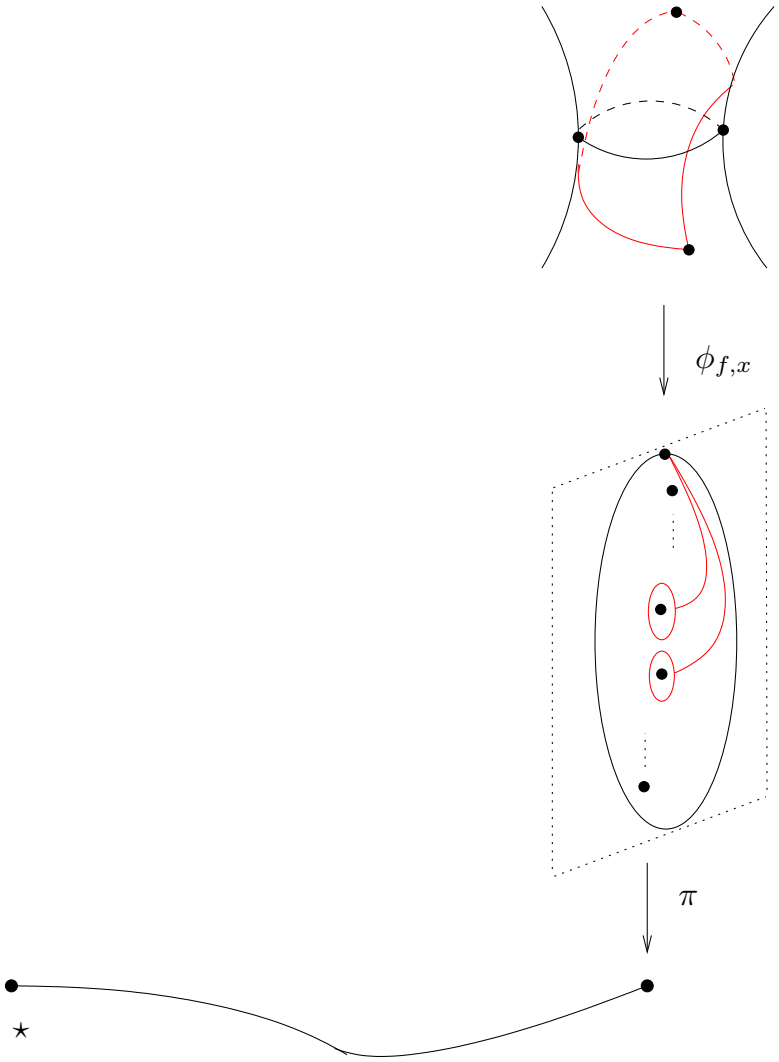
Sketch of the proof.

2. A distinguished basis.



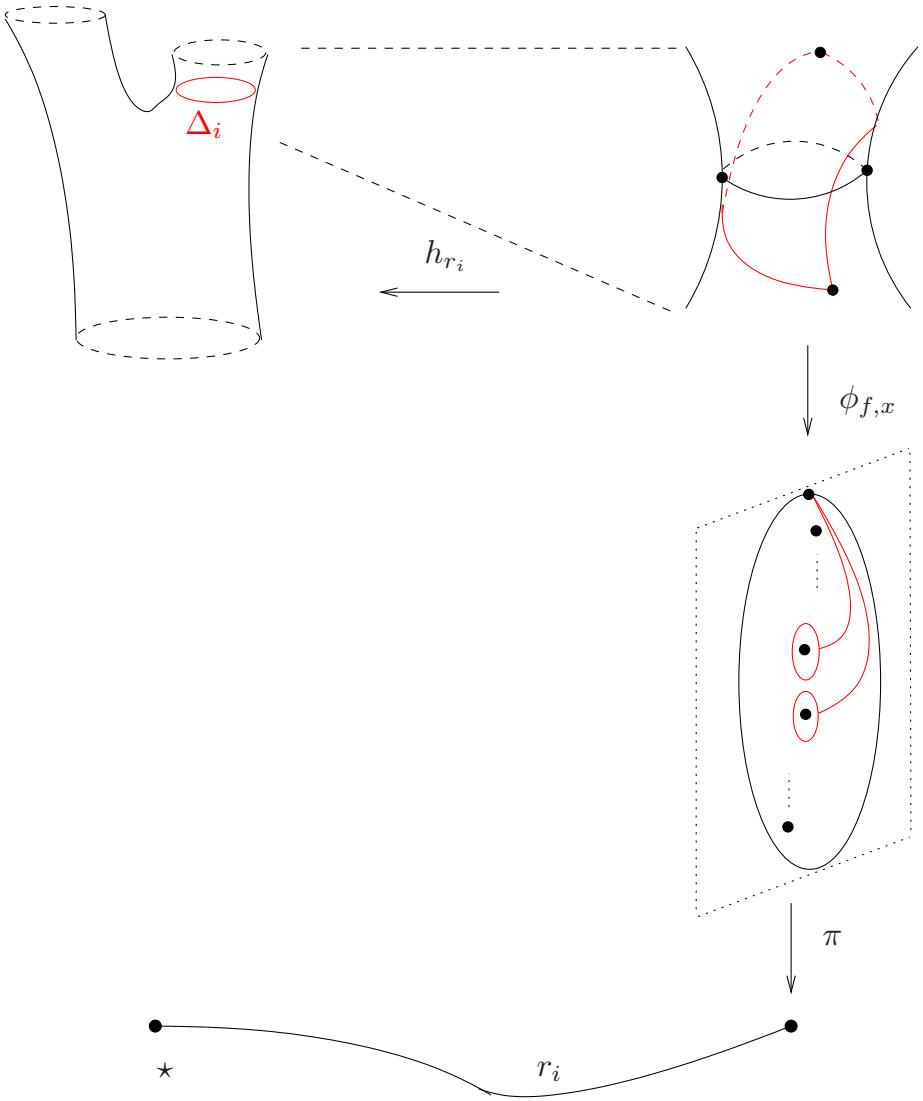
Sketch of the proof.

2. A distinguished basis.



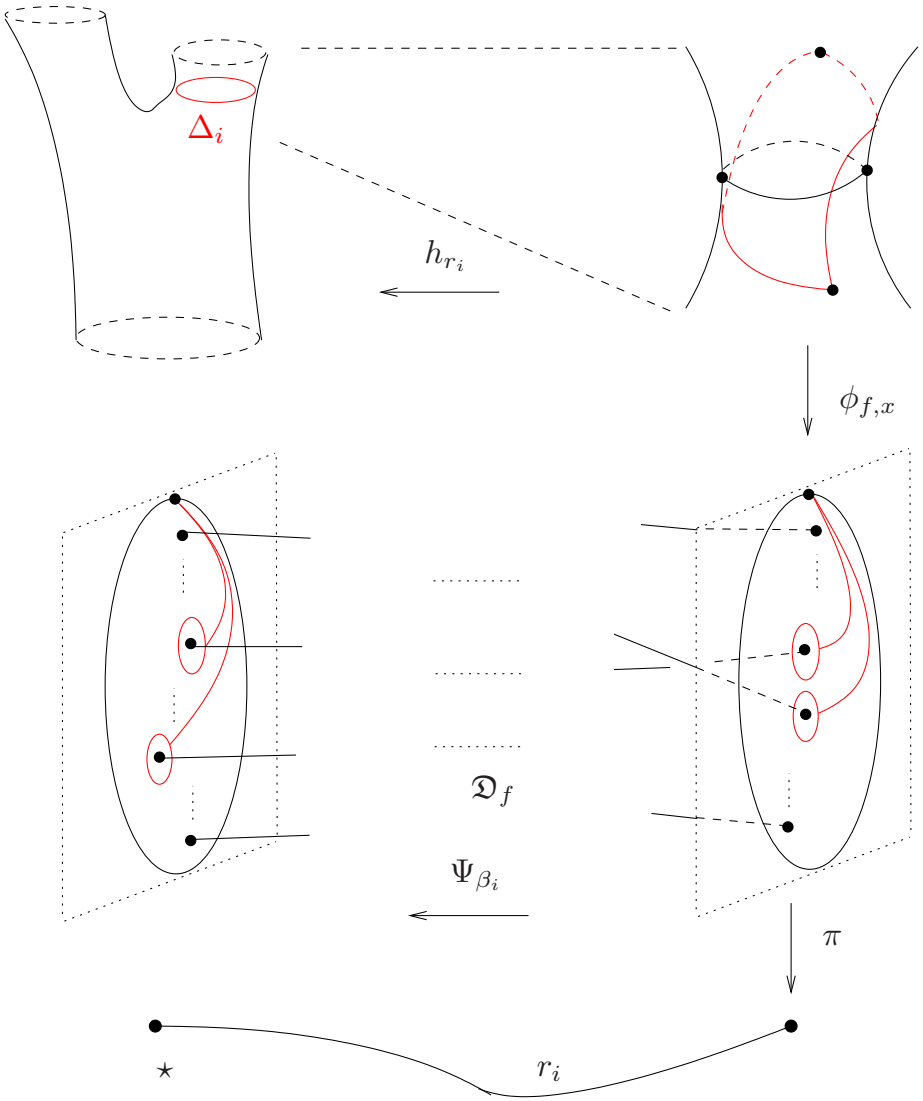
Sketch of the proof.

