Cryptographic Logical Relations

— What is the contextual equivalence for cryptographic protocols and how to prove it?

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Cryptography

Using cryptography to hide information:

Hello, buddy! $\rightarrow$ Encryption $\rightarrow$ Decryption $\rightarrow$ Hello, buddy!

$K_{enc}$

$K_{dec}$

But, how to distribute keys on Internet?
The Needham-Schroeder’s protocol

Alice

Bob

\{Secret\}_{NONCE_A}

NONCE_A becomes the session key
The Needham-Schroeder’s protocol

Alice \rightarrow \text{NONCE} \rightarrow \{ \text{NONCE}_A \}^{PK_{Charlie}} \rightarrow Charlie

Charlie \rightarrow \text{NONCE} \rightarrow \{ \text{NONCE}_B \}^{PK_{Alice}} \rightarrow Alice

Alice \rightarrow \text{NONCE} \rightarrow \{ \text{NONCE}_A \}^{PK_{Bob}} \rightarrow Bob

Bob \rightarrow \text{NONCE} \rightarrow \{ \text{NONCE}_B \}^{PK_{Alice}} \rightarrow Alice

Charlie \rightarrow \text{NONCE} \rightarrow \{ \text{NONCE}_B \}^{PK_{Bob}} \rightarrow Bob

Alice \rightarrow \text{NONCE} \rightarrow \{ \text{NONCE}_A \}^{PK_{Charlie}} \rightarrow Charlie

Cryptographic Logical Relations
Formal verification

1978 — The invention of the NS protocol [NS 78].

1995 — G. Lowe found the flaw [Lowe 95].

As a logician, I’d like to tell you very seriously: It’s NOT True!!!

Those who think that their problem can be solved by simply applying cryptography, don’t understand cryptography and don’t understand their problem.

---- R. Needham

What are you talking about? “Insecure”? We use CRYPTOGRAPHY here.

The protocol is secure, because I don’t find any attack!
Formal verification

1978 — The invention of the NS protocol [NS 78].

1995 — G. Lowe found the flaw [Lowe 95].

Verify security properties with formal methods.

Formal verification community

AVISPA

Cryptographic Logical Relations
Secrecy by contextual equivalence

Secrecy: for every messages $m_1$ and $m_2$, $\text{Protocol}(m_1) \approx \text{Protocol}(m_2)$.

Spi-Calculus: with bisimulations [Abadi & Gordon 97].

Cryptographic $\lambda$-calculus: with logical relations [Sumii & Pierce 02]. Higher-order functions are taken into account.
Motivation

We keep on using the $\lambda$-calculus approach.

Sumii and Pierce’s logical relations are somehow ad-hoc. Is there a systematic way to construct these logical relations?

And, to what extent can we rely on this method? If logical relations fail in proving the secrecy property, can we say that protocol is NOT secure?
## Related work and our contribution

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<th>Side-effects</th>
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<td>1992~93, categorical construction [Ma &amp; Reynolds 92, Mitchell &amp; Scedrov 93]</td>
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<td>1993~94, operational logical relations for name creation [Pitts &amp; Stark 93]</td>
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Outline

- The cryptographic metalanguage
- Denotational semantics
- Cryptographic logical relations
- Contextual equivalence
Cryptographic Logical Relations

☑ Introduction

❖ The cryptographic metalanguage

☐ Denotational semantics

☐ Cryptographic logical relations

☐ Contextual equivalence

☐ Conclusion
Syntax (i) — Types

Based on Moggi’s computational $\lambda$-calculus — a nice framework for reasoning about side-effects, including key generation.

\[
\begin{align*}
\tau & ::= \cdots \\
& | \text{key} \\
& | \text{msg} \\
& | \tau \tau
\end{align*}
\]

- A computation may generate fresh keys.

