Parametric Polynomial Minimal Surfaces of Degree Six with Isothermal Parameter

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Abstract. In this paper, parametric polynomial minimal surfaces of degree six with isothermal parameter are discussed. We firstly propose the sufficient and necessary condition of a harmonic polynomial parametric surface of degree six being a minimal surface. Then we obtain two kinds of new minimal surfaces from the condition. The new minimal surfaces have similar properties as Enneper's minimal surface, such as symmetry, self-intersection and containing straight lines. A new pair of conjugate minimal surfaces is also discovered in this paper. The new minimal surfaces can be represented by tensor product Bézier surface and triangular Bézier surface, and have several shape parameters. We also employ the new minimal surfaces for form-finding problem in membrane structure and present several modeling examples.

Keywords: minimal surface, harmonic surfaces, isothermal parametric surface, parametric polynomial minimal surface of degree six, membrane structure.

1 Introduction

Minimal surface is an important class of surfaces in differential geometry. Since Lagrange derived the minimal surface equation in $\mathbf{R}^3$ in 1762, minimal surfaces have a long history of over 200 years. Because of their attractive properties, the minimal surfaces have been extensively employed in many areas such as architecture, material science, aviation, ship manufacture, biology, crystallogeny and so on. For instance, the shape of the membrane structure, which has appeared frequently in modern architecture, is mainly based on the minimal surfaces [1]. Furthermore, triply periodic minimal surfaces naturally arise in a variety of systems, including block copolymers, nanocomposites , micellar materials, lipid-water systems and certain cell membranes[11]. So it is meaningful to introduce the minimal surfaces into CAGD/CAD systems.

However, most of the classic minimal surfaces, such as helicoid and catenoid, can not be represented by Bézier surface or B-spline surface, which are the basic modeling tools in CAGD/CAD systems. In order to introduce the minimal surfaces into CAGD/CAD systems, we must find some minimal surfaces in the parametric polynomial form. In practice, the highest degree of parametric surface used in CAD systems is six, hence, polynomial minimal surface of degree six with

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isothermal parameter is discussed in this paper. The new minimal surfaces have elegant properties and are valuable for architecture design.

1.1 Related Work

There has been many literatures on the minimal surface in the field of classical differential geometry [19,20]. The discrete minimal surface has been introduced in recent years in [2,4,12,21,22,27,30]. As the topics which are related with the minimal surface, the computational algorithms for conformal structure on discrete surface are presented in [7,8,10]; and some discrete approximation of smooth differential operators are proposed in[31,32]. Cosín and Monterde proved that Enneper's surface is the unique cubic parametric polynomial minimal surface [3]. Based on the nonlinear programming and the FEM(finite element method), the approximation to the solution of the minimal surface equation bounded by Bézier or B-spline curves is investigated in [14]. Monterde obtained the approximation solution of the Plateau-Bézier problem by replacing the area functional with the Dirichlet functional in [15,16]. The modeling schemes of harmonic and biharmonic Bézier surfaces to approximate the minimal surface are presented in [3,17,18,29]. The applications of minimal surface in aesthetic design, aviation and nano structures modeling have been presented in [6,25,26,28].

1.2 Contributions and Overview

In this paper, we employ the classical theory of minimal surfaces to obtain parametric polynomial minimal surfaces of degree six. Our main contribution are:

- We propose the sufficient and necessary condition of a harmonic polynomial parametric surface of degree six being a minimal surface. The coefficient relations are derived from the isothermal condition.
- Based on the sufficient and necessary condition, two kinds of new minimal surfaces with several shape parameters are presented. We analyze the properties of the new minimal surfaces, such as symmetry, self-intersection, containing straight lines and conjugate minimal surfaces.
- Using surface trimming method, we employ the new minimal surfaces for form-finding problems in membrane structure.

The remainder of this paper is organized as follows. Some preliminaries and notations are presented in Section 2. Section 3 presents the sufficient and necessary condition of a harmonic polynomial parametric surface of degree six being a minimal surface. From the condition, two kinds of new minimal surface and their properties are treated in Section 4. The topic of Section 5 is trimming of the new minimal surfaces and its application in membrane structure. Finally, we conclude and list some future works in Section 6.

2 Preliminary

In this section, we shall review some concepts and results related to minimal surfaces [5,19].

If the parametric form of a regular patch in $\mathbf{R}^3$ is given by

$$\boldsymbol{r}(u,v) = (x(u,v), y(u,v), z(u,v)), u \in (-\infty, +\infty), v \in (-\infty, +\infty),$$

Then the coefficients of the first fundamental form of $\boldsymbol{r}(u,v)$ are

$$E = \langle \boldsymbol{r}_u, \boldsymbol{r}_u \rangle, F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle, G = \langle \boldsymbol{r}_v, \boldsymbol{r}_v \rangle,$$

where $\boldsymbol{r}_u$, $\boldsymbol{r}_v$ are the first-order partial derivatives of $\boldsymbol{r}(u,v)$ with respect to u and v respectively and $\langle , \rangle$ defines the dot product of the vectors . The coefficients of the second fundamental form of $\boldsymbol{r}(u,v)$ are

$$L = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uu}), M = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uv}), N = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{vv}),$$

where $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{vv}$ and $\boldsymbol{r}_{uv}$ are the second-order partial derivatives of $\boldsymbol{r}(u,v)$ and $(,,)$ defines the mixed product of the vectors. Then the mean curvature H and the Gaussian curvature K of $\boldsymbol{r}(u,v)$ are

$$H = \frac{EN - 2FM + LG}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Definition 1. If parametric surface $\boldsymbol{r}(u,v)$ satisfies $E = G, F = 0$, then $\boldsymbol{r}(u,v)$ is called *surface with isothermal parameter.*

Definition 2. If parametric surface $\boldsymbol{r}(u,v)$ satisfies $\boldsymbol{r}_{uu} + \boldsymbol{r}_{vv} = 0$, then $\boldsymbol{r}(u,v)$ is called *harmonic surface.*

Definition 3. If $\boldsymbol{r}(u,v)$ satisfies $H = 0$, then $\boldsymbol{r}(u,v)$ is called *minimal surface.*

Lemma 1. *The surface with isothermal parameter is minimal surface if and only if it is harmonic surface.*

Definition 4. If two differentiable functions $p(u,v), q(u,v) : \boldsymbol{U} \mapsto \boldsymbol{R}$ satisfy the Cauchy-Riemann equations

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial v}, \frac{\partial p}{\partial v} = -\frac{\partial q}{\partial u}, \tag{1}$$

and both are harmonic. Then the functions are said to be *harmonic conjugate.*

Definition 5. If $\boldsymbol{P} = (p_1, p_2, p_3)$ and $\boldsymbol{Q} = (q_1, q_2, q_3)$ are isothermal parametrizations such that p_k and q_k are harmonic conjugate for $k = 1, 2, 3$, then $\boldsymbol{P}$ and $\boldsymbol{Q}$ are said to be *parametric conjugate minimal surfaces.*

For example, the helicoid and catenoid are conjugate minimal surface. Two conjugate minimal surfaces satisfy the following lemma.