2. A distinguished basis.



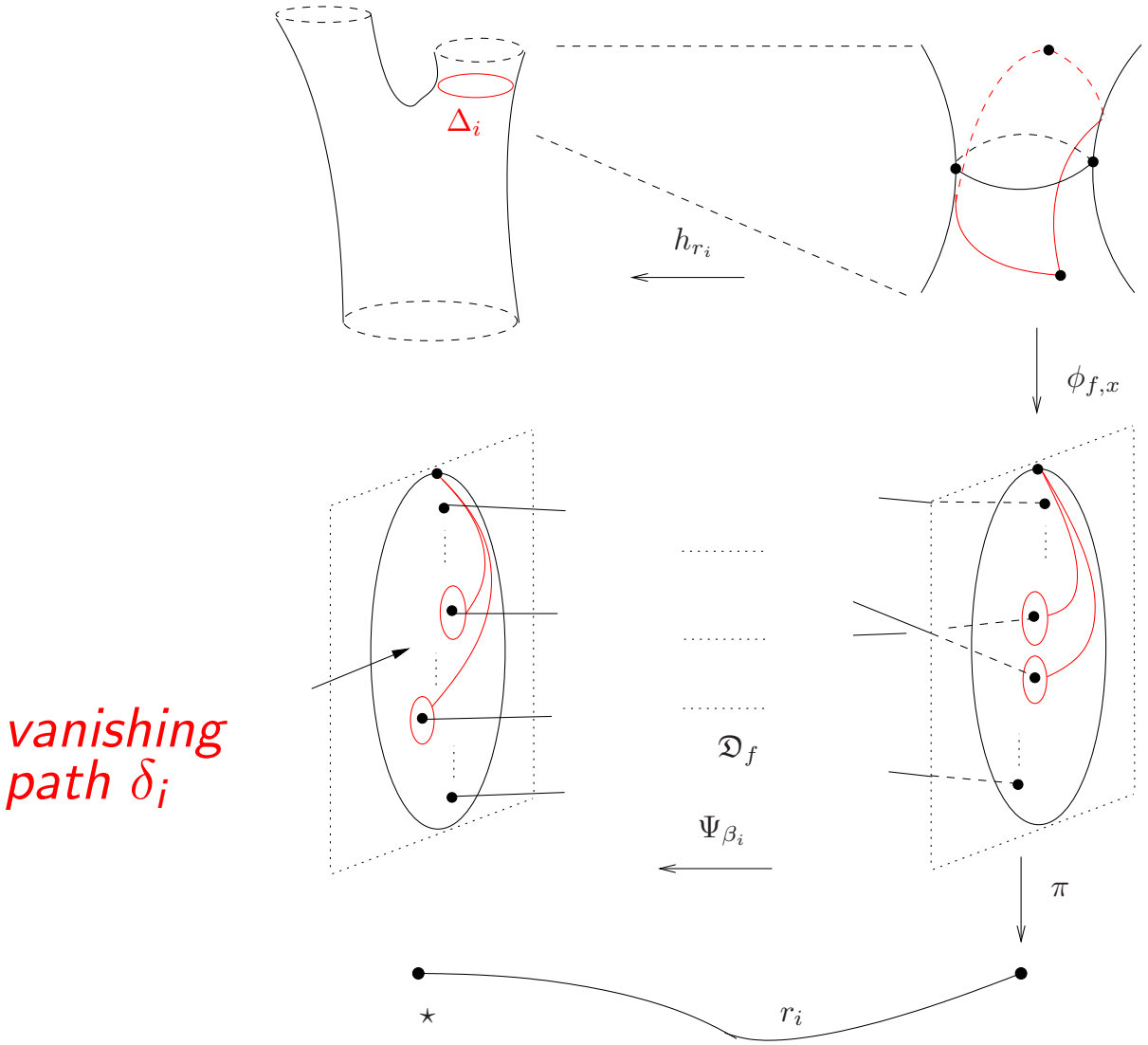
Sketch of the proof.

2. A distinguished basis.



Sketch of the proof.

2. A distinguished basis.

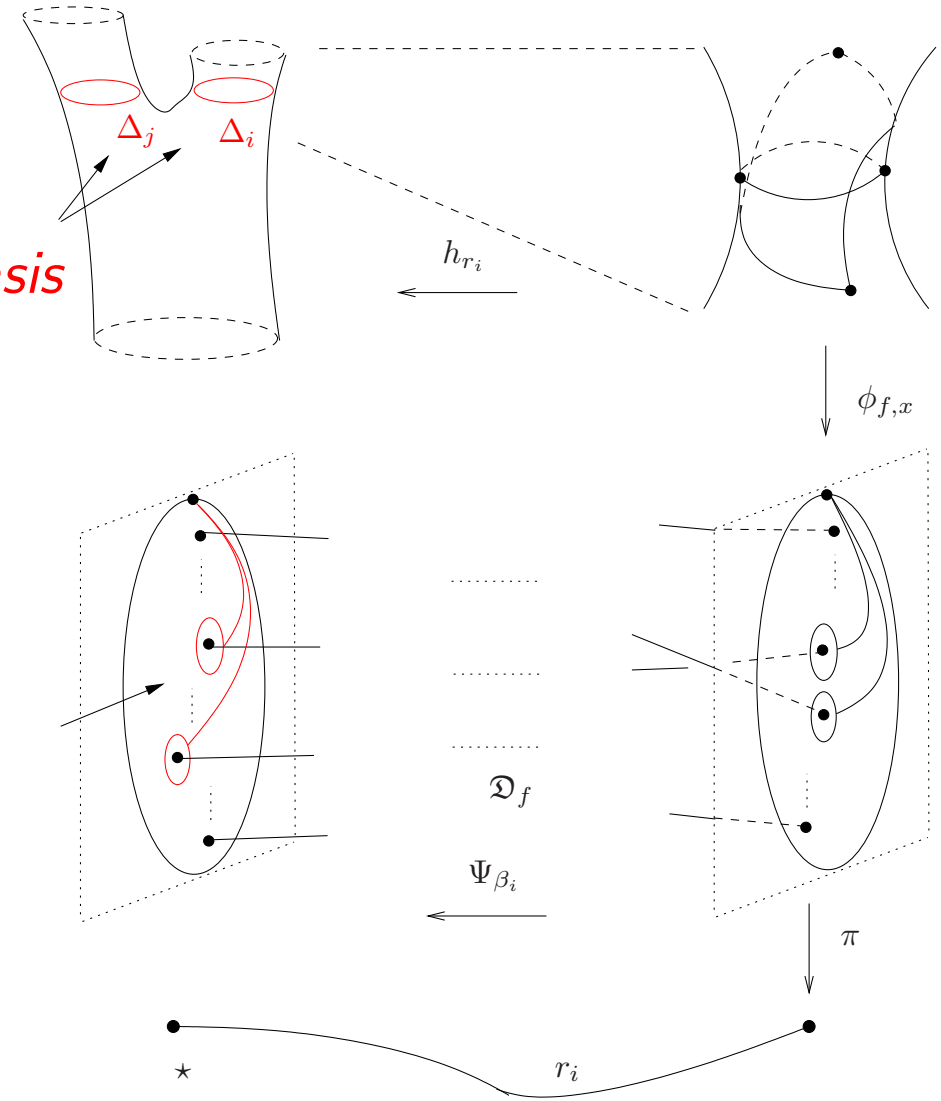


Sketch of the proof.

