The Assignment Problem in Constraint Programming

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based on joint work with:
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Outline

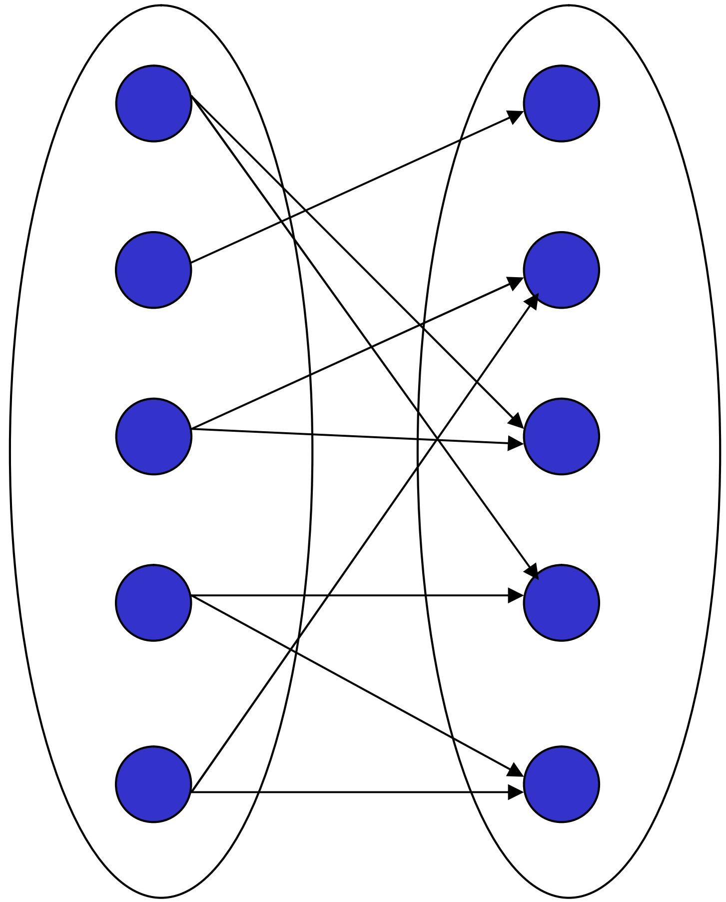
- The linear Assignment Problem (AP).

- Using the Assignment Problem:
  1. Optimization global constraints.

- Using the Assignment Problem:
  2. Discrepancy-based additive bounding techniques.

The Assignment Problem (1)

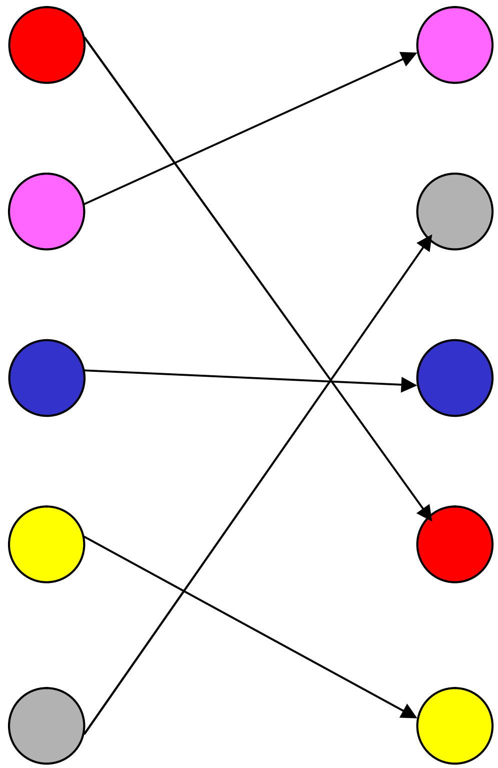
- We are given n machines and n jobs, and a square matrix c such that:
  - c_{ij} is the cost associated with job j if performed on machine i
- A graph theoretic model involves a bipartite, directed graph $\mathbf{G = (S \cup T; A)}$ with costs associated with arcs



The Assignment Problem (2)

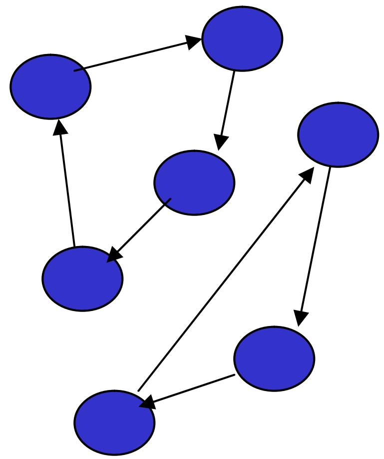
- A feasible solution is a perfect matching