Lemma 2. *Given two conjugate minimal surface $\boldsymbol{P}$ and $\boldsymbol{Q}$ and a real number t, all surfaces of the one-parameter family*

$$\boldsymbol{P}_t = (\cos t)\boldsymbol{P} + (\sin t)\boldsymbol{Q} \tag{2}$$

satisfy

(a) $\boldsymbol{P}_t$ *are minimal surfaces for all* $t \in \mathbf{R}$;
(b) $\boldsymbol{P}_t$ *have the same first fundamental forms for* $t \in \mathbf{R}$.

Thus, from above lemma, any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces, and the first fundamental form of this family is independent of t. In other words, these minimal surfaces are isometric and have the same Gaussian curvatures at corresponding points.

3 Sufficient and Necessary Condition

The main idea of construction of new minimal surfaces is based on Lemma 1. We firstly consider the harmonic parametric polynomial surface of degree six.

Lemma 3. *Harmonic polynomial surface of degree six* $\boldsymbol{r}(u,v)$ *must have the following form*

$$\begin{aligned}\boldsymbol{r}(u,v) = {} & \boldsymbol{a}(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + \boldsymbol{b}(3u^5v - 10u^3v^3 + 3uv^5) + \boldsymbol{c}(u^5 \\ & -10u^3v^2 + 5uv^4) + \boldsymbol{d}(v^5 - 10u^2v^3 + 5u^4v) + \boldsymbol{e}(u^4 - 6u^2v^2 + v^4) \\ & +\boldsymbol{f}\, uv(u^2 - v^2) + \boldsymbol{g}u(u^2 - 3v^2) + \boldsymbol{h}v(v^2 - 3u^2) + \boldsymbol{i}(u^2 - v^2) + \\ & \boldsymbol{j}uv + \boldsymbol{k}u + \boldsymbol{l}v + \boldsymbol{m},\end{aligned}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ *are coefficient vectors.*

Theorem 1. *Harmonic polynomial surface of degree six* $\boldsymbol{r}(u,v)$ *is a minimal surface if and only if its coefficient vectors satisfy the following system of equations*

$$\begin{cases} 4\boldsymbol{a}^2 = \boldsymbol{b}^2 \\ \boldsymbol{a}\cdot\boldsymbol{b} = 0 \\ 2\boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{d} = 0 \\ 2\boldsymbol{a}\cdot\boldsymbol{d} + \boldsymbol{b}\cdot\boldsymbol{c} = 0 \\ 25\boldsymbol{c}^2 - 25\boldsymbol{d}^2 + 48\boldsymbol{a}\cdot\boldsymbol{e} - 6\boldsymbol{b}\cdot\boldsymbol{f} = 0 \\ 25\boldsymbol{d}\cdot\boldsymbol{c} + 12\boldsymbol{b}\cdot\boldsymbol{e} + 6\boldsymbol{a}\cdot\boldsymbol{f} = 0 \\ 16\boldsymbol{e}^2 - \boldsymbol{f}^2 + 30\boldsymbol{c}\cdot\boldsymbol{g} - 30\boldsymbol{d}\cdot\boldsymbol{h} + 24\boldsymbol{a}\cdot\boldsymbol{i} - 6\boldsymbol{b}\cdot\boldsymbol{j} = 0 \\ 4\boldsymbol{e}\cdot\boldsymbol{f} - 15\boldsymbol{c}\cdot\boldsymbol{h} + 15\boldsymbol{d}\cdot\boldsymbol{g} + 6\boldsymbol{b}\cdot\boldsymbol{i} + 6\boldsymbol{a}\cdot\boldsymbol{j} = 0 \\ 9\boldsymbol{g}^2 - 9\boldsymbol{h}^2 + 16\boldsymbol{e}\cdot\boldsymbol{i} - 2\boldsymbol{f}\cdot\boldsymbol{j} + 10\boldsymbol{c}\cdot\boldsymbol{k} - 10\boldsymbol{l}\cdot\boldsymbol{d} = 0 \\ 9\boldsymbol{g}\cdot\boldsymbol{h} - 2\boldsymbol{f}\cdot\boldsymbol{i} - 4\boldsymbol{e}\cdot\boldsymbol{j} - 5\boldsymbol{d}\cdot\boldsymbol{k} - 5\boldsymbol{c}\cdot\boldsymbol{l} = 0 \\ 4\boldsymbol{i}^2 - \boldsymbol{j}^2 + 6\boldsymbol{g}\cdot\boldsymbol{k} + 6\boldsymbol{h}\cdot\boldsymbol{l} = 0 \\ 2\boldsymbol{i}\cdot\boldsymbol{j} - 3\boldsymbol{g}\cdot\boldsymbol{l} - 3\boldsymbol{h}\cdot\boldsymbol{k} = 0 \\ 18\boldsymbol{a}\cdot\boldsymbol{g} + 9\boldsymbol{b}\cdot\boldsymbol{h} + 20\boldsymbol{e}\cdot\boldsymbol{c} - 5\boldsymbol{f}\cdot\boldsymbol{d} = 0 \\ 18\boldsymbol{a}\cdot\boldsymbol{h} - 9\boldsymbol{b}\cdot\boldsymbol{g} - 20\boldsymbol{e}\cdot\boldsymbol{d} - 5\boldsymbol{f}\cdot\boldsymbol{c} = 0 \\ 6\boldsymbol{a}\cdot\boldsymbol{k} - 3\boldsymbol{b}\cdot\boldsymbol{l} + 10\boldsymbol{c}\cdot\boldsymbol{i} - 5\boldsymbol{d}\cdot\boldsymbol{j} + 12\boldsymbol{e}\cdot\boldsymbol{g} + 3\boldsymbol{f}\cdot\boldsymbol{h} = 0 \\ 6\boldsymbol{a}\cdot\boldsymbol{l} + 3\boldsymbol{b}\cdot\boldsymbol{k} + 5\boldsymbol{c}\cdot\boldsymbol{j} + 10\boldsymbol{d}\cdot\boldsymbol{i} + 3\boldsymbol{f}\cdot\boldsymbol{g} - 12\boldsymbol{e}\cdot\boldsymbol{h} = 0 \\ 4\boldsymbol{e}\cdot\boldsymbol{k} - \boldsymbol{f}\cdot\boldsymbol{l} + 3\boldsymbol{h}\cdot\boldsymbol{j} + 6\boldsymbol{g}\cdot\boldsymbol{i} = 0 \\ 4\boldsymbol{e}\cdot\boldsymbol{l} + \boldsymbol{f}\cdot\boldsymbol{k} + 3\boldsymbol{g}\cdot\boldsymbol{j} - 6\boldsymbol{h}\cdot\boldsymbol{i} = 0 \\ 2\boldsymbol{l}\cdot\boldsymbol{i} + \boldsymbol{k}\cdot\boldsymbol{j} = 0 \\ 2\boldsymbol{k}\cdot\boldsymbol{i} - \boldsymbol{l}\cdot\boldsymbol{j} = 0 \\ \boldsymbol{k}^2 = \boldsymbol{l}^2 \\ \boldsymbol{k}\cdot\boldsymbol{l} = 0 \end{cases} \tag{3}$$