2. A distinguished basis.

distinguished basis $(\Delta_1, \ldots, \Delta_{\mu(f)})$

vanishing path δ_i

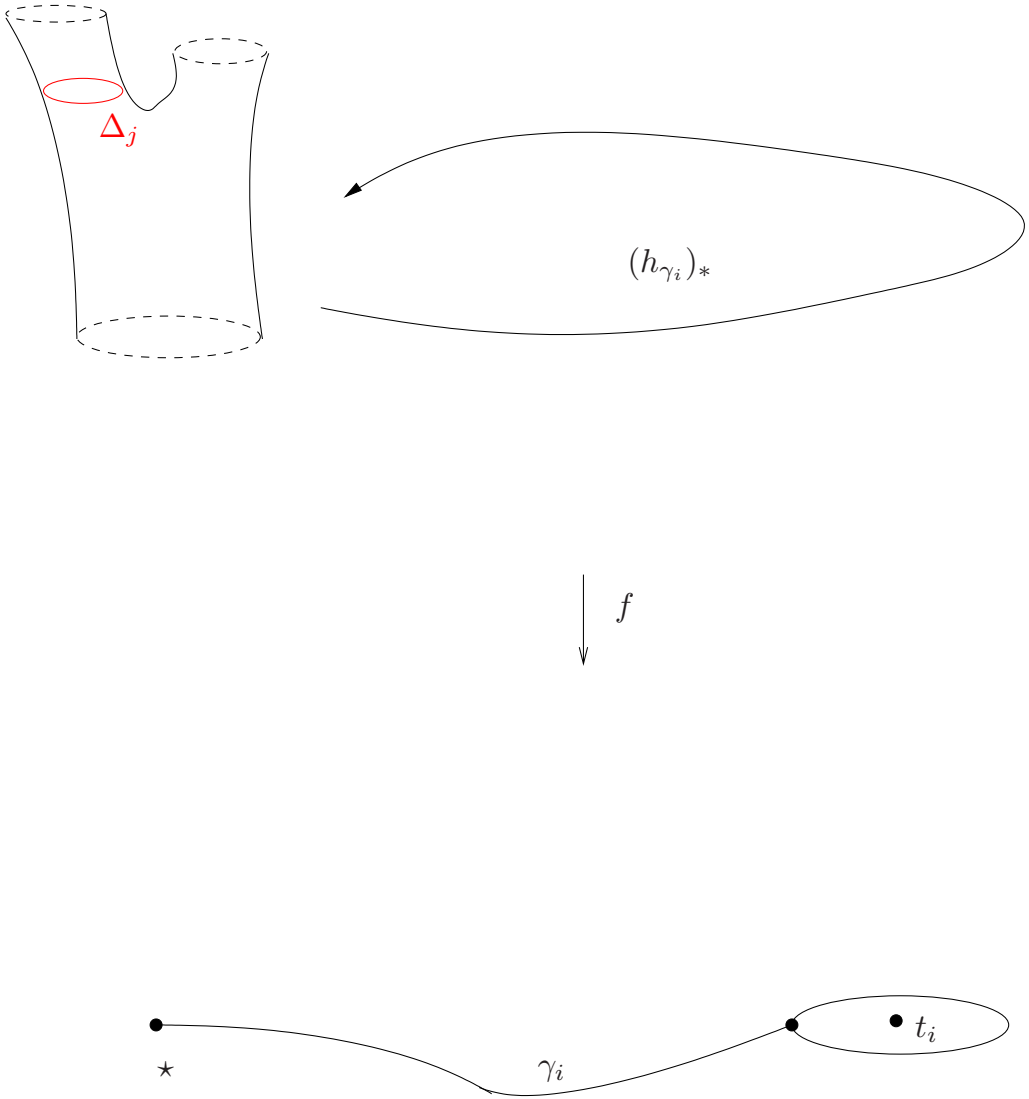


Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.

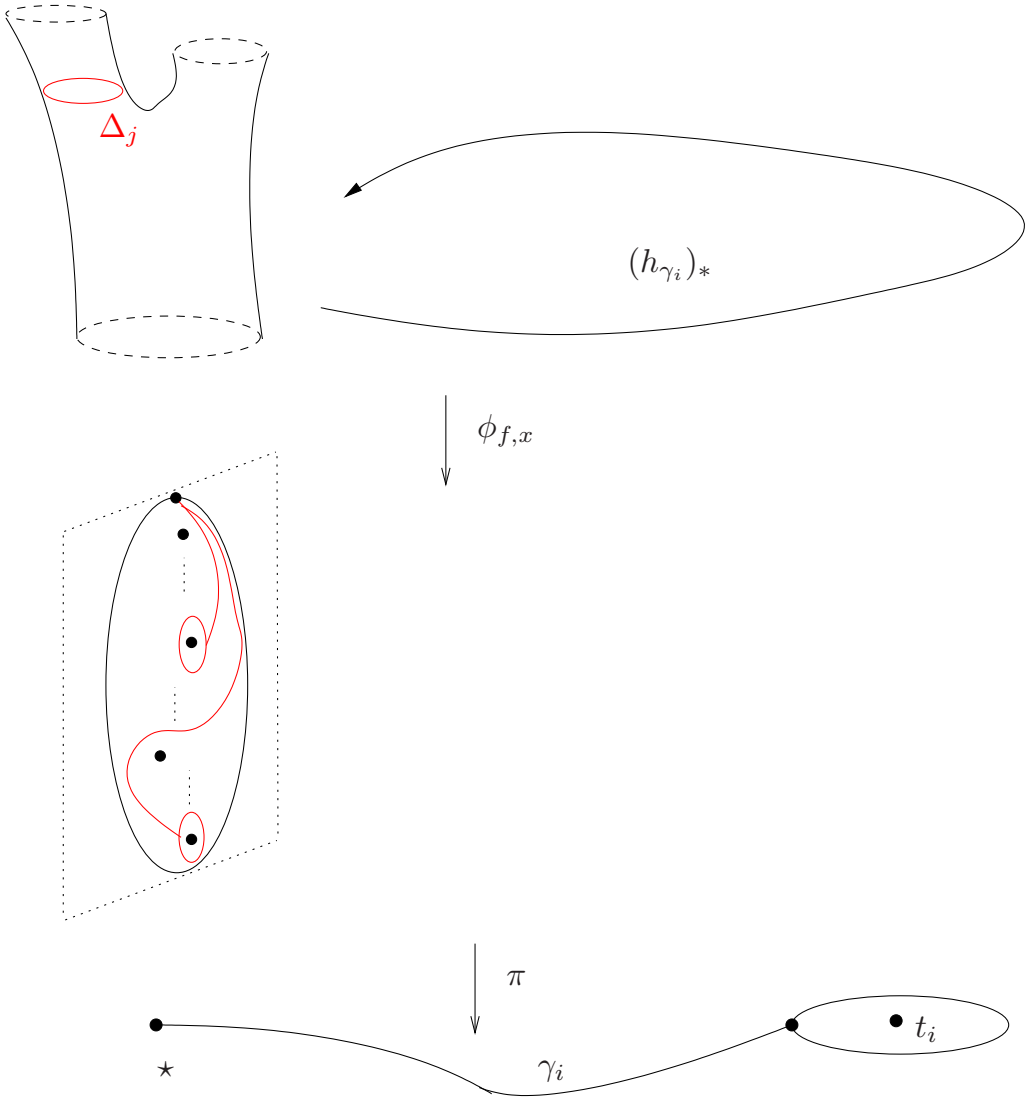
Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.