- An optimal solution is the least-cost perfect matching



The Assignment Problem (3)

$$\min Z = \sum_{i \in S} \sum_{j \in T} c_{ij} x_{ij}$$



$$\sum_{i \in S} x_{ij} = 1 \quad \forall j \in T$$

$$\sum_{j \in T} x_{ij} = 1 \quad \forall i \in S$$

$$x_{ij} \geq 0; \cancel{x_{ij} \text{ integer}} \quad \forall i \in S; j \in T$$

The Assignment Problem (4)

- The AP is a linear program, thus can be solved with general purpose techniques as the simplex method.

- However, AP has a rather special structure and there are efficient special purpose algorithms to solve it.

- In particular, we will make use of the so-called Hungarian algorithm which is a primal-dual method which can be implemented to run in $O(n^3)$ time.

- We will point out all over the talk why solving the problem through a combinatorial algorithm is so important.

1: Optimization global constraints

- Global Constraints for optimization problems:
    - global constraints in CP;
    - global constraints with an *optimization component:*
        - the *cost-based domain filtering* technique.
- Computational results on *TSP* and *TSPTW*.

Global Constraints

- Global Constraints:
  - capture sub-problems that frequently constitute a sub-structure of more general problems;

  - include *propagation* algorithms which perform pruning on domain variables on the basis of *feasibility reasoning*.

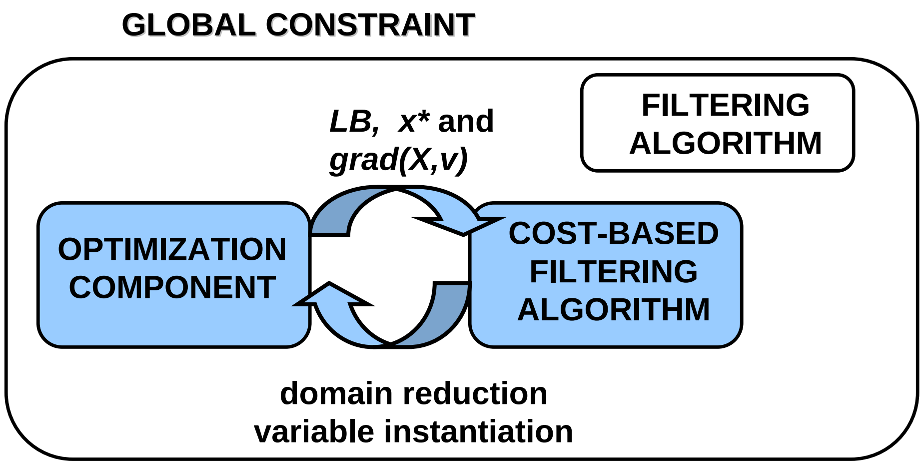
- Global Constraints for optimization problems:
  - we need a pruning based on *optimality reasoning*;
  - we embed an *optimization component*, i.e., a software component which solves to optimality a relaxation of the problem represented by the global constraint;
  - the relaxation depends on the *objective function*.

Global Constraints for Optimization Problems

- The optimization component is typically based on effective OR algorithms, thus a mapping between CP variables and the OR model is needed.

- The optimization component must provide:
  - ***LB***: the optimal solution value of the relaxation;
  - ***x****: the optimal solution of the relaxation in the OR model;
  - ***grad(X,v)***: a gradient function estimating the additional cost of variable-value assignments.

Global Constraints for Optimization Problems



Optimization Constraints (1)

- Lower Bound-based propagation:
  from ***LB*** towards objective function ***$Z::[Z_{min}..Z_{max}]$: $LB < Z_{max}$***

- cost-based propagation:
  from the gradient function towards decision variables:

  for each ***$X_i::[v_1,v_2,\ldots,v_m]$*** and ***$v_j$*** there is a gradient function ***$grad(X_i,v_j)$*** measuring the additional cost to pay if ***$X_i = v_j$***

$$\textbf{if} \quad LB + grad(X_i,v_j) \geq Z_{max} \quad \textbf{then} \quad X_i \neq v_j$$

which is the classic OR *variable fixing*.

