Parametric polynomial minimal surfaces of arbitrary degree

Gang Xu[a], Guozhao Wang[b]

[a] *Galaad, INRIA Sophia-Antipolis, 2004 Route des Lucioles, 06902 Cedex, France*
[b] *Department of Mathematics, Zhejiang University, Hangzhou, China*

Abstract

Weierstrass representation is a classical parameterization of minimal surfaces. However, two functions should be specified to construct the parametric form in Weierestrass representation. In this paper, we propose an explicit parametric form for a class of parametric polynomial minimal surfaces of arbitrary degree. It includes the classical Enneper surface for cubic case. The proposed minimal surfaces also have some interesting properties such as symmetry, containing straight lines and self-intersections. According to the shape properties, the proposed minimal surface can be classified into four categories with respect to $n = 4k - 1$ $n = 4k + 1$, $n = 4k$ and $n = 4k + 2$. The explicit parametric form of corresponding conjugate minimal surfaces is given and the isometric deformation is also implemented.

Keywords: minimal surface; parametric polynomial minimal surface; Enneper surface ; conjugate minimal surface

1. Introduction

Minimal surface is a kind of surface with vanishing mean curvature [1]. As the mean curvature is the variation of area functional, minimal surfaces include the surfaces minimizing the area with a fixed boundary [2, 3]. There have been many literatures on minimal surface in classical differential geometry [4, 5, 6]. Because of their attractive properties, minimal surfaces have been extensively employed in many areas such as architecture, material science, aviation, ship manufacture, biology and so on. For instance, the shape of the membrane structure, which has appeared frequently in modern architecture, is mainly based on minimal surfaces [8]. Furthermore, triply periodic minimal surfaces naturally arise in a variety of systems, including nano-composites, lipid-water systems and certain cell membranes [9].

Email address: gxu@sophia.inria.fr (Gang Xu)

In CAD systems, parametric polynomial representation is the standard form. For parametric polynomial minimal surface, plane is the unique quadratic parametric polynomial minimal surface, Enneper surface is the unique cubic parametric polynomial minimal surface. There are few research work on the parametric form of polynomial minimal surface with higher degree. Weierstrass representation is a classical parameterization of minimal surfaces. However, two functions should be specified to construct the parametric form in Weierestrass representation. In this paper, we discuss the answer to the following questions: What are the possible explicit parametric form of polynomial minimal surface of arbitrary degree and how about their properties? The proposed minimal surfaces include the classical Enneper surface for cubic case, and also have some interesting properties such as symmetry, containing straight lines and self-intersections. According to the shape properties, the proposed minimal surface can be classified into four categories with respect to $n = 4k-1$ $n = 4k$, $n = 4k+1$ and $n = 4k+2$. The explicit parametric form of corresponding conjugate minimal surfaces is given and the isometric deformation is also implemented.

The paper includes five sections. *Preliminary* introduces some notations and lemmas. *Main Results* presents the explicit parametric formula of parametric polynomial minimal surface of arbitrary degree. The next section, *Properties and Classification* presents the corresponding properties and classification of the proposed minimal surfaces. The following section focuses on corresponding conjugate counterpart of proposed minimal surface. Finally, in *Conclusions*, we summarize the main results.

2. Preliminary

In this section, we introduce the following two notations.

$$P_n = \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} (-1)^k \binom{n}{2k} u^{n-2k} v^{2k}, \tag{1}$$

$$Q_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} u^{n-2k-1} v^{2k+1} \tag{2}$$

P_n and Q_n have the following properties:

Lemma 1

$$\frac{\partial P_n}{\partial u} = nP_{n-1}, \quad \frac{\partial P_n}{\partial v} = -nQ_{n-1}$$
$$\frac{\partial Q_n}{\partial u} = nQ_{n-1}, \quad \frac{\partial Q_n}{\partial v} = nP_{n-1}$$

Lemma 2

$$\begin{aligned} P_n &= uP_{n-1} - vQ_{n-1} \\ Q_n &= vP_{n-1} + uQ_{n-1} \end{aligned}$$

Lemma 2 can be proved by using the following equation:

$$\binom{n}{2k} + \binom{n}{2k+1} = \binom{n+1}{2k+1}.$$

3. Main Results

Theorem 1 If the parametric representation of polynomial surface $\mathbf{r}(u,v)$ with arbitrary degree n, is given by $\mathbf{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$, where

$$\begin{aligned} X(u,v) &= -P_n + \omega P_{n-2}, \\ Y(u,v) &= Q_n + \omega Q_{n-2}, \\ Z(u,v) &= \frac{2\sqrt{n(n-2)\omega}}{n-1} P_{n-1}, \end{aligned} \tag{3}$$

then $\mathbf{r}(u,v)$ is a minimal surface.