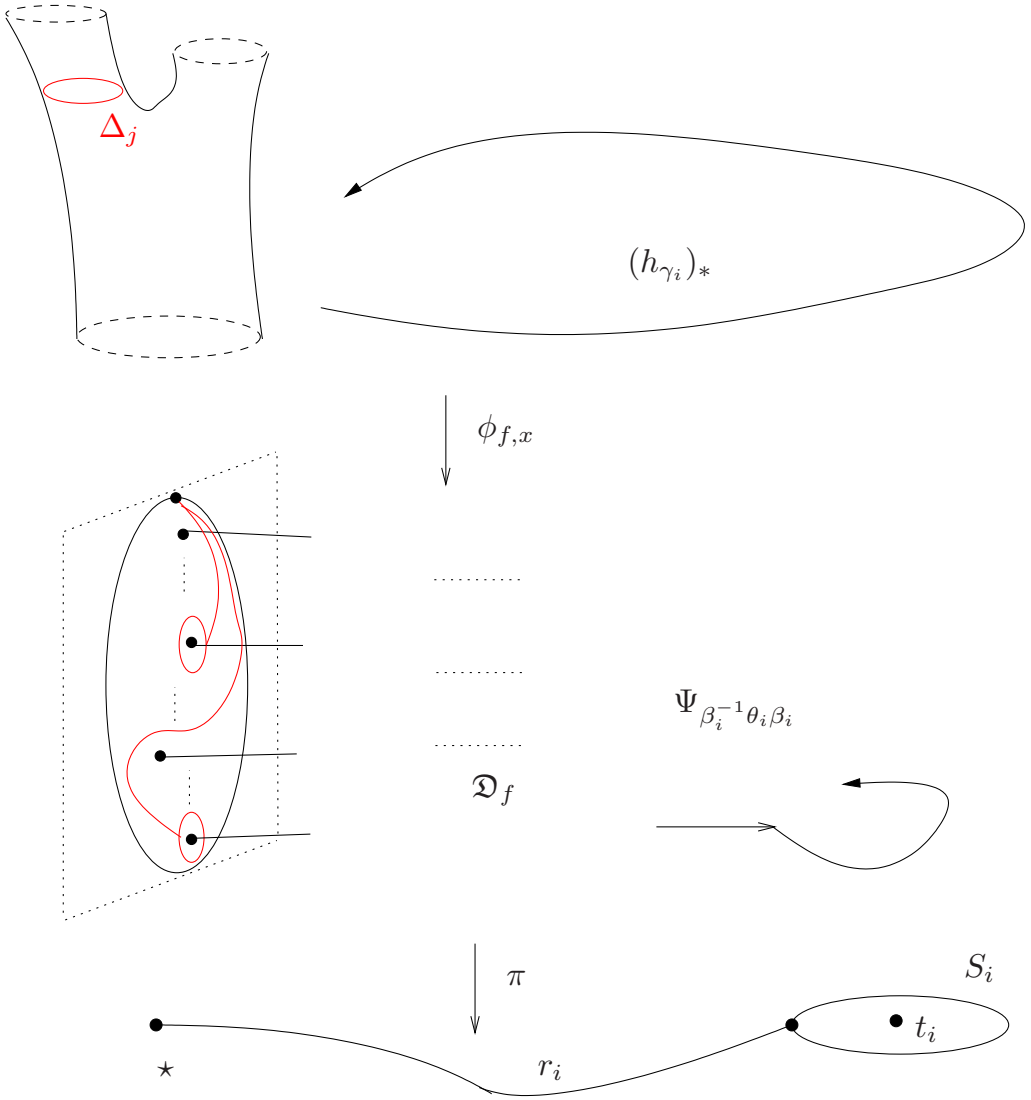
Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.



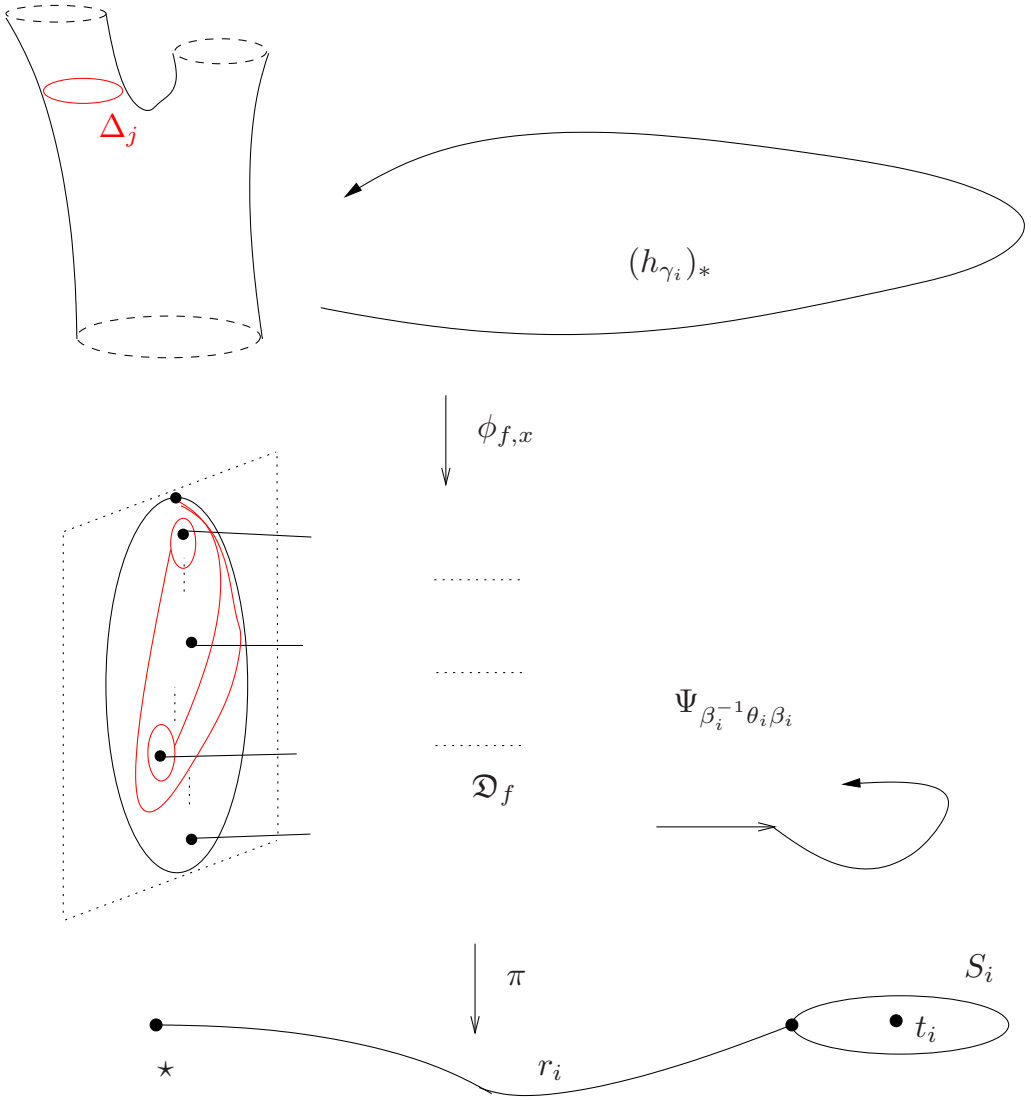
Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.