Optimization Constraints (2)

- Finally, the optimal solution of the relaxation in the OR model may help, through the mapping, to guide the search.

- The simplest example of gradient function are the linear programming *reduced costs* which can be computed for some special cases by combinatorial algorithms.

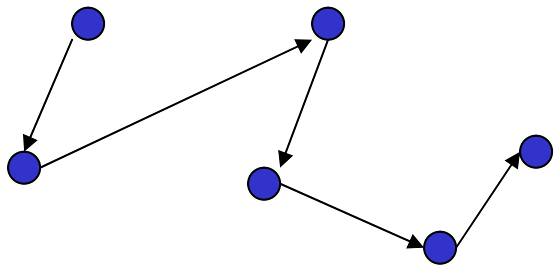
- We consider in the following:
  the **Assignment Problem** as a relaxation of the path constraint, and the **Hungarian Algorithm** as (combinatorial) optimization component.

Path constraint: an optimization component (1)

Given a directed graph $G=(V,A)$ with $|V| = n$, and associated with each node $\boldsymbol{i}$ a variable $\boldsymbol{X_i}$ whose domain contains the next possible nodes in a path, the CP **path constraint**:

$$\boldsymbol{X_0::D_0,\ X_1::D_1,\ldots,\ X_k::D_k}$$

$$\boldsymbol{path([X_0,X_1,\ldots,X_k])}$$



holds if and only if the assignment of variables $\boldsymbol{X_0,X_1,\ldots,X_k}$ defines a simple path involving all nodes $\boldsymbol{0,\ldots,k}$.

Path constraint: an optimization component (2)

If a ***cost*** is associated to each arc, and we want to model the **Asymmetric Traveling Salesman Problem** (ATSP), we can use the path constraint as follows:

- one of the node, say ***0***, is duplicated generating node ***n***;
- node ***n*** reaches only node ***0*** with zero cost, while it is reached from each node (but ***0***) with the same cost paid to reach node ***0***;
- the constraint ***path([X_0,X_1,…,X_n])*** is imposed.

AP can then be used as *optimization component* for **path()**.

Mapping

CP- Model:

$X_i::[v_1,v_2,\ldots,v_n]$ i=0..n-1

$path([X_0,X_1,\ldots,X_n])$

$C_i::[c_{i1},c_{i2},\ldots,c_{in}]$ i=0..n-1

$C_n = 0;\ X_n = 0;$

$C_0+\ldots+C_{n-1} = Z$

minimize(Z)

Mapping $\Longleftrightarrow$

IP- Model

$$\min Z = \sum_{i\in V}\sum_{j\in V} c_{ij}\ x_{ij}$$

$$\sum_{i\in V} x_{ij} = 1 \quad j\in V \ \text{(A)}$$

$$\sum_{j\in V} x_{ij} = 1 \quad i\in V \ \text{(B)}$$

$$\sum_{i\in S}\ \sum_{j\in V\setminus S} x_{ij} \geq 1 \quad S\subset V \quad S\neq\emptyset$$

$x_{ij} \geq 0$ and integer

$X_i = v_j \longleftrightarrow x_{ij}=1$

$path([X_0, X_1, \ldots, X_n]) \xleftrightarrow{\textit{relaxed in}}$ (A) + (B)

$C_i=c_{ij} \longleftrightarrow c_{ij}$ iff $x_{ij}=1$

Path constraint: an optimization component (4)