Type for computations, from Moggi’s language
Syntax (ii) — Terms

\[
t ::= \ldots
  | \text{new}
  | \text{enc}(t, k)
  | \text{dec}(t, k)
  \ldots
  | \text{val}(t)
  | \text{let } x \leftarrow t_1 \text{ in } t_2
\]

- generation of fresh key, from Stark's metalanguage
- trivial computation and sequential computation, from Moggi's language
Syntax (ii) — Typing rules

\[
\frac{}{\Gamma \vdash \text{new} : T_{\text{key}}} \quad (\text{New})
\]

\[
\frac{}{\Gamma \vdash \text{val}(t) : T_\tau} \quad (\text{Val})
\]

\[
\frac{}{\Gamma, x : \tau \vdash t_2 : T_{\tau'}} \quad (\text{Let})
\]

\[
\frac{}{\Gamma \vdash \text{enc}(t, k) : \text{msg}} \quad (\text{Enc})
\]

\[
\frac{}{\Gamma \vdash \text{dec}(t, k) : \text{opt}[\text{msg}]} \quad (\text{Dec})
\]
Public key cryptography can be modeled using functions [Sumii & Pierce 02]:

- If $k$ is a private key, then the public key is:

  $$PK(k) = \lambda x.\text{enc}(x, k)$$

- Encrypt a message with a public key:

  $$Enc(m, PK(k)) = (\lambda x.\text{enc}(x, k))m$$
Encoding of protocols

- **Principals as functions.**
- **Interactions as function applications.**

\[
\lambda \{ \cdot \} \cdot \langle B, f_B^1 \rangle = f_B^0 \cdot B \\
\lambda \{ x, A \}_{k_B} \cdot \nu N_b \cdot \langle \{ x, N_b \}_{k_a} \cdot f_B^2 \rangle = f_B^1 \cdot \langle \{ N_a, N_b \}_{k_a} \rangle \\
\lambda \{ N_b \}_{k_b} \cdot \{ \text{msg} \}_{N_b} = f_B^2 \cdot \langle \text{msg} \rangle_{N_b} \\
\lambda X. \nu N_a \cdot \langle \{ N_a, A \}_{k_x} \cdot f_A^2 \rangle
\]

- The protocol is a tuple of functions:
  \[ P(\text{secret}) = \langle f_{\text{Alice}}, f_{\text{Bob}}, \ldots \rangle \]

- An attack is a function \( F \):
  \[ F(P(\text{secret})) = \text{secret} \]
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Modeling cryptography

\[ [\text{key}] \] — a set of keys.

An encrypted message is written as \( e(v, k) \).

\[
\begin{align*}
[\text{enc}(t_1, t_2)] &= e([t_1], [t_2]) \\
[\text{dec}(t_1, t_2)] &= \begin{cases} 
  v, & \text{if } [t_1] = e(v, k) \text{ and } [t_2] = k \\
  \bot, & \text{otherwise.}
\end{cases}
\end{align*}
\]
Computations as monads

- According to Moggi, side-effects can be modeled by monads [Moggi 89].
  - Concrete monads: exceptions, non-determinism, ...
- Fresh key generation is seen as a side-effect.
- Key generation monad: computations might generate fresh keys.
  - Stark uses this monad to interpret his language for name creation [Stark 94].
Stark’s model

A functor category $\text{Set}^\mathcal{I}$ with a monad $\mathbf{T}$:

- $\mathcal{I}$ — category of finite sets and injections.
  - A set represents a computation stage.
- Denotations are defined over a set of keys.
- Computations are interpreted as

$$\mathbf{T}[\tau]s = \{[s', a] \mid s' \in \mathcal{I}, a \in \llbracket \tau \rrbracket(s + s')\}$$

We use Stark’s model to interpret our metalanguage.
Cryptographic Logical Relations

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What is a logical relation?

- A logical relation is a family of relations, each indexed by a type.

\[ n_1 \mathcal{R}_{\text{int}} n_2 \iff n_1 = n_2 \]

\[ \langle a_1, b_1 \rangle \mathcal{R}_{\tau \times \tau}, \langle a_2, b_2 \rangle \iff a_1 \mathcal{R}_\tau a_2 \& b_1 \mathcal{R}_\tau b_2 \]

- Two functions \( f_1 \) and \( f_2 \) are related iff

\[ (\forall a_1, a_2) \ a_1 \mathcal{R}_\tau a_2 \Rightarrow f_1(a_1) \mathcal{R}_\tau f_2(a_2) \]

- Basic Lemma
  - If the denotation of each constant is related to itself, denotations of every term in related environments are related.
  - Basic Lemma helps us to prove contextual equivalence.

What is a cryptographic logical relation?