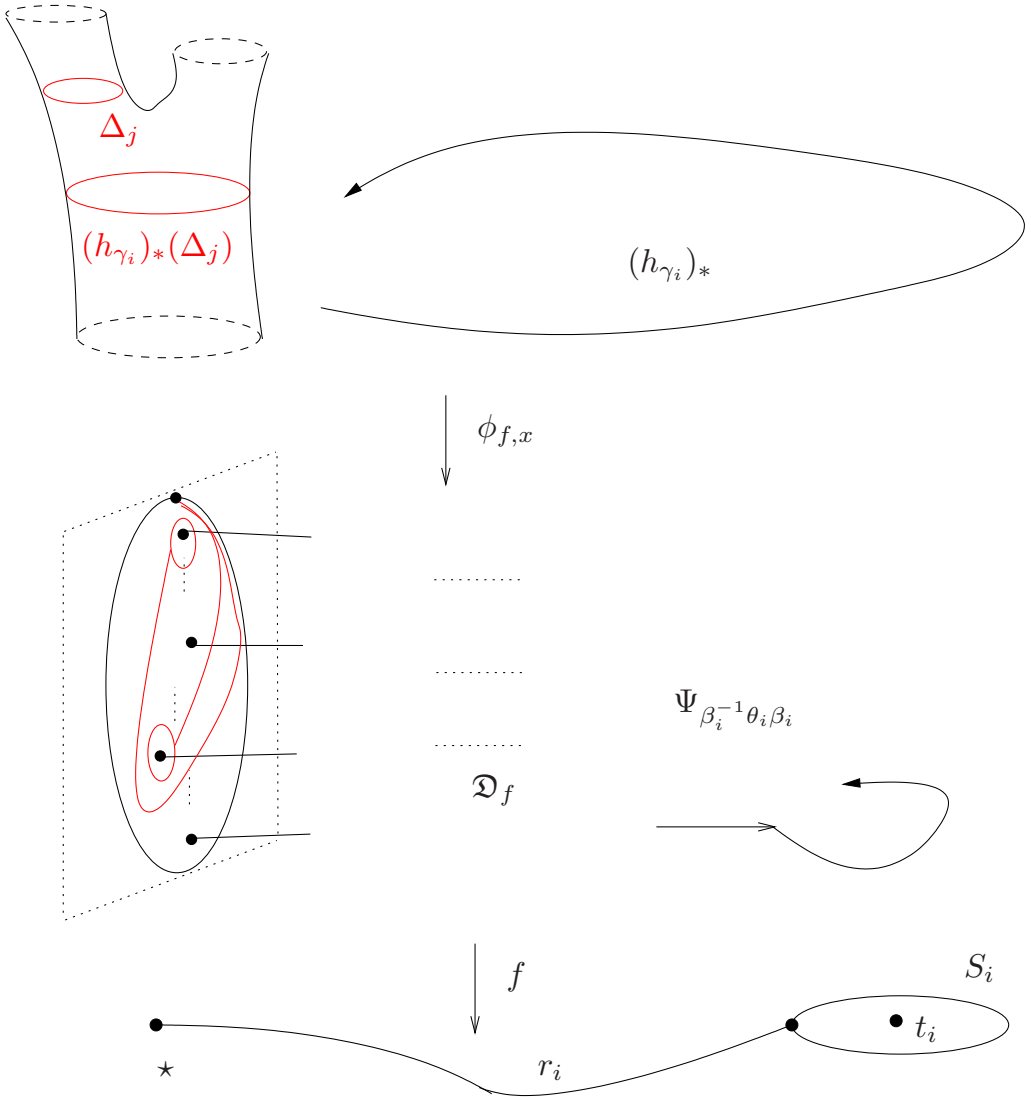
Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.



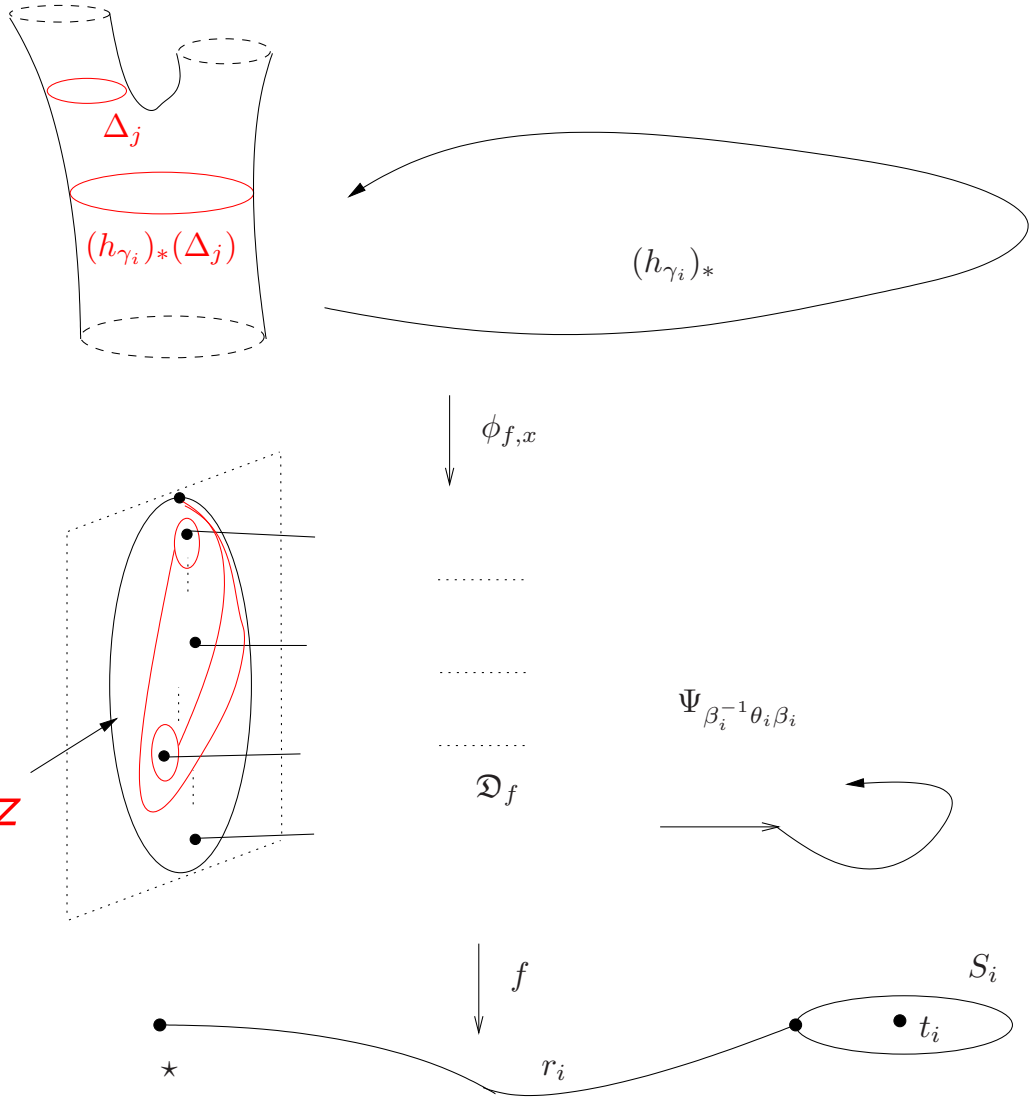
Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.



Sketch of the proof.

3. Homological monodromy $(h_{\gamma_i})_*$.



Picard-Lefschetz transformation of δ_j

Sketch of the proof.

4. Algebraic expression.

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation
$\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j.

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j. Let β_i be an element which conjugates, for example, σ_1.