Remark. The proof of this theorem will be given in the Appendix.

4 Examples and Properties

Obviously, it is difficult to find the general solution for the system (3). But we can construct some special solutions from the condition. In order to simplify the system (3), we firstly make some assumptions about the coefficient vectors,

$\boldsymbol{a} = (a_1, -a_2, 0), \boldsymbol{b} = (2a_2, 2a_1, 0), \boldsymbol{c} = (c_1, c_2, c_3), \boldsymbol{d} = (d_1, d_2, d_3), \boldsymbol{e} = (e_1, e_2, e_3), \boldsymbol{f} = (f_1, f_2, f_3),$
$\boldsymbol{g} = (g_1, g_2, g_3), \boldsymbol{h} = (h_1, h_2, h_3), \boldsymbol{i} = (i_1, i_2, i_3), \boldsymbol{j} = (j_1, j_2, j_3), \boldsymbol{k} = (k_1, k_2, k_3), \boldsymbol{l} = (l_1, l_2, l_3),$

From $2\boldsymbol{a}\cdot\ \boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{d} = 0$ and $2\boldsymbol{a}\cdot\boldsymbol{d} + \boldsymbol{b}\cdot\boldsymbol{c} = 0$, we have

$$a_1(c_1 - d_2) - a_2(c_2 + d_1) = 0, \tag{4}$$

$$a_2(c_1 - d_2) + a_1(c_2 + d_1) = 0, \tag{5}$$

From (4)$\times a_1$+(5)$\times a_2$ and (5)$\times a_1$−(4)$\times a_2$, we have

$$(a_1^2 + a_2^2)(c_1 - d_2) = 0, (a_1^2 + a_2^2)(c_2 + d_1) = 0.$$

Hence, we obtain

$$d_2 = c_1, d_1 = -c_2. \tag{6}$$

In the following subsections, we shall use this method to obtain the solutions.

4.1 Example 1

Supposing $\boldsymbol{c} = \boldsymbol{d} = \boldsymbol{g} = \boldsymbol{h} = \boldsymbol{k} = \boldsymbol{l} = 0, j_1 = 2i_2, j_2 = -2i_1$ in (3), we obtain

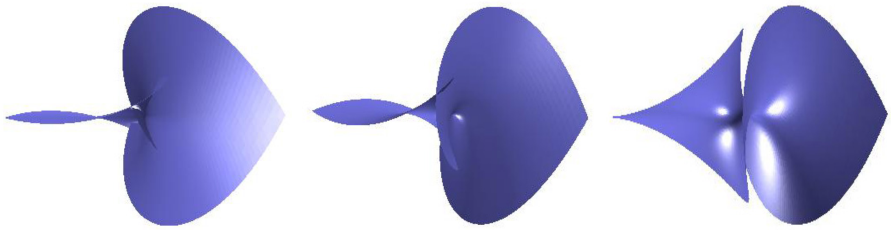
$$\begin{cases} 8\boldsymbol{a}\cdot\boldsymbol{e} - \boldsymbol{b}\cdot\boldsymbol{f} = 0 \\ 2\boldsymbol{b}\cdot\boldsymbol{e} + \boldsymbol{a}\cdot\boldsymbol{f} = 0 \\ 16\boldsymbol{e}^2 - \boldsymbol{f}^2 + 24\boldsymbol{a}\cdot\boldsymbol{i} - 6\boldsymbol{b}\cdot\boldsymbol{j} = 0 \\ 4\boldsymbol{e}\cdot\boldsymbol{f} + 6\boldsymbol{b}\cdot\boldsymbol{i} + 6\boldsymbol{a}\cdot\boldsymbol{j} = 0 \\ 8\boldsymbol{e}\cdot\boldsymbol{i} - \boldsymbol{f}\cdot\boldsymbol{j} = 0 \\ \boldsymbol{f}\cdot\boldsymbol{i} + 2\boldsymbol{e}\cdot\boldsymbol{j} = 0 \\ 4i_3^2 - j_3^2 = 0 \\ i_3 \cdot j_3 = 0 \end{cases} \tag{7}$$

From (7), we have

$$\begin{aligned} f_1 = f_2 = e_1 = e_2 = i_3 = j_3 = 0, \\ e_3 = \frac{\sqrt{6}}{2}\sqrt{\sqrt{(a_1^2 + a_2^2)(i_1^2 + i_2^2)} + (a_2 i_2 - a_1 i_1)}, \\ f_3 = -2\sqrt{6}\sqrt{\sqrt{(a_1^2 + a_2^2)(i_1^2 + i_2^2)} - (a_2 i_2 - a_1 i_1)}. \end{aligned}$$

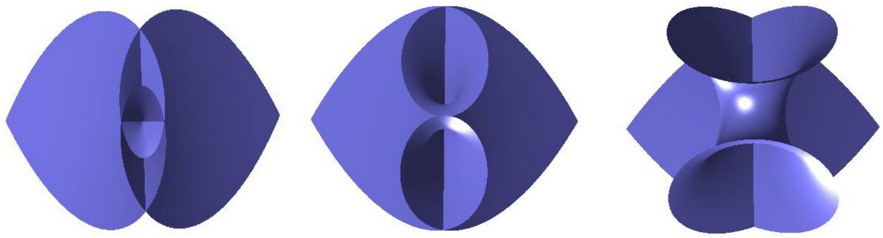
Then we obtain a class of minimal surface with four shape parameters a_1, a_2, i_1 and i_2:

$$\boldsymbol{r}(u, v) = (X(u, v), Y(u, v), Z(u, v)) \tag{8}$$



a) $i_1 = 40$ b) $i_1 = 800$ c) $i_1 = 2000$

Fig. 1. The effect of shape parameter i_1 in $\boldsymbol{r}_1(u,v)$. Here $a_1 = 4, u, v \in [-4, 4]$.



