Category Theory: A Motivational Speech

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Motivation

1. Categories are present in many domains of computer science
2. Some of the central concepts can be found in very different contexts
3. Close relationship between category theory and type theory

The Basic Concepts

A Category $\mathcal{C}$ consists of:

1. A *Class* of *Objects* $\mathcal{O}$
2. For every pair of objects A and B, a *Set* of *Morphisms* or *Arrows* $\mathcal{M}or(A, B)$

Which verify the following proprieties:

- For every object A there exists an *identity morphism* 1_A
- There exists a *composition operation* $\circ\colon \mathcal{M}or(A, B) \times \mathcal{M}or(B, C) \rightarrow \mathcal{M}or(A, C)$
- The composition is *associative* and respects identity i.e.:
  - $\forall A, B, C, D \in \mathcal{O}, f \in \mathcal{M}or(A, B), g \in \mathcal{M}or(B, C), f \in \mathcal{M}or(B, C), h \in \mathcal{M}or(C, D), h \circ (g \circ f) = (h \circ g) \circ f$
  - $f \circ 1_A = 1_B \circ f = f$

Some Examples

1. category *Set* of *sets*:
   1. objects are sets
   2. morphisms are applications between sets
2. for a given field $\mathbb{k}$, the category $\mathbb{k}$-*Vect* of *vector spaces* on $\mathbb{k}$
3. for each poset, an associated category
   1. the objects are the points of the posets
   2. given two points a and b, $\mathcal{M}or(a, b) = \{*\}$ if $a \leq b$ else $\emptyset$
4. category *Pos* of posets
   1. objects are posets
   2. morphisms are order-preserving maps
5. category of types
   1. objects are types
   2. given two types A and B morphisms are terms of type $A \rightarrow B$

A Few Concepts

1. Given a category $\mathcal{C}$ and two objects A and B, a *product* $A \times B$ is an object of $\mathcal{C}$ verifying:

$$\begin{tikzcd}
O \arrow[rd, dotted, "\exists!"] \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
 & A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \\
 & B &
\end{tikzcd}$$

   Products are unique *up to isomorphism*.

2. The dual notion of coproduct, an object $A + B$ that verifies the dual diagram.

3. An *initial object* is an object I with the property: $\begin{tikzcd} I \arrow[r, dotted, "\exists!"] & A \end{tikzcd}$

   An initial object is also unique up to iso.

Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* F between $\mathcal{C}$ and $\mathcal{D}$ is:

① a map between objects of $\mathcal{C}$ and objects of $\mathcal{D}$:

$$X \longmapsto F(X)$$

② a map between morphisms of $\mathcal{C}$ and morphisms of $\mathcal{D}$:

$$f \longmapsto F(f)$$

Which verifies:

$$\forall \text{ composable } f, g: \; F(f \circ g) = F(f) \circ F(g)$$

Examples of Functors

1. The "forgetfull functor" $\mathcal{U} : Vect \longrightarrow Set$ which "forgets" the vector field structure
2. The "free functor" $\mathcal{F} : Set \longrightarrow Vect$ which, given a set E gives the free vector space generated by E.

other interesting functors include

1. Topological spaces to groups
2. increasing functions between posets
3. etc!

More Examples

Given a category $\mathcal{C}$ with products and coproducts, and given an object A of $\mathcal{C}$

1. $X \mapsto X \times A$ is a functor, written $\_ \times A$
2. $\_ + A$

An Application:The Notion of F-Algebras

Given A category $\mathcal{C}$ and an endofunctor $F : \mathcal{C} \to \mathcal{C}$ an *F-algebra* is

- An object A of $\mathcal{C}$
- A map $F(A) \to A$

Given A and B two F-algebras, a *morphism of F-algebras* from A to B is a map $f : A \to B$ verifying:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d] & F(B) \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

F-algebras with such morphisms form a category

Initial Objects = Inductive Objects

An initial object in such a category is an F-algebra O such that forall F-algebra A, there exists a unique ϕ such that:

$$\begin{tikzcd}
F(O) \arrow[r, "F(\phi)"] \arrow[d] & F(A) \arrow[d] \\
O \arrow[r, "\phi"] & A
\end{tikzcd}$$

1. If such an object exists, then $O \cong F(O)$
2. O is called an *inductive object*
3. O is unique up to iso, as are all initial objects

Example:The Natural Number Object

Take $\mathcal{C} = Set$, $F = 1 + \_$ with $1 = \{\emptyset\}$. Then the following are the case:

1. For any inductive object Ω, $\Omega \cong \mathbb{N}$
2. The induction principle is equivalent to the existence of aforementioned ϕ
3. The recursion principle is equivalent to the unicity of ϕ

Application:An Inductive Definition, A Recursive Proof

Inductive definition of the function $f : \mathbb{N} \to \mathbb{Q}$ defined by $f : n \mapsto 2^{-n}$ The definition $f(0) = 1$ and $f(n+1) = \frac{1}{2}n$ corresponds to the diagram:

$$\begin{tikzcd}
1 + \mathbb{N} \arrow[r, dotted, "id+f"] \arrow[d, "{[0,S]}"'] & 1 + \mathbb{Q} \arrow[d, "{[1,\frac{1}{2}(\_)]}"] \\
\mathbb{N} \arrow[r, dotted, "f"] & \mathbb{Q}
\end{tikzcd}$$

Recursive proof of $\forall n \in \mathbb{N}, n + n = 2n$ corresponds to the unicity in the diagram:

$$\begin{tikzcd}
1 + \mathbb{N} \arrow[r, shift left, "id+2\times(-)"] \arrow[r, shift right, "id+\big((-)+_{\mathbb{N}}(-)\big)"'] \arrow[d, "{[0,S]}"'] & 1 + \mathbb{N} \arrow[d, "{[0,S\circ S]}"] \\
\mathbb{N} \arrow[r, shift left, "2\times(-)"] \arrow[r, shift right, "(-)+_{\mathbb{N}}(-)"'] & \mathbb{N}
\end{tikzcd}$$

What About Coinduction?

For every notion or theorem in category theory, there is a *dual* notion or theorem. The dual notion for inductive objects in a category $\mathcal{C}$ is the notion of *coinductive* objects, *i.e.* a final object in the category of co-F-algebras. Therefore:

Proving a theorem about coinductive types $\Leftrightarrow$ Proving a theorem about inductive types!