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j. Let β_i be an element which conjugates, for example, σ_1.
- $\Phi : \pi_1(X \setminus \mathbf{k}; *) \times \mathbb{B}_k \to \pi_1(X \setminus \mathbf{k}; *)$ such that Basis μ_i

$$\mu_j^{\sigma_i} = \begin{cases} \mu_{i+1} & \text{if } j = i \\ \mu_{i+1} \cdot \mu_i \cdot \mu_{i+1}^{-1} & \text{if } j = i+1 \\ \mu_j & \text{if } j \neq i, i+1, \end{cases}$$

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j. Let β_i be an element which conjugates, for example, σ_1.
- $\Phi : \pi_1(X \setminus \mathbf{k}; *) \times \mathbb{B}_k \to \pi_1(X \setminus \mathbf{k}; *)$ such that Basis μ_i

$$\mu_j^{\sigma_i} = \begin{cases} \mu_{i+1} & \text{if } j = i \\ \mu_{i+1} \cdot \mu_i \cdot \mu_{i+1}^{-1} & \text{if } j = i+1 \\ \mu_j & \text{if } j \neq i, i+1, \end{cases}$$

- We compute $(\mu_2 \cdot \mu_1)^{\beta_i}$ and $(\mu_2 \cdot \mu_1)^{\beta_j \nabla_\tau(\gamma_i)}$

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j. Let β_i be an element which conjugates, for example, σ_1.
- $\Phi : \pi_1(X \setminus \mathbf{k}; *) \times \mathbb{B}_k \to \pi_1(X \setminus \mathbf{k}; *)$ such that Basis μ_i

$$\mu_j^{\sigma_i} = \begin{cases} \mu_{i+1} & \text{if } j = i \\ \mu_{i+1} \cdot \mu_i \cdot \mu_{i+1}^{-1} & \text{if } j = i+1 \\ \mu_j & \text{if } j \neq i, i+1, \end{cases}$$

- We compute $(\mu_2 \cdot \mu_1)^{\beta_i}$ and $(\mu_2 \cdot \mu_1)^{\beta_j \nabla_\tau(\gamma_i)}$ $\Rightarrow$ The vanishing path δ_i and the Picard-Lefschetz transformation of δ_j are the image by the Hurwitz move Ψ_τ, respectively. Hurwitz

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j. Let β_i be an element which conjugates, for example, σ_1.
- $\Phi : \pi_1(X \setminus \mathbf{k}; *) \times \mathbb{B}_k \to \pi_1(X \setminus \mathbf{k}; *)$ such that Basis μ_i

$$\mu_j^{\sigma_i} = \begin{cases} \mu_{i+1} & \text{if } j = i \\ \mu_{i+1} \cdot \mu_i \cdot \mu_{i+1}^{-1} & \text{if } j = i+1 \\ \mu_j & \text{if } j \neq i, i+1, \end{cases}$$

- We compute $(\mu_2 \cdot \mu_1)^{\beta_i}$ and $(\mu_2 \cdot \mu_1)^{\beta_j \nabla_\tau(\gamma_i)}$ $\Rightarrow$ The vanishing path δ_i and the Picard-Lefschetz transformation of δ_j are the image by the Hurwitz move Ψ_τ, respectively. Hurwitz
- A model of $H_1(V_\star(f))$ which depends only of $(m(\mu_1^\tau), \ldots, m(\mu_k^\tau))$

Sketch of the proof.

4. Algebraic expression.

- $\mathbb{B}_k$ has the following presentation $\langle \sigma_1, \ldots, \sigma_{k-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$.
- $\nabla_\tau(\gamma_i)$ is a conjugate of any σ_j. Let β_i be an element which conjugates, for example, σ_1.
- $\Phi : \pi_1(X \setminus \mathbf{k}; *) \times \mathbb{B}_k \to \pi_1(X \setminus \mathbf{k}; *)$ such that Basis μ_i

$$\mu_j^{\sigma_i} = \begin{cases} \mu_{i+1} & \text{if } j = i \\ \mu_{i+1} \cdot \mu_i \cdot \mu_{i+1}^{-1} & \text{if } j = i+1 \\ \mu_j & \text{if } j \neq i, i+1, \end{cases}$$

- We compute $(\mu_2 \cdot \mu_1)^{\beta_i}$ and $(\mu_2 \cdot \mu_1)^{\beta_j \nabla_\tau(\gamma_i)}$ $\Rightarrow$ The vanishing path δ_i and the Picard-Lefschetz transformation of δ_j are the image by the Hurwitz move Ψ_τ, respectively. Hurwitz
- A model of $H_1(V_\star(f))$ which depends only of $(m(\mu_1^\tau), \ldots, m(\mu_k^\tau))$ $\Rightarrow$ Δ_i and $(h_{\gamma_i})_*(\Delta_j)$. ■

Remarks

- Since our method strongly uses the discriminant curve $\mathfrak{D}_f$ we call it *the discriminant method*.
- The computation of first and second data can be done with the help of computer programs such as [3] and [5]. Since m and ∇_τ are homotopy invariants we can use any representatives of μ_i^τ and γ_i.
- Different elections of β_i in $\nabla_\tau(\gamma_i)$ result in the same Δ_i and $(h_{\gamma_i})_*(\Delta_j)$ up to orientation.
- The discriminant method is currently implemented in `MAPLE 8` (-) and `SINGULAR 3` (J. Martín).

Applications

- Global and local homological monodromy of two-variable singularities can be effectively computed using the discriminant method.
- The intersection matrix of any Yomdine surface can be computed using Gabrielov's [10] and discriminant methods. This is currently implemented in `MAPLE 8` (-) and `SINGULAR 3` (J. Martín).
- Topological properties of polynomial maps can be detected by means the discriminant method.

Applications

- Global and local homological monodromy of two-variable singularities can be effectively computed using the discriminant method.
- The intersection matrix of any Yomdine surface can be computed using Gabrielov's [10] and discriminant methods. This is currently implemented in `MAPLE 8` (-) and `SINGULAR 3` (J. Martín).
- Topological properties of polynomial maps can be detected by means the discriminant method.
  Example: $\left(y^2x-(y+1)^3\right)\left(s^2\left(2\,s-3\right)y+x-3\,s^2\right)$ with $s\in\{3+2\sqrt{3},3-2\sqrt{3}\}$ are conjugated by the Galois isomorphism $a+b\sqrt{3}\mapsto a-b\sqrt{3}$, $a,b\in\mathbb{Q}$ but they are not topologically equivalent polynomials.

N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes. I*, Math. Ann. **213** (1975), 1–32.

———, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes. II*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 395–404.

D. Bessis and J. Michel, *VKCURVE package for GAP3*, (`www.math.jussieu.fr/~jmichel/vkcurve/vkcurve.html`) (2002).

S. A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math. **92** (1988), no. 2, 217–241.

J. Carmona, *Aproximación numérica de trenzas algebraicas*, Séptimo encuentro de Álgebra computacional y aplicaciones, EACA 2001.

A. Dimca and A. Némethi, *On the monodromy of complex polynomials*, Duke Math. J. **108** (2001), no. 2, 199–209.

_____, *Thom-Sebastiani construction and monodromy of polynomials*, Tr. Mat. Inst. Steklova **238** (2002), no. Monodromiya v Zadachakh Algebr. Geom. i Differ. Uravn., 106–123.

A. M. Gabriélov, *Intersection matrices for certain singularities*, Funkcional. Anal. i Priložen. **7** (1973), no. 3, 18–32.

_____, *Dynkin diagrams of unimodal singularities*, Funkcional. Anal. i Priložen. **8** (1974), no. 3, 1–6, Engl translation in Funct. Anal. Appl. **8** (1974), 192–196.

———, *Polar curves and intersection matrices of singularities*, Invent. Math. **54** (1979), no. 1, 15–22.

S. M. Guseĭn-Zade, *Dynkin diagrams of the singularities of functions of two variables*, Funkcional. Anal. i Priložen. **8** (1974), no. 4, 23–30.

———, *Intersection matrices for certain singularities of functions of two variables*, Funkcional. Anal. i Priložen. **8** (1974), no. 1, 11–15.

A. Hefez and F. Lazzeri, *The intersection matrix of Brieskorn singularities*, Invent. Math. **25** (1974), 143–157.

W. D. Neumann and P. Norbury, *Unfolding polynomial maps at infinity*, Math. Ann. **318** (2000), no. 1, 149–180.

———, *Vanishing cycles and monodromy of complex polynomials*, Duke Math. J. **101** (2000), 487–497.