Proof of Theorem 1. From Lemma 2, we have

$$\frac{\partial^2 \boldsymbol{r}(u,v)}{\partial^2 u} + \frac{\partial^2 \boldsymbol{r}(u,v)}{\partial^2 v} = 0$$

Hence, $\boldsymbol{r}(u,v)$ is harmonic surface.

By using Lemma 2, we have

$$\begin{aligned} F &= \frac{\partial \boldsymbol{r}(u,v)}{\partial u} \frac{\partial \boldsymbol{r}(u,v)}{\partial v} \\ &= 2n(n-2)\omega(Q_{n-3}P_{n-1} + P_{n-3}Q_{n-1} - 2Q_{n-2}P_{n-2}) \end{aligned} \tag{4}$$

From Lemma 2, we have

$$P_{n-2} = uP_{n-3} - vQ_{n-3}, \tag{5}$$

$$Q_{n-2} = vP_{n-3} + uQ_{n-3}, \tag{6}$$

$$P_{n-1} = (u^2 - v^2)P_{n-3} - 2uvQ_{n-3}, \tag{7}$$

$$Q_{n-1} = (u^2 - v^2)Q_{n-3} + 2uvP_{n-3}, \tag{8}$$

Substituting (5)(6)(7)(8) into (4), we can obtain $F = 0$. Similarly, we have

$$\begin{aligned} E - G &= \frac{\partial \boldsymbol{r}(u,v)}{\partial u}\frac{\partial \boldsymbol{r}(u,v)}{\partial u} - \frac{\partial \boldsymbol{r}(u,v)}{\partial v}\frac{\partial \boldsymbol{r}(u,v)}{\partial v} \\ &= 4n(n-2)\omega(Q_{n-1}Q_{n-3} - P_{n-3}P_{n-1} + P_{n-2}^2 - Q_{n-2}^2) \\ &= 0 \end{aligned}$$

Hence, ***r(u,v)*** is a parametric surface with isothermal parameterization. From [1], if a parametric surface with isothermal parameterization is harmonic, then it is a minimal surface. The proof is completed.

4. Properties and Classification

From *Theorem 1*, if $n = 3$, we can get the Enneper surface, which is the unique cubic parametric polynomial minimal surface. It has the following parametric form

$$\boldsymbol{E}(u,v) = (-(u^3 - 3uv^2) + \omega u, -(v^3 - 3vu^2) + \omega v, \sqrt{3\omega}(u^2 - v^2)).$$

Enneper surface has several interesting properties, such as symmetry, self-intersection, and containing orthogonal straight lines on it. For the new proposed minimal surface, we can also prove that it has also has these properties.

If $n = 5$, a kind of quintic polynomial minimal surface proposed in [10] can be obtained as follows

$$\boldsymbol{Q}(u,v) = (X(u,v), Y(u,v), Z(u,v)). \tag{9}$$

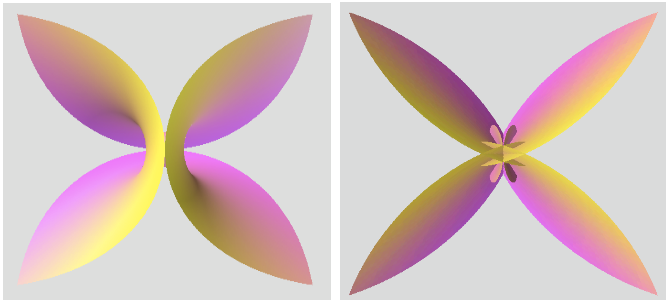
where

$$\begin{aligned} X(u,v) &= -(u^5 - 10u^3v^2 + 5uv^4) + \omega u(u^2 - 3v^2), \\ Y(u,v) &= -(v^5 - 10v^3u^2 + 5vu^4) + \omega v(v^2 - 3u^2), \\ Z(u,v) &= \frac{\sqrt{15\omega}}{2}(u^4 - 6u^2v^2 + v^4). \end{aligned}$$