a) $i_2 = 40$ b) $i_2 = 800$ c) $i_2 = 200$

Fig. 2. The effect of shape parameter i_2 in $\boldsymbol{r}_2(u,v)$. Here $a_2 = 4, u, v \in [-4, 4]$.

where

$$\begin{aligned}
X(u,v) &= a_1(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + 2a_2(3u^5v - 10u^3v^3 + 3uv^5) \\
&\quad + i_1(u^2 - v^2) + 2i_2uv, \\
Y(u,v) &= -a_2(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + 2a_1(3u^5v - 10u^3v^3 + 3uv^5) \\
&\quad + i_2(u^2 - v^2) - 2i_1uv, \\
Z(u,v) &= \frac{\sqrt{6}}{2}\sqrt{\sqrt{(a_1^2 + a_2^2)(i_1^2 + i_2^2)} + (a_2i_2 - a_1i_1)}(u^4 - 6u^2v^2 + v^4) - \\
&\quad 2\sqrt{6}\sqrt{\sqrt{(a_1^2 + a_2^2)(i_1^2 + i_2^2)} - (a_2i_2 - a_1i_1)}uv(u^2 - v^2).
\end{aligned}$$

When $a_2 = i_2 = 0$, we denote the minimal surface in (8) by $\boldsymbol{r}_1(u,v)$. The Gaussian curvature of $\boldsymbol{r}_1(u,v)$ is

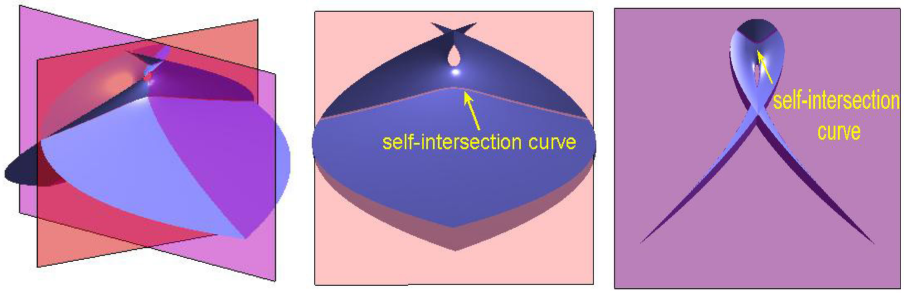
$$K = -192a_1i_1(u^2 + v^2)^2. \tag{9}$$

Fig 1 shows the effect of the shape parameter i_1.

In the case $a_1 = i_1 = 0$, the minimal surface in (8) is denoted by $\boldsymbol{r}_2(u,v)$. The Gaussian curvature of $\boldsymbol{r}_2(u,v)$ is

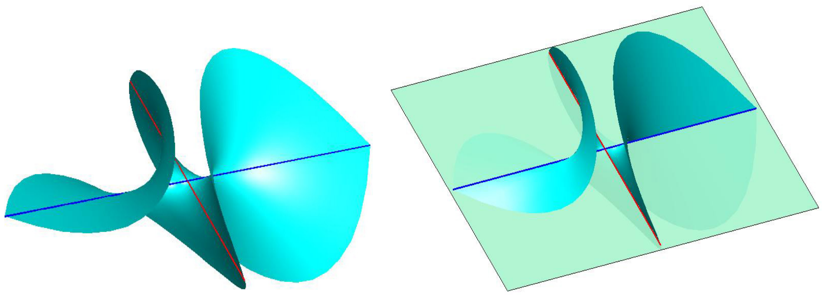
$$K = -192a_2i_2(u^2 + v^2)^2. \tag{10}$$

Fig 2 illustrates the effect of the shape parameter i_2.



a) b) c)

Fig. 3. The minimal surface $\boldsymbol{r}_2(u, v)$ and its symmetric planes: (a) $\boldsymbol{r}_2(u, v)$ with $a_2 = 4$ and $i_2 = 500$, $u, v \in [-4, 4]$, its symmetric planes $X = 0$ and $Y = 0$; (b) self-intersection curve on the plane $X = 0$;(c) self-intersection curve on the plane $Y = 0$



a) b)

Fig. 4. The minimal surface $\boldsymbol{r}_1(u, v)$ and the straight lines on it. Here $a_1 = 4, i_1 = 2000$ $u, v \in [-4, 4]$.

Enneper surface is the unique cubic isothermal parametric polynomial minimal surface, and it has several interesting properties, such as symmetry, self-intersection, and containing straight lines on it. For $\boldsymbol{r}(u, v)$, we have the following propositions.

Proposition 1. *The minimal surface $\boldsymbol{r}_2(u, v)$ is symmetric about the plane $X = 0$ and the plane $Y = 0$.*

Furthermore, there exists two self-intersection curves of $\boldsymbol{r}_2(u, v)$ on the plane $X = 0$ and the plane $Y = 0$; besides the two self-intersection curves, there are no other self-intersection points on $\boldsymbol{r}_2(u, v)$. Fig 3 shows the symmetric planes and self-intersection curves when $a_2 = 4$ and $i_2 = 500$.

Proposition 2. *The minimal surface $\boldsymbol{r}_1(u, v)$ contains two orthogonal straight lines $x = \pm y$ on the plane $Z = 0$.*

Fig 4 shows the minimal surface and the straight lines on it. It is consistent with the fact that if a piece of a minimal surface has a straight line segment on its boundary, then 180° rotation around this segment is the analytic continuation of the surface across this edge.

Helicoid and catenoid are a pair of conjugate minimal surfaces. For $\boldsymbol{r}(u,v)$, we find out a new pair of conjugate minimal surfaces as follows.