- The sprit of Sumii and Pierce’s logical relations: A cryptographic logical relation
  must relate encryption with itself, and relate decryption with itself.
Relations for base types

- Only keys that are accessible to attackers are related [Sumii & Pierce 02, Abadi & Gordon 97]:

\[ k_1 \mathcal{R}_{\text{key}}^{fr} k_2 \iff k_1 = k_2 \land k_1, k_2 \in fr \]

\[ fr \subseteq \llbracket \text{key} \rrbracket \] - the set of disclosed keys.

- Encrypted messages are then divided into two parts

\[ \{ e(v, k) \mid v \in \llbracket \text{msg} \rrbracket, k \in fr \} \cup \{ e(v, k) \mid v \in \llbracket \text{msg} \rrbracket, k \not\in fr \} \]

\[ e(v_1, k_1) \mathcal{R}_{\text{msg}}^{fr, \varphi} e(v_2, k_2) \iff v_1 \mathcal{R}_{\text{msg}}^{fr, \varphi} v_2 \land k_1 \mathcal{R}_{\text{key}}^{fr, \varphi} k_2 \]

or

\[ k_1, k_2 \not\in fr \land (v_1, v_2) \in \varphi(k_1, k_2) \]

\[ \varphi \] — parameter of the logical relation, fixing the relation between secret messages [Sumii & Pierce 02].
Logical relations for monadic types

- Categorical construction of logical relation for monadic types [Goubault-Larrecq et al. 02].

  But what is the category for constructing logical relations?

- A logical relation constructed over $\text{Set}^\mathcal{I}$:
  \[ \mathcal{R}_s^\tau \subseteq [\tau]^s \times [\tau]^s \]

  - Kripke logical relation — logical relations defined over functor categories [Mitchell & Moggi 91].
  - $s \in \mathcal{I}$ is called a “world”, representing a computation stage.
  - Two functions are related iff they take related arguments at any larger world to related results.

- Logical relations derived over $\text{Set}^\mathcal{I}$ are too weak with naïve relations for keys:
  \[ k_1 \mathcal{R}_{\text{key}}^s k_2 \iff k_1 = k_2 \]

  How to represent the parameter $fr$?
The “frame” category

Formalize the parameter $fr$ in the category $\mathcal{I} \rightarrow [ZN\ 03]$:

- objects are tuples $\langle w, i, s \rangle$;
- morphisms are pairs of injections $(j, l)$ such that the following diagram commutes:

$$
\begin{array}{c}
w \\ \downarrow j \\ w' \\
\end{array} \xrightarrow{i} 
\begin{array}{c}
w \\ \downarrow l \\ s' \\
\end{array} =
\begin{array}{c}
s \\ \downarrow \ \ \\
\end{array} =
\begin{array}{c}
s \\ \downarrow \ \ \\
\end{array}
$$

$fr$ Becomes $i(w)$:

$$k_1 \mathcal{R}_{key}^{\langle w, i, s \rangle} k_2 \iff k_1 = k_2 \& k_1, k_2 \in i(w)$$
Logical relations over $\mathbb{Set}^{\tau\rightarrow}$

$\mathcal{R}_{\tau}^{\langle w,i,s \rangle, \varphi} \subseteq [\tau]_s \times [\tau]_s$ (using the general construction of [GLLN02]).

- Basic Lemma holds, but only for a very limited set of $\varphi$.
- This logical relation fails in relating equivalent programs:

  let $k \leftarrow \text{new in val}((\{\text{true}\}_k, \{\text{false}\}_k, \lambda \{x\}_k.x))$
  let $k \leftarrow \text{new in val}((\{\text{false}\}_k, \{\text{true}\}_k, \lambda \{x\}_k.\text{not}(x)))$

$w \xrightarrow{} s \xrightarrow{} s'$

$\text{disclosed keys}$

$w' \xrightarrow{} s' \xrightarrow{} k \notin w$

$\text{Secret keys get known by attackers at a larger "world".}$
The “frame” category (revised)

- In our model, secret keys must NOT be exposed at any larger “world”.
  - A “world” represents a stage based on keys, not on time.