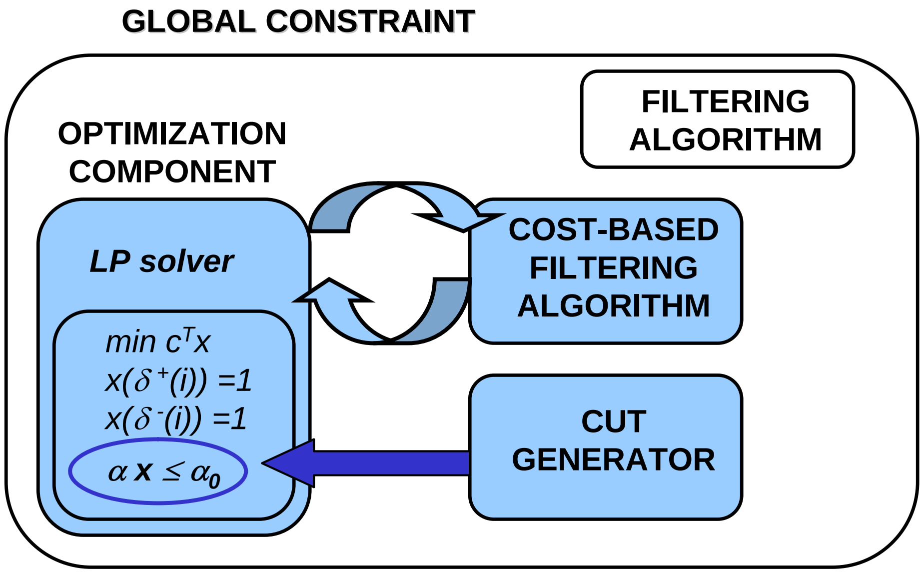
- The connectivity constraints are relaxed, and by the Hungarian algorithm we obtain a lower bound value Z_{AP}, an **integer** solution x^*, and the reduced costs. In addition the Hungarian algorithm is incremental ($O(n^3)$ first solution, $O(n^2)$ each re-computation).

- However, the bound could be very poor, mainly for pure problems as TSP, and a classical OR method for improving it is **cutting planes** generation.

- The simplest cutting planes are the Subtour Elimination Constraints (SECs) whose separation is polynomially solvable.

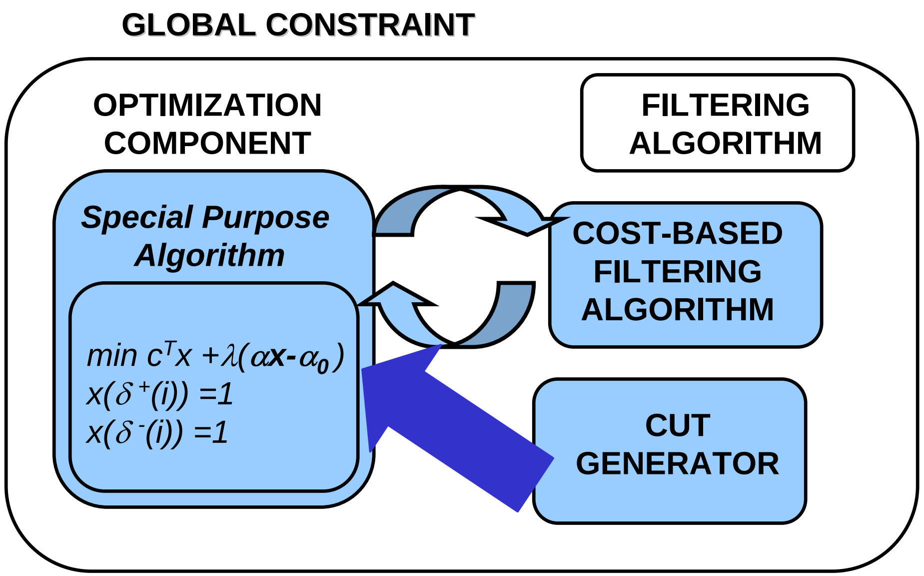
Cutting planes in global constraints

- The cut generator is again a black-box in the global constraints, but the optimization component is now a general LP solver (the AP structure is lost), whereas the cost-based propagation remains unchanged.



Lagrangean relaxation of cuts

- The drawback of using a general LP solver (not incremental, not integer solution) can be partially overcome by dualizing in Lagrangean fashion the generated cuts.



Lagrangean multipliers

- Algorithm:
  - optimally solve the original structured relaxation → LB_{AP};
  - repeat

    generate violated cuts;

    add cuts to the current formulation;

    solve the corresponding LP;
  - until a given point (e.g., the end of the root node) → LB_r;
  - extract the dual values associated to tight cuts:

    they are the optimal Lagrangean multipliers of the cuts;
  - dualize tight cuts and update the cost matrix;
  - solve the structured relaxation → LB_{APm}.

Through duality theory: $LB_{AP} \leq LB_r = LB_{APm}$

AP+Lagrangean vs AP+cuts

- AP + cuts + Lagrangean Relaxation:
  - \+ still an AP, i.e. a structured problem;
  - \+ $O(n^2)$ incrementally;
  - \+ x^* is integer;
  - \- λ are optimal only at root node;
  - \- dynamically purging trivially satisfied cuts.

- AP + cuts:
  - \+ *LB* always accurate;
  - \- resulting LPs may be huge;
  - \- only partially incremental.