Proposition 3. *When $a_1 = a_2, i_1 = i_2$, $\boldsymbol{r}_1(u,v)$ and $\boldsymbol{r}_2(u,v)$ are conjugate minimal surfaces.*

Proof. Suppose that $\boldsymbol{r}_1(u,v)=(X_1(u,v),Y_1(u,v),Z_1(u,v))$, $\boldsymbol{r}_2(u,v)=(X_2(u,v), Y_2(u,v),Z_2(u,v))$. After some computation, we have

$$\begin{aligned}
\frac{\partial X_1(u,v)}{\partial u} &= a_1(6u^5 - 60u^3v^2 + 30uv^4) + 2i_1u,\\
\frac{\partial X_1(u,v)}{\partial v} &= a_1(60u^2v^3 - 30u^4v - 6v^5) - 2i_1v\\
\frac{\partial X_2(u,v)}{\partial u} &= a_2(30u^4v - 60u^2v^3 + 6v^5) + 2i_2v,\\
\frac{\partial X_2(u,v)}{\partial v} &= a_2(6u^5 - 60u^3v^2 + 30uv^4) + 2i_2u.
\end{aligned}$$

When $a_1 = a_2, i_1 = i_2$, $\frac{\partial X_1(u,v)}{\partial u} = \frac{\partial X_2(u,v)}{\partial v}, \frac{\partial X_1(u,v)}{\partial v} = -\frac{\partial X_2(u,v)}{\partial u}$. That is, $X_1(u,v)$ and $X_2(u,v)$ are harmonic conjugate. Similarly, $Y_1(u,v)$ and $Y_2(u,v)$, $Z_1(u,v)$ and $Z_2(u,v)$ are also harmonic conjugate respectively. From Definition 5, the proof is completed. □

From Lemma 2, when $a_1 = a_2, i_1 = i_2$, the surfaces of one-parametric family

$$\boldsymbol{r}_t(u,v) = (\cos t)\boldsymbol{r}_1(u,v) + (\sin t)\boldsymbol{r}_2(u,v) \tag{11}$$

are minimal surfaces with the same first fundamental form. These minimal surfaces are isometric and have the same Gaussian curvature at corresponding points. It is consistent with (9)and (10).

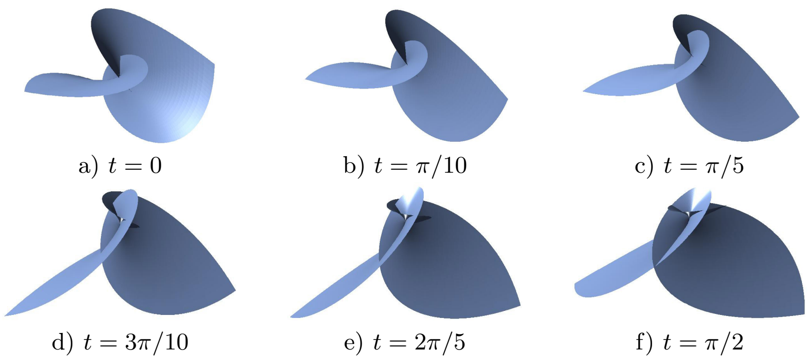


Fig. 5. Dynamic deformation between $\boldsymbol{r}_1(u,v)$ and $\boldsymbol{r}_2(u,v)$. Here $a_1 = a_2 = 500$, $i_1 = i_2 = 5, u, v \in [-4, 4]$.

Let $t \in [0, \pi/2]$. When $a_1 = a_2$ and $i_1 = i_2$, for $t = 0$, the minimal surface $\boldsymbol{r}_t$ reduces to $\boldsymbol{r}_1(u, v)$; for $t = \pi/2$, it reduces to $\boldsymbol{r}_2(u, v)$. Then when t varies from 0 to $\pi/2$, $\boldsymbol{r}_1(u, v)$ can be continuously deformed into $\boldsymbol{r}_2(u, v)$, and each intermediate surface is also minimal surface. Fig 5 illustrates the dynamic deformation when $a_1 = a_2 = 500, i_1 = i_2 = 5$. It is similar with the dynamic deformation between helicoid and catenoid.

4.2 Example 2

Supposing $\boldsymbol{k} = \boldsymbol{l} = \boldsymbol{i} = \boldsymbol{j} = 0, c_3 = d_3 = g_3 = h_3 = 0$ in (3), we have

$$\begin{cases} 8\boldsymbol{a} \cdot \boldsymbol{e} - \boldsymbol{b} \cdot \boldsymbol{f} = 0 \\ 2\boldsymbol{b} \cdot \boldsymbol{e} + \boldsymbol{a} \cdot \boldsymbol{f} = 0 \\ 16\boldsymbol{e}^2 - \boldsymbol{f}^2 + 30\boldsymbol{c} \cdot \boldsymbol{g} - 30\boldsymbol{d} \cdot \boldsymbol{h} = 0 \\ 4\boldsymbol{e} \cdot \boldsymbol{f} - 15\boldsymbol{c} \cdot \boldsymbol{h} + 15\boldsymbol{d} \cdot \boldsymbol{g} = 0 \\ \boldsymbol{g}^2 - \boldsymbol{h}^2 = 0 \\ \boldsymbol{g} \cdot \boldsymbol{h} = 0 \\ 18\boldsymbol{a} \cdot \boldsymbol{g} + 9\boldsymbol{b} \cdot \boldsymbol{h} + 20\boldsymbol{e} \cdot \boldsymbol{c} - 5\boldsymbol{f} \cdot \boldsymbol{d} = 0 \\ 18\boldsymbol{a} \cdot \boldsymbol{h} - 9\boldsymbol{b} \cdot \boldsymbol{g} - 20\boldsymbol{e} \cdot \boldsymbol{d} - 5\boldsymbol{f} \cdot \boldsymbol{c} = 0 \\ 4\boldsymbol{e} \cdot \boldsymbol{g} + \boldsymbol{f} \cdot \boldsymbol{h} = 0 \\ \boldsymbol{f} \cdot \boldsymbol{g} - 4\boldsymbol{e} \cdot \boldsymbol{h} = 0 \end{cases} \tag{12}$$

From (6) and (12), two solutions can be obtained: if $c_1 g_1 + c_2 g_2 > 0$, then

$$f_2 = 4e_1, f_1 = -4e_2, h_1 = g_2, h_2 = -g_1,$$
$$e_3 = 0, f_3 = 2\sqrt{15}\sqrt{c_1 g_1 + c_2 g_2};$$

if $c_1 g_1 + c_2 g_2 < 0$, then

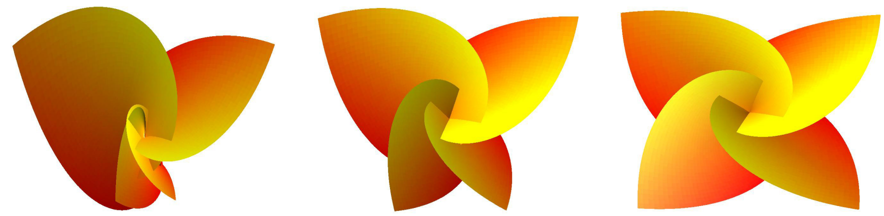
$$f_2 = 4e_1, f_1 = -4e_2, h_1 = g_2, h_2 = -g_1,$$
$$f_3 = 0, e_3 = \frac{\sqrt{15}}{2}\sqrt{-c_1 g_1 - c_2 g_2};$$