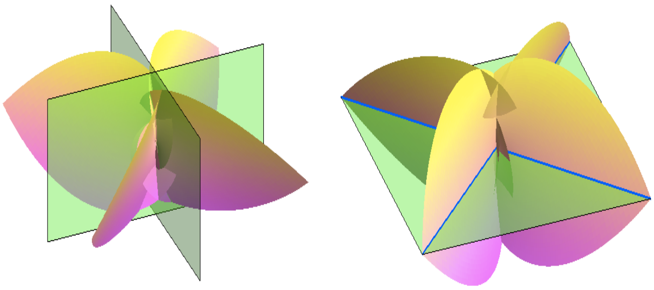
According to the shape properties, the proposed minimal surface in *Theorem 1* can be classified into four classes with $n = 4k - 1$ $n = 4k$, $n = 4k + 1$, $n = 4k + 2$.

Proposition 1. In case of $n = 4k - 1$, the corresponding proposed minimal surface $\boldsymbol{r}(u,v)$ has the following properties:

- $\boldsymbol{r}(u,v)$ is symmetric about the plane $X = 0$ and the plane $Y = 0$,



(a) Ennerper surface (b) Polynomial surface of degree 7



(a) Symmetry plane (b) Straight lines

Figure 1: Enneper surface and minimal surface of degree seven. $\omega = 1, [-1, 1] \times [-1, 1]$.

- $\boldsymbol{r}(u, v)$ contains two orthogonal straight lines $x = \pm y$ on the plane $Z = 0$

Fig. 1(a) shows an example of Enneper surface, Fig. 1 (b) shows an example of proposed minimal surface with $n = 7$. The symmetry plane and straight lines of minimal surface in Fig. 1 (b) are shown in Fig. 1 (c) and Fig. 1 (d).

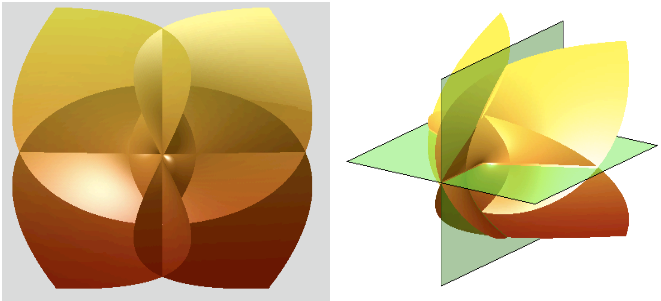
Proposition 2. In case of $n = 4k$, the corresponding proposed minimal surface $\boldsymbol{r}(u, v)$ is symmetric about the plane $Z = 0$ and the plane $Y = 0$.

Fig. 2 (a) present an example of proposed quartic minimal surface and the corresponding symmetry planes are shown in Fig. 2 (b).

Proposition 3. In case of $n = 4k + 1$, the corresponding proposed minimal surface $\boldsymbol{r}(u, v)$ has the following properties:

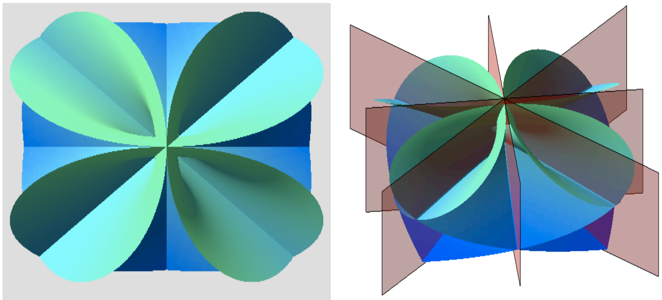
- $\boldsymbol{r}(u, v)$ is symmetric about the plane $X = 0$, the plane $Y = 0$, the plane $X = Y$ and the plane $X = -Y$.