Results

- Although CP is not competitive to cope with problems like TSP and ATSP, the addition of an optimization component allows the solution of bigger-size instances.

TSP and ATSP instances						
Instance	***pure AP***			***AP + Lagrangean relaxation of cuts***		
	Opt	***Time***	***Fails***	***Opt***	***Time***	***Fails***
Gr17	2085	0.39	511	2085	0.49	30
Fri26	937	0.82	725	937	0.71	80
Bays29	2020	4.12	4185	2020	1.20	403
Dantzig42	699	>300	-	699	5.55	1081
RY48P	14854*	>300	-	14422	130.00	50K

- Pure CP gets stuck even on problems of this size.

TSP with Time Windows (TSPTW)

- On less pure problems it is possible to exploit the flexibility of CP.

- TSPTW is the TSP variant in which the visit of each city must be done within a prefixed *Time Window*.

- TSPTW has two main components:
  - a **routing** component which is basically *optimization*, i.e. find the tour of *minimum cost*;
  - a **scheduling** component which is mainly a *feasibility* issue.

TSPTW: aggregated experimental results

- symmetric TSPTW:
  - outperforming state of the art methods Pesant et al. 1998
  - Lagrangean relaxation of cuts is very effective

- asymmetric TSPTW
  - competitive results with state of the art branch-and-cut methods Ascheuer, et al. 2001
  - Lagrangean relaxation does not pay off since AP bound already effective

2: Linking search and bound

- Discrepancy-based additive bounding:
  - limited discrepancy search;
  - additive bounding techniques;
  - speeding up the proof of optimality.
- Computational results.

Limited discrepancy search

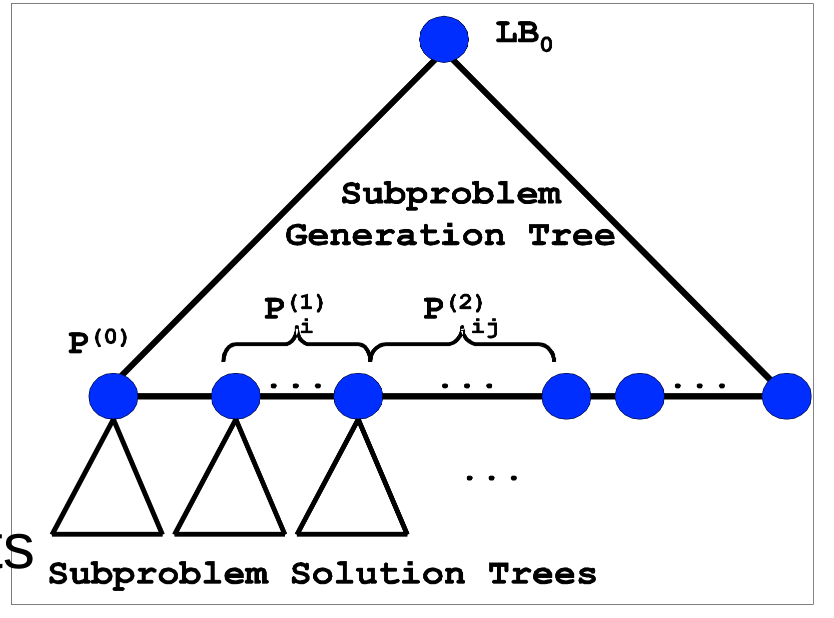
- Explores the most promising branches of a search tree first
  - At each node a heuristic *recommends a branch*
  - Tolerate a maximum number of *discrepancies* from the heuristic's recommendation

- Limited Discrepancy Search (LDS) is effective to rapidly identify good solutions

- LDS has no real incentive to accelerate the proof of optimality

Discrepancy based search

- Introduced by Milano and van Hoeve at CP 2002
- First split the domain of each variable into:
  - a *Good* set containing the promising values
  - a *Bad* set containing the rest of the values.



- Perform LDS with these sets
  - At each node a discrepancy is counted if a variable takes a value in its *Bad* set.