Then we obtain two classes of minimal surface with eight shape parameters $a_1, a_2, c_1, c_2, e_1, e_2, g_1$ and g_2:

$$\boldsymbol{r}(u, v) = (X(u, v), Y(u, v), Z(u, v)) \tag{13}$$

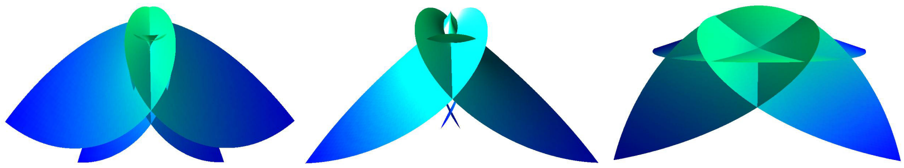
where

$$\begin{aligned} X(u, v) = {} & a_1(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + 2a_2(3u^5v - 10u^3v^3 + 3uv^5) \\ & + c_1(u^5 - 10u^3v^2 + 5uv^4) - c_2(v^5 - 10u^2v^3 + 5u^4v) + e_1(u^4 - 6u^2v^2 + v^4) \\ & - 4e_2uv(u^2 - v^2) + g_1u(u^2 - 3v^2) + g_2v(v^2 - 3u^2), \\ Y(u, v) = {} & -a_2(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + 2a_1(3u^5v - 10u^3v^3 + 3uv^5) \\ & + c_2(u^5 - 10u^3v^2 + 5uv^4) + c_1(v^5 - 10u^2v^3 + 5u^4v) + e_2(u^4 - 6u^2v^2 + v^4) \\ & + 4e_1uv(u^2 - v^2) + g_2u(u^2 - 3v^2) - g_1v(v^2 - 3u^2), \\ Z(u, v) = {} & 2\sqrt{15}\sqrt{c_1g_1 + c_2g_2}\,uv(u^2 - v^2), \end{aligned}$$



a) $c_1 = 10$ b)$c_1 = 40$ c)$c_1 = 200$

Fig. 6. The effect of shape parameter c_1 in $\boldsymbol{r}_3(u,v)$. Here $a_1 = e_1 = g_1 = 4, u, v \in [-2, 2]$.



a) $g_2 = 10$ b) $g_2 = 80$ c)$g_2 = 800$

Fig. 7. The effect of shape parameter g_2 in $\boldsymbol{r}_6(u,v)$. Here $a_2 = e_2 = 4$, $c_2 = -4$, $u, v \in [-2, 2]$.

or

$$\bar{\boldsymbol{r}}(u,v) = (\bar{X}(u,v), \bar{Y}(u,v), \bar{Z}(u,v)) \tag{14}$$

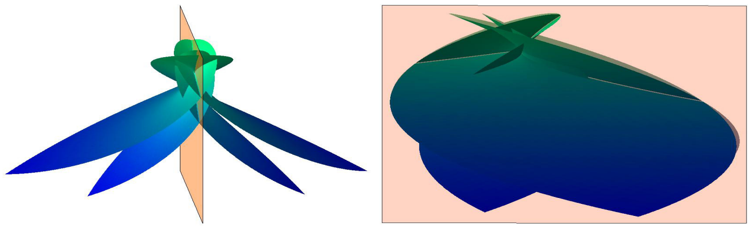
where

$$\begin{aligned}
\bar{X}(u,v) = {}& a_1(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + 2a_2(3u^5v - 10u^3v^3 + 3uv^5) \\
&+c_1(u^5 - 10u^3v^2 + 5uv^4) - c_2(v^5 - 10u^2v^3 + 5u^4v) + e_1(u^4 - 6u^2v^2 + v^4) \\
&-4e_2uv(u^2 - v^2) + g_1u(u^2 - 3v^2) + g_2v(v^2 - 3u^2), \\
\bar{Y}(u,v) = {}& -a_2(u^6 - 15u^4v^2 + 15u^2v^4 - v^6) + 2a_1(3u^5v - 10u^3v^3 + 3uv^5) \\
&+c_2(u^5 - 10u^3v^2 + 5uv^4) + c_1(v^5 - 10u^2v^3 + 5u^4v) + e_2(u^4 - 6u^2v^2 + v^4) \\
&+4e_1uv(u^2 - v^2) + g_2u(u^2 - 3v^2) - g_1v(v^2 - 3u^2), \\
\bar{Z}(u,v) = {}& \frac{\sqrt{15}}{2}\sqrt{-c_1g_1 - c_2g_2}(u^4 - 6u^2v^2 + v^4).
\end{aligned}$$

When $a_2 = c_2 = e_2 = g_2 = 0$, we denote the minimal surface $\boldsymbol{r}(u,v)$ in (13) by $\boldsymbol{r}_3(u,v)$; similarly, in the case $a_1 = c_1 = e_1 = g_1 = 0$, $\boldsymbol{r}(u,v)$ in (13) is denoted by $\boldsymbol{r}_4(u,v)$. When $a_1 = a_2, c_1 = c_2, e_1 = e_2, g_1 = g_2$, we can obtain $\boldsymbol{r}_4(u,v)$ from $\boldsymbol{r}_3(u,v)$ by rotation transformation. Fig 6 illustrates the effect of c_1 of $\boldsymbol{r}_3(u,v)$.

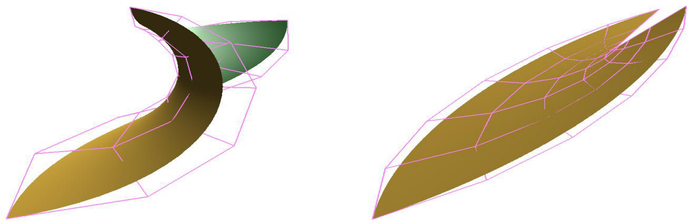
In the case $a_2 = c_2 = e_2 = g_2 = 0$, the minimal surface $\bar{\boldsymbol{r}}(u,v)$ in (14) is denoted by $\boldsymbol{r}_5(u,v)$; similarly, when $a_1 = c_1 = e_1 = g_1 = 0$, we denote $\bar{\boldsymbol{r}}(u,v)$ in (14) by $\boldsymbol{r}_6(u,v)$. In the case $a_1 = a_2, c_1 = c_2, e_1 = e_2, g_1 = g_2$, $\boldsymbol{r}_6(u,v)$ can be obtained from $\boldsymbol{r}_5(u,v)$ by rotation transformation. The effect of g_2 of $\boldsymbol{r}_6(u,v)$ is shown in Fig 7.