- Category $PI\rightarrow$:
  - the subcategory of $I\rightarrow$ where every $\downarrow$ is a pull-back.
Cryptographic logical relations

- **Cryptographic logical relations** derived over $\text{Set}^{PI\rightarrow}$:

$$\mathcal{R}_{\tau}^{w,i,s},\varphi \subseteq [\tau]s \times [\tau]s$$

- Cipher function $\varphi$ — a group of “world”-indexed functions, each determining the relation between secret cipher-texts at the “world”.
- Basic Lemma holds for a non-trivial set of cipher functions.
- Recognize Pitts and Stark’s operational logical relations for name creation.
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  - Contextual equivalence
- Conclusion
Contexts for cryptographic protocols

- In the computational lambda-calculus, contexts are allowed to do computations:

\[ x : \tau \vdash C : To \]

- Contexts represent exactly the knowledge of attackers:

- Contexts have access to disclosed keys.

- Contexts have access to secret cipher-texts, although they cannot decrypt them.

\[ w : \text{key}, m : \text{msg}, x : \tau \vdash C : To \]

Cryptographic Logical Relations
Cryptographic contextual equivalence

\[ \approx \langle w, i, s \rangle, \kappa \subseteq [\tau]_S \times [\tau]_S \]

defined using category \( \mathcal{PIT} \):

- \( w' : \text{key}, m : \text{msg}, x : \tau \vdash C : T_0 \) holds;

- \( \kappa \) — context knowledge, sets of secret cipher-texts that contexts can access;

- \( \kappa \) - honest environment, mapping every message variable to a cipher-text in \( \kappa \).
Verifying the secrecy property

- **Secrecy property:**
  \[ \forall \text{msg}_1, \text{msg}_2, \quad \text{Protocol}(\text{msg}_1) \approx \text{Protocol}(\text{msg}_2) \]

- **Theorem:**
  Cryptographic logical relations are **sound**:
  \[ R_{\tau}^{\langle w,i,s \rangle} \Rightarrow \approx_{\tau}^{\langle w,i,s \rangle} \]

- **Proposition:**
  This technique shows that Lowe’s fixed version of the Needham-Shroeder protocol satisfies the secrecy property (for multi-sessions).
Completeness

• A logical relation \( R_\tau \) is complete if \( \approx_\tau \implies R_\tau \)
• Completeness for monadic logical relations is hard to achieve, even for first-order types.

Our results:
• The cryptographic logical relations are complete for types:

\[
\tau^1_c ::= \text{key} \mid \text{msg} \mid \text{Tkey} \mid \text{key} \to \tau^1_c \mid \text{msg} \to \tau^1_c
\]
• A lax logical relation that is complete for all types.
Decidability

- In general, contextual equivalence in the cryptographic metalanguage is undecidable.
- Cryptographic logical relations are decidable for types:
  \[ \tau_d^1 ::= \text{key} \mid \text{msg} \mid T\tau_d^1 \mid \text{key} \rightarrow \tau_d^1 \]
- Contextual equivalence is decidable for types:
  \[ \tau_{\approx}^1 ::= \text{key} \mid \text{msg} \mid T\text{key} \mid \text{key} \rightarrow \tau_{\approx}^1 \]
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Main results

- The category \( \text{Set}^{\mathcal{P} \mathcal{i}} \) for deriving cryptographic logical relations.
- A proper notion of contextual equivalence for cryptographic protocols.
- Cryptographic logical relations:
  - sound (can deduce contextual equivalence);
  - complete for types:
    \[
    \tau_c^1 ::= \text{key} \mid \text{msg} \mid \text{Tkey} \mid \text{key} \to \tau_c^1 \mid \text{msg} \to \tau_c^1
    \]
- A complete lax logical relation.
- Decidability for contextual equivalence for types:
  \[
  \tau_{\approx}^1 ::= \text{key} \mid \text{msg} \mid \text{Tkey} \mid \text{key} \to \tau_{\approx}^1
  \]
Future work

• On programming languages:
  - Extend the model for dealing with recursion.
  - Freshness: nominal techniques based on FM-sets (name-swapping) [Pitts et al.].

• On security:
  - Protocols aiming at other security properties, e.g., anonymity.
  - The computational model:
    • Lambda-calculi might be a better language for expressing games, oracle calls, etc.
    • Typing systems enforcing complexity constraints [Hofmann 1997, Mitchell et al. 1998]
    • Logical relations might help!