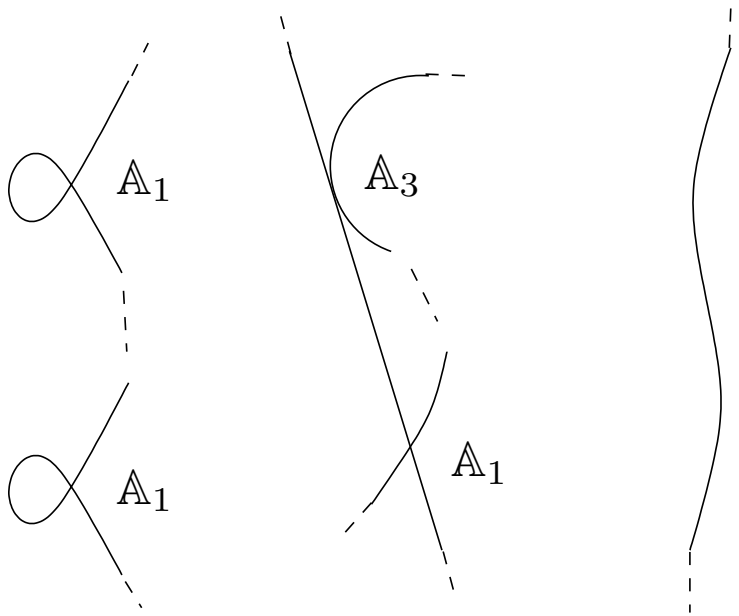
D. Siersma and M. Tibăr, *Singularities at infinity and their vanishing cycles*, Duke Math. J. **80** (1995), no. 3, 771–783.

Example

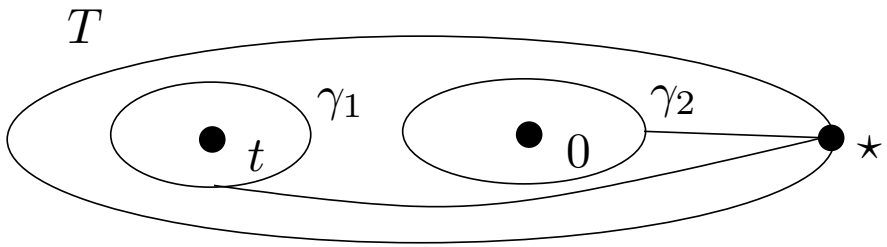
$$\left(y^2x-(y+1)^3\right)\left(s^2\,(2\,s-3)\,y+x-3\,s^2\right)\ \text{with}\ s=3-2\sqrt{3}.$$

$$(h_{\gamma_1})_* = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$(h_{\gamma_2})_* = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 1 \end{array}\right]$$



$$\mathbb{I} = \left[\begin{array}{cc|cccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{array}\right]$$



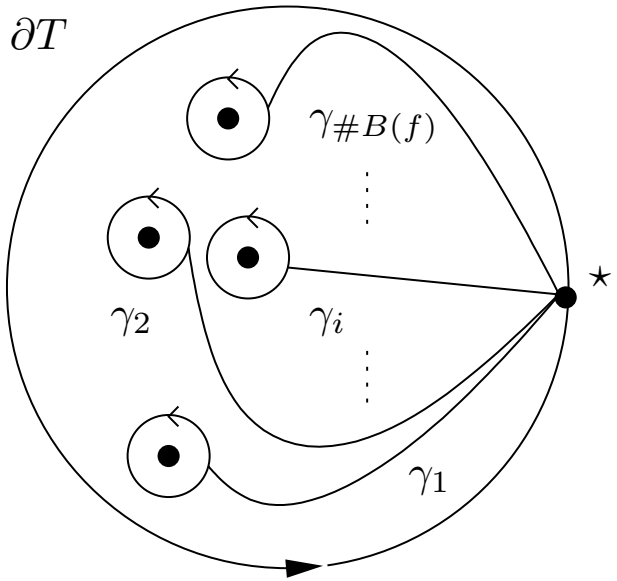
Geometric basis

Let $T \subset \mathbb{C}$ be a geometric disk such that $B(f) = \{t_i\} \subset \mathrm{Int}(T)$ and $\star \in \partial T$.

Definition 1

A *geometric basis* of the group $\pi_1(T \setminus B(f); \star)$ is an ordered list $(\gamma_1, \gamma_2, \ldots, \gamma_{\#B(f)})$ such that:

- γ_i is a simple meridian in $T \setminus B(f)$ based at $\star$.
- $\mathrm{Supp}(\gamma_i) \cap \mathrm{Supp}(\gamma_j) = \{\star\}$ for all i, j with $i \neq j$.
- $\gamma_{\#B(f)} \cdot \ldots \cdot \gamma_1$ is homotopic to ∂T which is positively oriented (product from left to right).

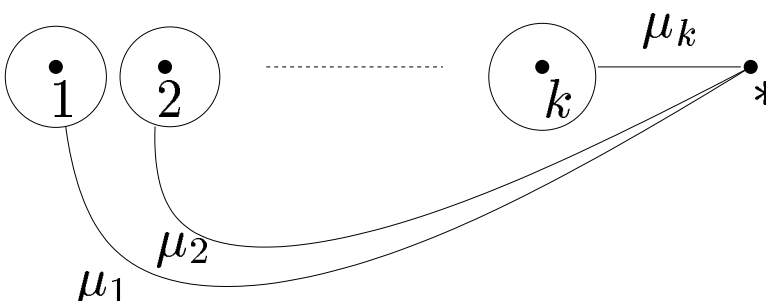


First datum

- Let X be a big geometric disk such that

$$\mathbf{x}^\star, \mathbf{k} := \{1, \ldots, k\} \subset \mathrm{Int}(X).$$

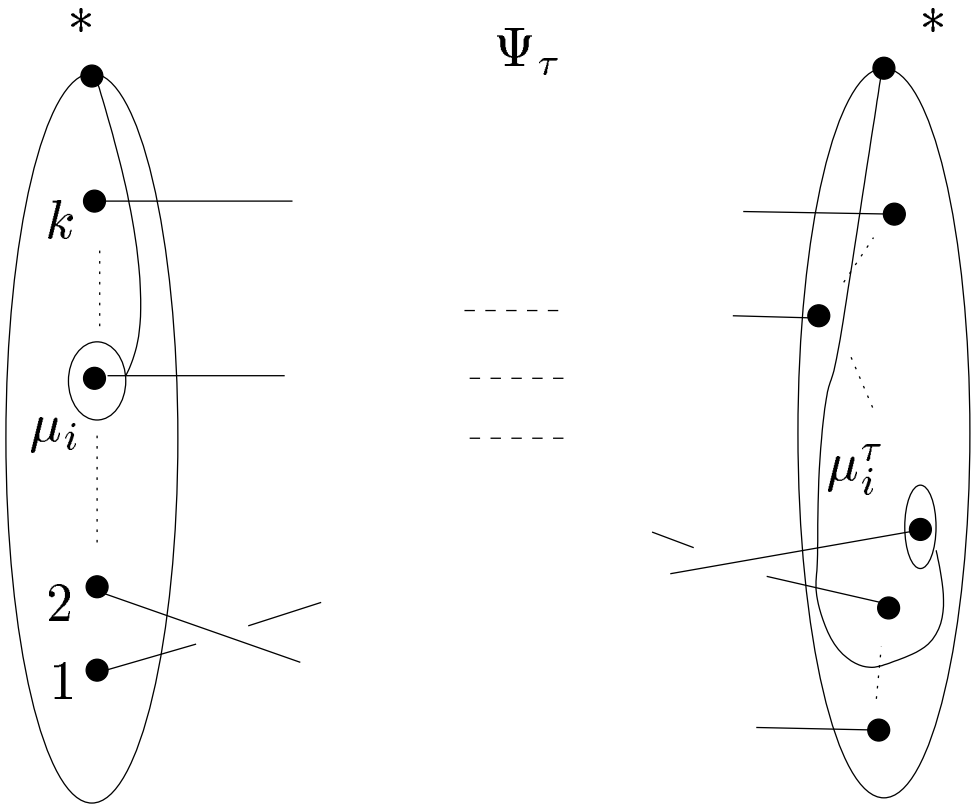
- Let $(\mu_1, \ldots, \mu_k)$ be the following geometric basis of $\pi_1(X \setminus \mathbf{k}; *)$:



- Let us fix $\tau \in \mathbb{B}(\mathbf{k}, \mathbf{x}^\star)$.