For $\boldsymbol{r}_5(u,v)$ and $\boldsymbol{r}_6(u,v)$, we have the following proposition.



a) b)

Fig. 8. Two different views of the minimal surface $\boldsymbol{r}_6(u, v)$ and its symmetric plane. Here $a_2 = e_2 = g_2 = 4, c_2 = -4, u, v \in [-2, 2]$.



a) b)

Fig. 9. Tensor product Bézier surface representation of $\boldsymbol{r}(u, v)$ in (8) and (13):(a)$\boldsymbol{r}(u, v)$ in (8) and its control mesh, $a_1 = 3, a_2 = 500, i_1 = i_2 = 1, u, v \in [0, 1]$ (b)$\boldsymbol{r}(u, v)$ in (13) and its control mesh, $a_1 = c_1 = e_1 = g_1 = a_2 = c_2 = e_2 = 4, g_2 = 400, u, v \in [0, 1]$

Proposition 4. *The minimal surface* $\boldsymbol{r}_5(u, v)$ *is symmetric about the plane* $Y = 0$; $\boldsymbol{r}_6(u, v)$ *is symmetric about the plane* $X = 0$.

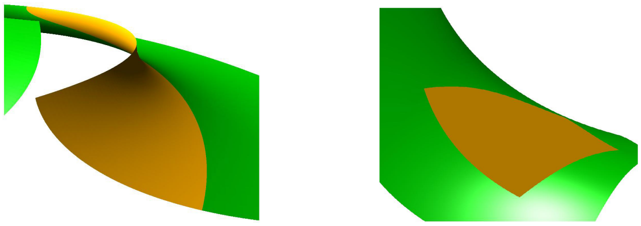
Fig 8 presents the symmetric plane of $\boldsymbol{r}_6(u, v)$ with $a_2 = e_2 = g_2 = 4$ and $c_2 = -4$.

5 Application in Architecture

Obviously, the proposed minimal surfaces can be represented by tensor product Bézier surface or triangular Bézier surface. Fig 9 shows the tensor product Bézier surface representation of $\boldsymbol{r}(u, v)$ in (8) and (13).

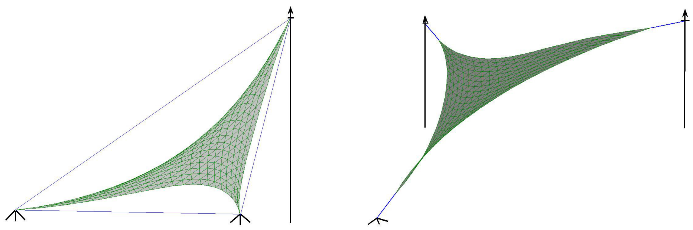
Geometric design and computation in architecture has been a hotspot in recent years [13,23,24]. In the surface of membrane structure, we need that the resultant nodal forces (i.e. residual forces) must be reduced to zero, so that there is no pressure difference across the surface. Hence, minimal surface is the ideal shape of the membrane structure. In particular, the minimal surfaces proposed in the current paper can be used for the form-finding problem, which is the first stage in the construction process of membrane structure.

From the classical theory of minimal surface, any trimmed surfaces on minimal surface are also minimal surfaces. Hence, the traditional surface trimming



a) b)

Fig. 10. Trimmed surfaces (yellow) on minimal surfaces $\boldsymbol{r}(u,v)$(green) in (8) with $i_1 = a_1 = 1, i_2 = a_2 = 5$



a) b)

Fig. 11. Two modeling examples of membrane structures by using the triangular trimmed surfaces on minimal surface $\boldsymbol{r}_1(u,v)$ with $a_1 = i_1 = 50$

methods in [9] are employed for form finding problems. Fig 10 illustrates two trimmed minimal surface. Modeling examples of membrane structure are shown in Fig 11.

6 Conclusion and Future Work

In order to introduce minimal surfaces into CAGD/CAD systems, the parametric polynomial minimal surface of degree six is studied in this paper. We propose the sufficient and necessary condition of a harmonic polynomial parametric surface of degree six being a minimal surface. Two kinds of new minimal surface with several shape parameters are obtained from the condition. We analyze the properties of the new minimal surface, such as symmetry, self-intersection, containing straights lines and conjugate property. Hence, the new minimal surfaces have the similar properties as the classical Enneper surface. In particular, the conjugate property is similar with the catenoid and the helicoid.

The Weierstrass representation is another method to obtain new minimal surfaces. However, it is difficult to choose the proper initial functions to obtain the parametric polynomial minimal surfaces. The method presented in this paper can directly derive the parametric polynomial minimal surface.

In the future, we will investigate the other applications of the new minimal surfaces. How to give the general formula of the parametric polynomial minimal surfaces, is also our future work.

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References

1. Bletzinger, K.-W.: Form finding of membrane structures and minimal surfaces by numerical continuation. In: Proceeding of the IASS Congress on Structural Morphology: Towards the New Millennium, Nottingham, pp. 68–75 (1997) ISBN 0-85358-064-2
2. Bobenko, A.I., Hoffmann, T., Springborn, B.A.: Minimal surfaces from circle patterns: Geometry from combinatorics. Annals of Mathematics 164, 231–264 (2006)
3. Cosín, C., Monterde, J.: Bézier Surfaces of Minimal Area. In: Sloot, P.M.A., Tan, C.J.K., Dongarra, J., Hoekstra, A.G. (eds.) ICCS-ComputSci 2002. LNCS, vol. 2330, pp. 72–81. Springer, Heidelberg (2002)
4. Desbrun, M., Grinspun, E., Schroder, P.: Discrete differential geometry: an applied introduction. In: SIGGRAPH Course Notes (2005)
5. Do Carmo, M.: Differential geometry of curves and surfaces. Prentice-Hall, Englewood Cliffs (1976)
6. Grandine, S., Del Valle, T., Moeller, S.K., Natarajan, G., Pencheva, J., Sherman, S.: Wise. Designing airplane struts using minimal surfaces IMA Preprint 1866 (2002)
7. Gu, X., Yau, S.: Surface classification using conformal structures. ICCV, 701-708 (2003)
8. Gu, X., Yau, S.: Computing conformal structure of surfaces CoRR cs.GR/0212043 (2002)
9. Hoscheck, J., Schneider, F.: Spline conversion for trimmed rational Bezier and B-spline surfaces. Computer Aided Design 9, 580–590 (1990)
10. Jin, M., Luo, F., Gu, X.F.: Computing general geometric structures on surfaces using Ricci flow. Computer-Aided Design 8, 663–675 (2007)
11. Jung, K., Chu, K.T., Torquato, S.: A variational level set approach for surface area minimization of triply-periodic surfaces. Journal of Computational Physics 2, 711–730 (2007)
12. Li, X., Guo, X.H., Wang, H.Y., He, Y., Gu, X.F., Qin, H.: Harmonic volumetric mapping for solid modeling applications. In: Symposium on Solid and Physical Modeling, pp. 109–120 (2007)
13. Liu, Y., Pottmann, H., Wallner, J., Yang, Y.L., Wang, W.P.: Geometric modeling with conical meshes and developable surfaces. ACM Trans. Graphics 3, 681–689 (2006)
14. Man, J.J., Wang, G.Z.: Approximating to nonparameterzied minimal surface with B-spline surface. Journal of Software 4, 824–829 (2003)
15. Monterde, J.: The Plateau-Bézier Problem. In: Wilson, M.J., Martin, R.R. (eds.) Mathematics of Surfaces. LNCS, vol. 2768, pp. 262–273. Springer, Heidelberg (2003)