Discrepancy constraint

- Introduce a *Discrepancy Constraint* in the model of any problem solved via the Discrepancy Based Search (DBS)
  - *X* is the vector of finite domain variables
  - $ß_V$ is the bad set of variable V
  - *k* is the current accepted level of discrepancy
  - *Discrepancy_Cst(X,ß,k)*

$$\sum_{i \in N} (X_i \in \beta_{X_i}) = k$$

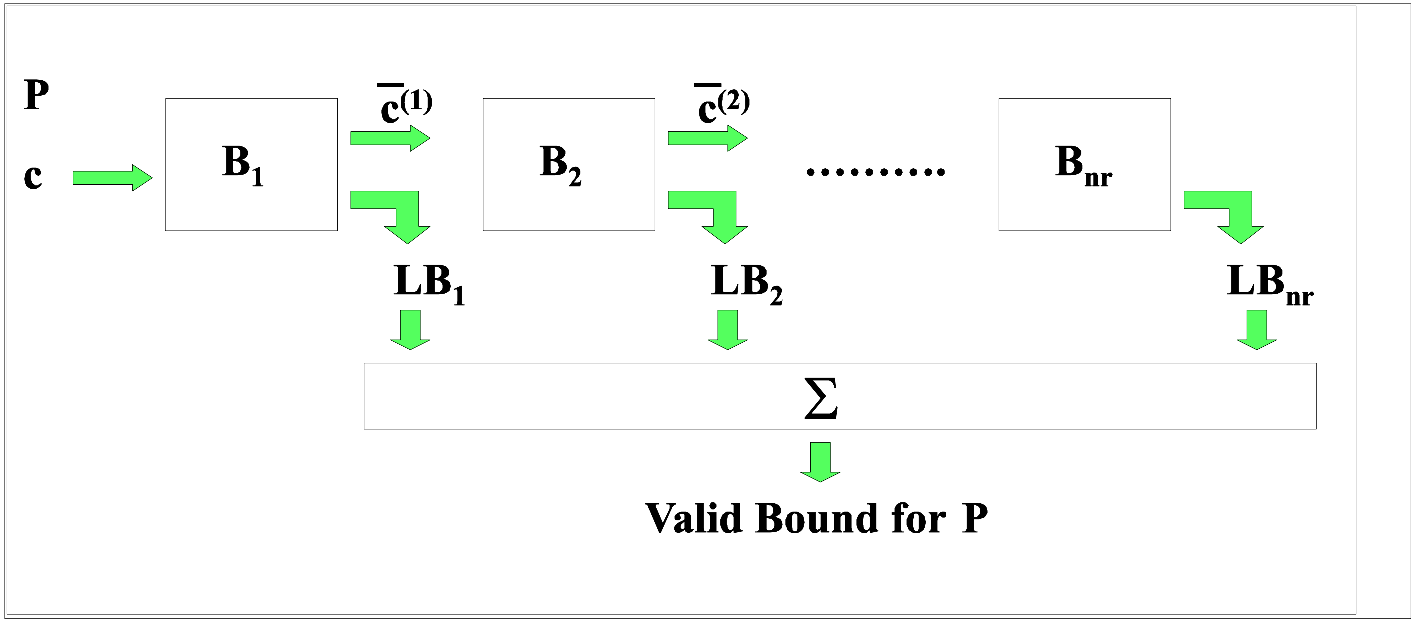
Reduced costs

- The *reduced costs* computed as a result of the solution of a linear program:
  - Associated with a variable V
  - Represent the cost to add to the optimal solution if the variable V becomes basic at value 1
  - Denoted here by: c_V

- If for a given problem:
  - More than one “bound” is available: $B^1,\dots,B^{nr}$
  - Each bound takes as input a cost vector: $B^k(c^{k-1})$
  - All bounds return a value LB^k
  - All bounds output a reduced cost vector: c^k

Additive Bounding Procedure (1)

- The Additive Bounding Procedure (ABP) is:
  - Compute: $LB^1 = B^1(c^0)$ where c^0 is the original cost vector
  - for all k:$2,\ldots,nr$: $LB^k = B^k(c^{k-1})$
  - $LB = LB^1 + LB^2 + ,\ldots, + LB^{nr}$