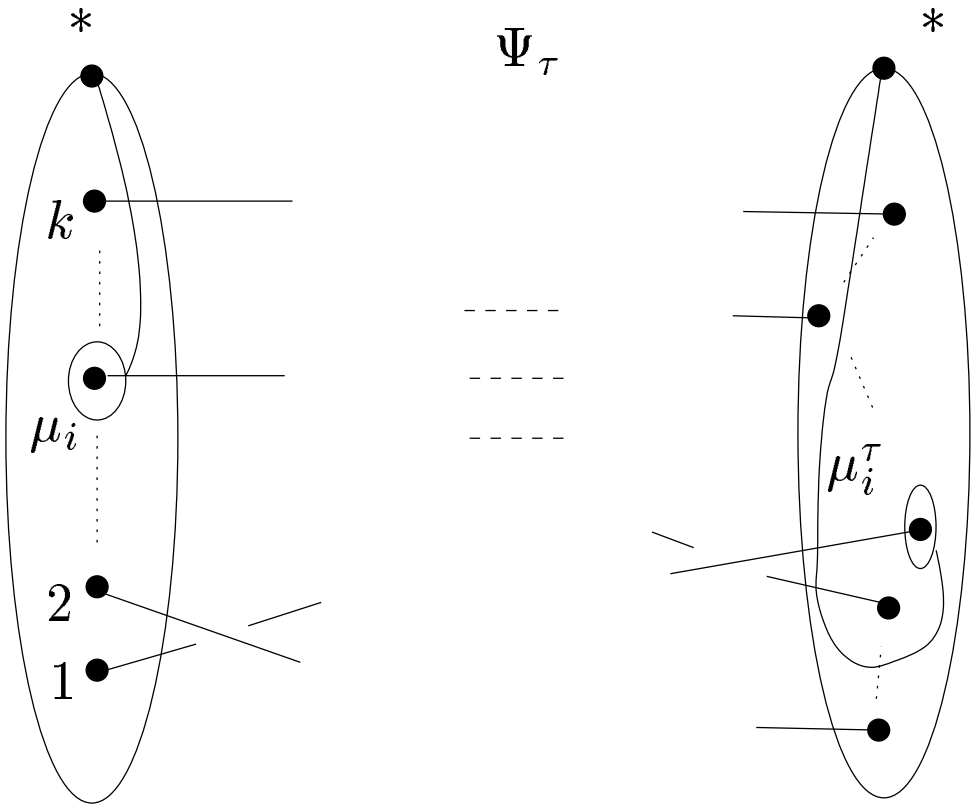
First datum

The Hurwitz move $\Psi_\tau : \pi_1(X \setminus \mathbf{k}; *) \longrightarrow \pi_1(X \setminus \mathbf{x}^\star; *)$ gives us the geometric basis $(\mu_1^\tau, \ldots, \mu_k^\tau)$ of $\pi_1(X \setminus \mathbf{x}^\star; *)$.



First datum

The Hurwitz move $\Psi_\tau : \pi_1(X \setminus \mathbf{k}; *) \longrightarrow \pi_1(X \setminus \mathbf{x}^\star; *)$ gives us the geometric basis $(\mu_1^\tau, \ldots, \mu_k^\tau)$ of $\pi_1(X \setminus \mathbf{x}^\star; *)$.



The first datum is: $(m(\mu_1^\tau), \ldots, m(\mu_k^\tau)) \in \Sigma_N^k$

Second datum

- Consider the projection map

$$\pi : (\mathbb{C}^2, \mathfrak{D}_f) \longrightarrow \mathbb{C}, (t, x) \mapsto t$$

- The second member of the pair is a k-fold covering ramified on a finite set of points $\mathcal{T}$.
- The fundamental group of the base, $\pi_1(\mathbb{C} \setminus \mathcal{T}; \star)$, induces the braid monodromy ∇_τ of the pair $(\mathbb{C}^2, \mathfrak{D}_f)$ with respect to the projection π:

$$\begin{array}{rccccc} \nabla_\tau : & \pi_1(\mathbb{C} \setminus \mathcal{T}; \star) & \longrightarrow & \mathbb{B}(\mathbf{x}^\star, \mathbf{x}^\star) & \longrightarrow & \mathbb{B}_k \\ & \gamma & \mapsto & \pi|_{\mathfrak{D}_f}^{-1}(\gamma) =: \gamma^\star & \mapsto & \tau \cdot \gamma^\star \cdot \tau^{-1} \end{array}$$

(recall $\mathbf{x}^\star = \mathfrak{D}_f \cap \{t = \star\}$ and $\tau \in \mathbb{B}(\mathbf{k}, \mathbf{x}^\star)$)

Second datum

$$\begin{array}{rccccc} \nabla_\tau : & \pi_1(\mathbb{C} \setminus \mathcal{T}; \star) & \longrightarrow & \mathbb{B}(\mathbf{x}^\star, \mathbf{x}^\star) & \longrightarrow & \mathbb{B}_k \\ & \gamma & \mapsto & \pi|_{\mathfrak{D}_f}^{-1}(\gamma) =: \gamma^\star & \mapsto & \tau \cdot \gamma^\star \cdot \tau^{-1} \end{array}$$

- **Remark:** Since f is good at infinity, its discriminant curve has no vertical asymptotes $\Rightarrow \mathcal{T} = f(P) \cup \pi(\mathrm{Sing}(\mathfrak{D}_f))$ (disjoint since (x, y) generic).

Second datum

$$\begin{array}{rccccc} \nabla_\tau : & \pi_1(\mathbb{C} \setminus \mathcal{T}; \star) & \longrightarrow & \mathbb{B}(\mathbf{x}^\star, \mathbf{x}^\star) & \longrightarrow & \mathbb{B}_k \\ & \gamma & \mapsto & \pi|_{\mathfrak{D}_f}^{-1}(\gamma) =: \gamma^\star & \mapsto & \tau \cdot \gamma^\star \cdot \tau^{-1} \end{array}$$

- **Remark:** Since f is good at infinity, its discriminant curve has no vertical asymptotes $\Rightarrow \mathcal{T} = f(P) \cup \pi(\mathrm{Sing}(\mathfrak{D}_f))$ (disjoint since (x, y) generic).
- We can assume that $\mathrm{Supp}(\gamma_i) \cap \pi(\mathrm{Sing}(\mathfrak{D}_f)) = \emptyset$.

Second datum

$$\begin{array}{rccccc} \nabla_\tau : & \pi_1(\mathbb{C} \setminus \mathcal{T}; \star) & \longrightarrow & \mathbb{B}(\mathbf{x}^\star, \mathbf{x}^\star) & \longrightarrow & \mathbb{B}_k \\ & \gamma & \mapsto & \pi|_{\mathfrak{D}_f}^{-1}(\gamma) =: \gamma^\star & \mapsto & \tau \cdot \gamma^\star \cdot \tau^{-1} \end{array}$$

- **Remark:** Since f is good at infinity, its discriminant curve has no vertical asymptotes $\Rightarrow \mathcal{T} = f(P) \cup \pi(\mathrm{Sing}(\mathfrak{D}_f))$ (disjoint since (x, y) generic).
- We can assume that $\mathrm{Supp}(\gamma_i) \cap \pi(\mathrm{Sing}(\mathfrak{D}_f)) = \emptyset$.

The second datum is

$$(\nabla_\tau(\gamma_1), \ldots, \nabla_\tau(\gamma_{\mu(f)})) \in \mathbb{B}_k^{\mu(f)}.$$

Tame polynomials: reduction to the Morse case