16. Monterde, J.: Bézier surfaces of minimal area: The Dirichlet approach. Computer Aided Geometric Design 1, 117–136 (2004)
17. Monterde, J., Ugail., H.: On harmonic and biharmonic Bézier surfaces. Computer Aided Geometric Design 7, 697–715 (2004)
18. Monterde, J., Ugail., H.: A general 4th-order PDE method to generate Bézier surfaces from boundary. Computer Aided Geometric Design 2, 208–225 (2006)
19. Nitsche, J.C.C.: Lectures on minimal surfaces, vol. 1. Cambridge Univ. Press, Cambridge (1989)
20. Osserman, R.: A survey of minimal surfaces, 2nd edn. Dover publ., New York((1986)
21. Pinkall, U., Polthier, K.: Computing discrete minimal surface and their conjugates. Experiment Mathematics 1, 15–36 (1993)
22. Polthier, K.: Polyhedral surface of constant mean curvature. Habilitationsschrift TU Berlin (2002)
23. Pottmann, H., Liu, Y.: Discrete surfaces in isotropic geometry. In: Mathematics of Surfaces, vol. XII, pp. 341-363 (2007)
24. Pottmann, H., Liu, Y., Wallner, J., Bobenko, A., Wang, W.P.: Geometry of multilayer freeform structures for architecture. ACM Transactions on Graphics 3, 1–11 (2007)
25. Séquin, C.H.: CAD Tools for Aesthetic Engineering. Computer Aided Design 7, 737–750 (2005)
26. Sullivan, J.: The aesthetic value of optimal geometry. In: Emmer, M. (ed.) The Visual Mind II, pp. 547–563. MIT Press, Cambridge (2005)
27. Wallner, J., Pottmann, H.: Infinitesimally flexible meshes and discrete minimal surfaces. Monatsh. Math (to appear, 2006)
28. Wang, Y.: Periodic surface modeling for computer aided nano design. Computer Aided Design 3, 179–189 (2007)
29. Xu, G., Wang, G.Z., Harmonic, B.-B.: surfaces over triangular domain. Journal of Computers 12, 2180–2185 (2006)
30. Xu, G., Zhang, Q.: G^2 surface modeling using minimal mean-curvature-variation flow. Computer-Aided Design 5, 342–351 (2007)
31. Xu, G.L.: Discrete Laplace-Beltrami operators and their convergence. Computer Aided Geometric Design 10, 767–784 (2004)
32. Xu, G.L.: Convergence analysis of a discretization scheme for Gaussian curvature over triangular surfaces. Computer Aided Geometric Design 2, 193–207 (2006)

Appendix: The Proof of Theorem 1

Theorem 1 can be proved from the isothermal condition and the linear independence of the power basis. The partial derivatives of the harmonic surface $\boldsymbol{r}(u,v)$ in Lemma 3 has the following forms:

$$\boldsymbol{r}_u(u,v) = 6\boldsymbol{a}A_5^o + 3\boldsymbol{b}A_5^e + 5\boldsymbol{c}A_4^e + 10\boldsymbol{d}A_4^o + 4\boldsymbol{e}A_3^o + \boldsymbol{f}A_3^e + 3\boldsymbol{g}A_2^e - 6\boldsymbol{h}A_2^o + 2\boldsymbol{i}A_1^o + \boldsymbol{j}A_1^e + \boldsymbol{k}$$
$$\boldsymbol{r}_v(u,v) = -6\boldsymbol{a}A_5^e + 3\boldsymbol{b}A_5^o + 5\boldsymbol{d}A_4^e - 10\boldsymbol{c}A_4^o + \boldsymbol{f}A_3^o - 4\boldsymbol{e}A_3^e - 3\boldsymbol{h}A_2^e - 6\boldsymbol{g}A_2^o + \boldsymbol{j}A_1^o - 2\boldsymbol{i}A_1^e + \boldsymbol{l}$$

where $A_5^o = u^5 - 10u^3v^2 + 5uv^4, A_5^e = 5vu^4 - 10v^3u^2 + v^5, A_4^e = u^4 - 6u^2v^2 + v^4, A_4^o = 2u^3v - 2uv^3, A_3^o = u^3 - 3uv^2, A_3^e = 3u^2v - v^3, A_2^e = u^2 - v^2, A_2^o = uv, A_1^o = u, A_1^e = v.$

Hence, from $F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle$, the term u^{10} in F is related with A_5^o, then we obtain $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ from $F = 0$. The term $u^9 v$ is related with A_5^e and A_5^o, then we get $4\boldsymbol{a}^2 = \boldsymbol{b}^2$. Similarly, the other equations in (3) can be obtain from $F = 0$ and $E = G$.

It is noted that we obtain only two equation for the terms $u^i v^j, i + j = k, k = 0, 1, 2, \cdots 9, 10$. One is for the case of i is even, and the other one is for the case of i is odd. The equations derived from $F = 0$ are the same as the case of $E = G$ except for the equations $\boldsymbol{k}^2 = \boldsymbol{l}^2$ and $\boldsymbol{k} \cdot \boldsymbol{l} = 0$. Hence, the number of equations in system (3) is $2 \times 2 \times (6 - 1) + 2 = 22$. Thus, the proof is completed. □