- Self-intersection points of $\boldsymbol{r}(u, v)$ are only on the symmetric planes, i.e., there are no other



(a) Quartic minimal surface (b) Symmetry plane

Figure 2: Quartic minimal surface and its symmetry plane. $\omega = 1, [-1,1] \times [-1,1]$.



(a) Quintic minimal surface (b) Symmetry plane

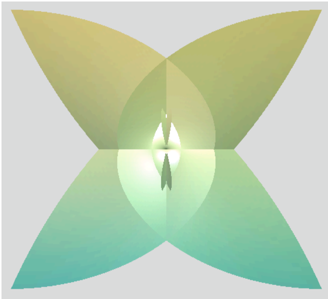
Figure 3: Quintic minimal surface and its symmetry plane. $\omega = 1, [-1,1] \times [-1,1]$.

self-intersection points on $\underline{\boldsymbol{r}}(u,v)$, and the self-intersection curve has the same symmetric plane as the minimal surface.

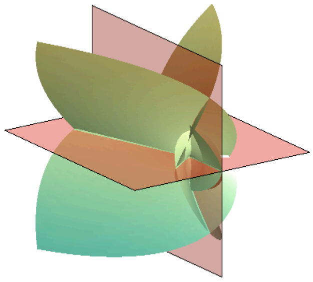
Fig. 3 (a) present an example of proposed quintic minimal surface and the corresponding symmetry planes are shown in Fig. 3 (b).

Proposition 4. In case of $n = 4k+2$, the corresponding proposed minimal surface $\boldsymbol{r}(u,v)$ is symmetric about the plane $Z = 0$ and the plane $Y = 0$.

For the case of $n = 6$, it has been studied in [11]. Fig.4 (a) present an example of proposed minimal surface with $n = 6$ and the corresponding symmetry planes are shown in Fig. 4 (b).



(a) Minimal surface of degree six



(b) Symmetry plane

Figure 4: Minimal surface of degree six and its symmetry plane. $\omega = 1, [-1, 1] \times [-1, 1]$.

5. Conjugate Minimal Surface

Definition 1. If two differentiable functions $p(u, v), q(u, v) : \boldsymbol{U} \mapsto \boldsymbol{R}$ satisfy the Cauchy-Riemann equations

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial v}, \frac{\partial p}{\partial v} = -\frac{\partial q}{\partial u},$$

and both are harmonic, then the functions are said to be *harmonic conjugate*.

Definition 2. If $\boldsymbol{P} = (p_1, p_2, p_3)$ and $\boldsymbol{Q} = (q_1, q_2, q_3)$ are with isothermal parameterizations such that p_k and q_k are harmonic conjugate for $k = 1, 2, 3$, then $\boldsymbol{P}$ and $\boldsymbol{Q}$ are said to be *parametric conjugate minimal surfaces*.

Helicoid and catenoid are a pair of conjugate minimal surfaces. For $\boldsymbol{r}(u, v)$, we can find out a new pair of conjugate minimal surfaces as follows.

Theorem 2 The conjugate minimal surface of $\mathbf{r}(u, v)$ has the following parametric form

$$\mathbf{s}(u, v) = (X_s(u, v), Y_s(u, v), Z_s(u, v))$$

where

$$\begin{aligned} X_s(u, v) &= -Q_n + \omega Q_{n-2}, \\ Y_s(u, v) &= -P_n - \omega P_{n-2}, \\ Z_s(u, v) &= \frac{2\sqrt{n(n-2)\omega}}{n-1} Q_{n-1}, \end{aligned} \tag{10}$$

It can be proved directly by Lemma 2. From [2], the surfaces of one-parametric family

$$\boldsymbol{C}_t(u, v) = (\cos t)\boldsymbol{r}(u, v) + (\sin t)\boldsymbol{s}(u, v)$$