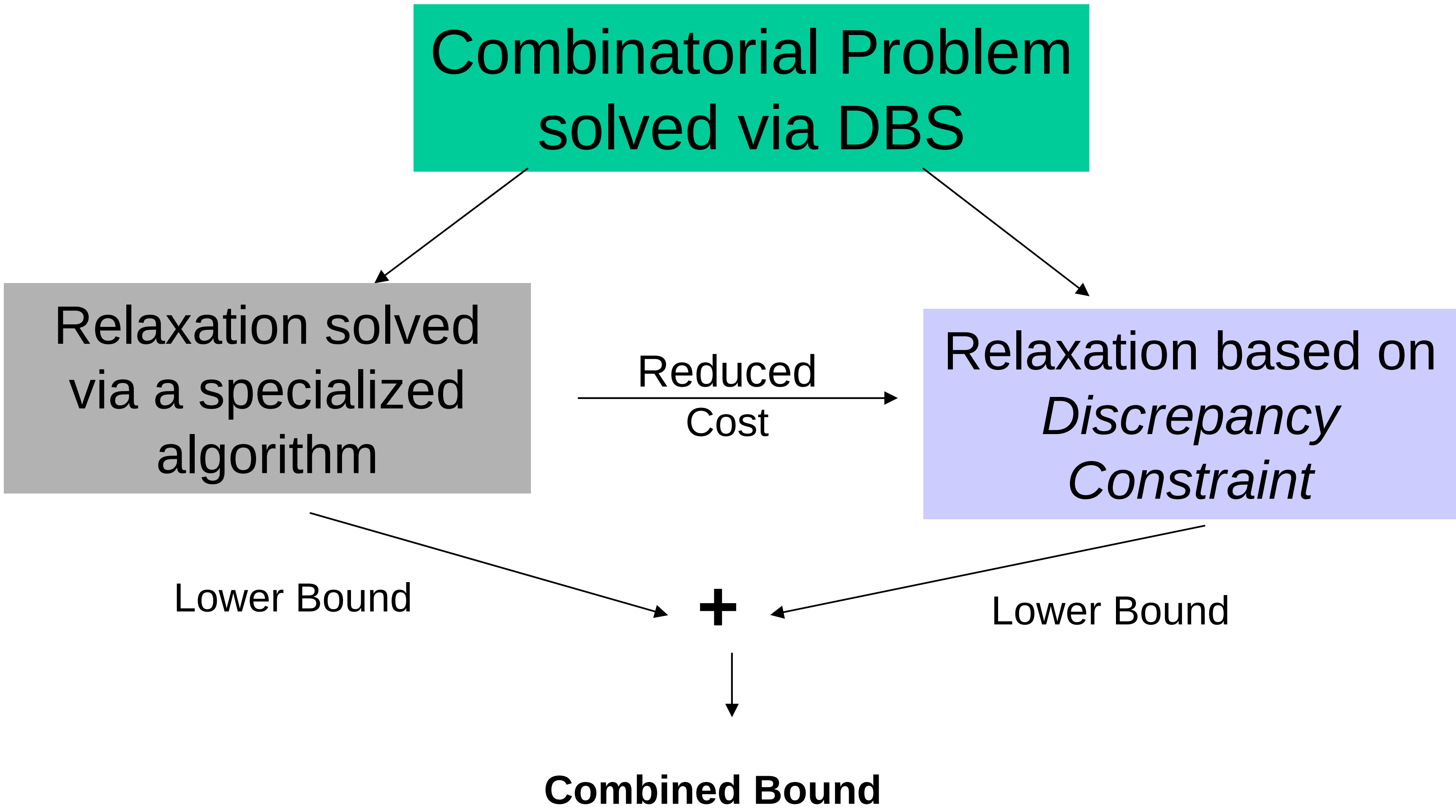
Additive Bounding Procedure (2)

- This remarkable technique has been introduced by Fischetti & Toth as a general framework and successfully applied in the context of the Traveling Salesman Problem.