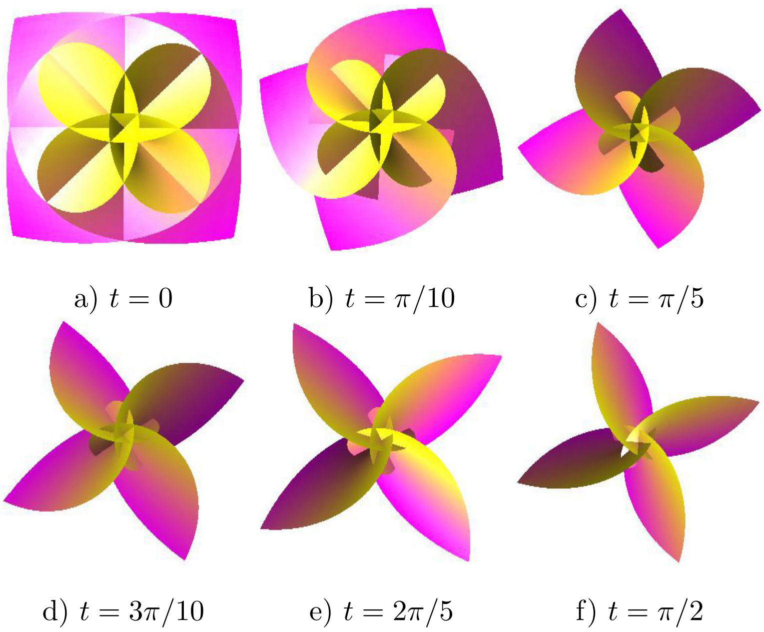


Figure 5: Dynamic deformation between $\boldsymbol{r}(u,v)$ and $\boldsymbol{s}(u,v)$, $u,v \in [-4,4]$.

are minimal surfaces with the same first fundamental form. These minimal surfaces are isometric and have the same Gaussian curvature at corresponding points. Fig. 5 illustrates the isometric deformation between $\boldsymbol{r}(u,v)$ and $\boldsymbol{s}(u,v)$. It is similar with the isometric deformation between helicoid and catenoid.

6. Conclusion

The explicit parametric formula of polynomial minimal surface is presented. It can be considered as the generalization of Enneper surface in cubic case. The corresponding properties and classification of the proposed minimal surface are investigated. The corresponding conjugate minimal surface are constructed and the dynamic isometric deformation between them are also implemented.

Acknowlegments

This work was partially supported by the National Nature Science Foundation of China (No.60970079, 60933008), Foundation of State Key Basic Research 973 Development Programming Item of China (No.2004CB318000), and the Nature Science Foundation of Zhejiang Province, China(No. Y1090718).

References

[1] Nitsche J.C.C. 1989. Lectures on minimal surfaces, vol. 1. Cambridge Univ. Press, Cambridge

[2] Frank Morgan. Minimal surfaces, crystals, and norms on Rn, Proc. 7th Annual Symp. on Computational Geom., June, 1991, N. Conway, NH.

[3] Frank Morgan. Area-minimizing surfaces, faces of Grassmannians, and calibrations. The Am. Math. Monthly 95 (1988), 813-822

[4] Osserman R.. 1986. A survey of minimal surfaces, Dover publ. 2nd ed., New York

[5] W. H. Meeks III, L. Simon and S. T. Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature . Ann. of Math. (2) 116 (1982), no. 3, 621-659.

[6] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness . Math. Z. 179(1982), 151-168.

[7] T.H. Colding and W.P.Minicozzi II. Shapes of embedded minimal surfaces, Proc. Nat. Acad. Sciences, July 25, 2006, vol. 103, no. 30, 11106-11111

[8] Bletzinger K.W.. 1997. Form finding of membrane structures and minimal surfaces by numerical continuation. Proceeding of the IASS Congress on Structural Morphology: Towards the New Millennium, Nottingham. ISBN 0-85358-064-2, 68-75

[9] Jung K., Chu, K.T., Torquato, S.. 2007. A variational level set approach for surface area minimization of triply-periodic surfaces. Journal of Computational Physics, 2, 711-730

[10] G. Xu, G.Z Wang. Quintic parametric polynomial minimal surfaces and their properties. Differential Geometry and its Applications, to appear

[11] G. Xu, G.Z Wang. Parametric polynomial minimal surfaces of degree six with isothermal parameter Proc. of Geometric Modeling and Processing (GMP 2008), 2008, LNCS 4975, 329-343