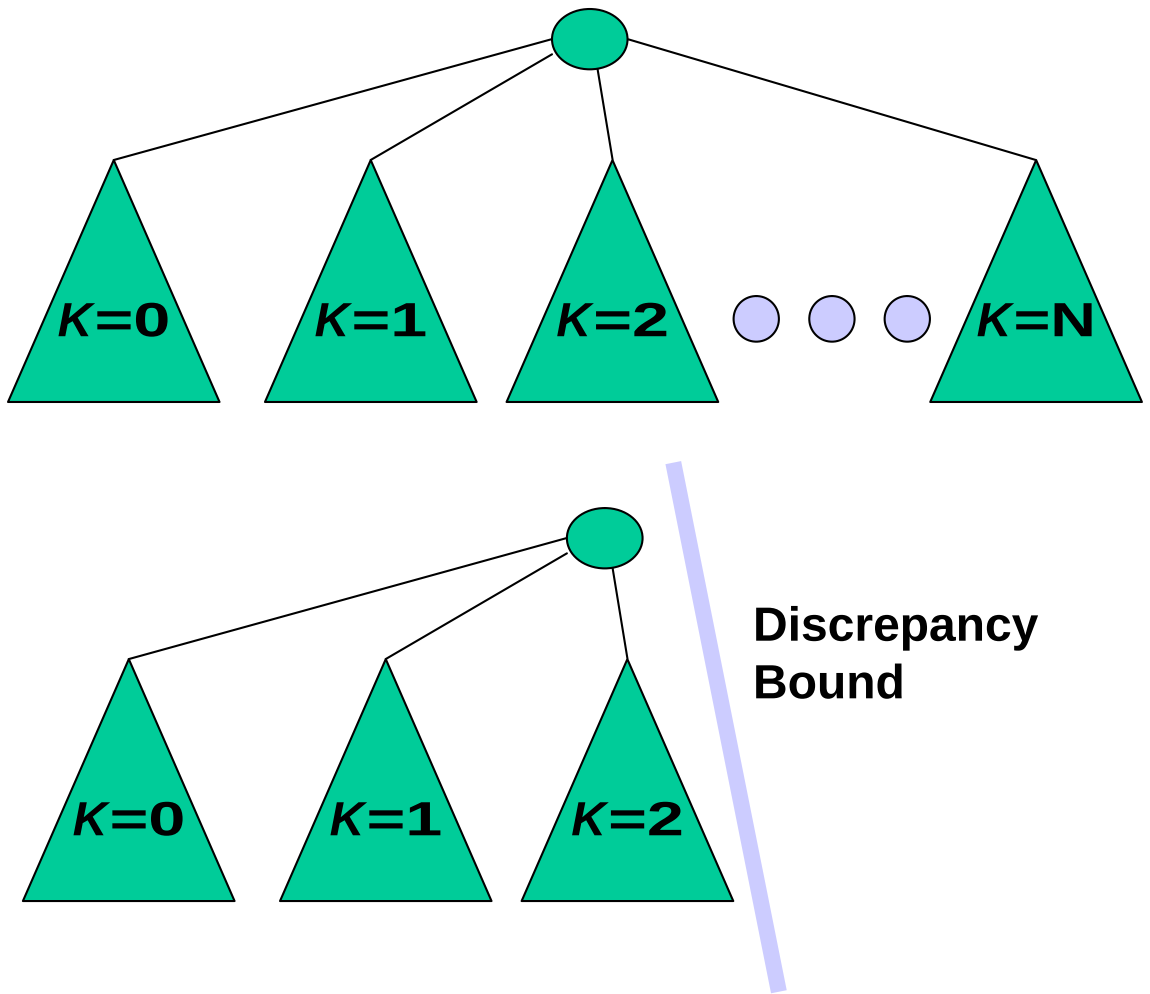
- Enhancing LDS proof of optimality by improving the quality of the bounds.

- Additional motivation: use ABP in conjunction with DBS to establish a **stronger** *link between search* and *bound*.

ABP and DBS


Combinatorial Problem solved via DBS
Relaxation solved via a specialized algorithm
Reduced Cost
Relaxation based on *Discrepancy Constraint*
Lower Bound
+
Lower Bound
Combined Bound

ABP and DBS: what is to be gained



Discrepancy-Based Additive Bounding

Can be seen as an Assignment Problem

$$Minimize: \sum_{i \in N} C_{iX_i}$$

$$Subject\ To: \text{AllDifferent}(X)$$

$$\text{AnySideConstraint}(X)$$

$$X_{i \in N} \in N$$

$$\text{Discrepancy\_Cst}(X, \beta, k)$$

Side Constraint which makes the problem *NP*-hard

Discrepancy constraint

Discrepancy-Based Additive Bounding
A first additive bound

$$\text{Minimize} : \sum_{i \in N} C_{iX_i}$$

$$\text{Subject To} : \text{AllDifferent}(X)$$

$$X_{i \in N} \in N$$

$$\text{Discrepancy\_Cst}(X, \beta, k)$$

$$\text{AnySideConstraint}(X)$$

First bound from the Assignment Problem

Second bound from Discrepancy Constraint

Discrepancy-Based Additive Bounding
A first additive bound

$$\textit{Minimize} : \sum_{i \in N} C_{iX_i}$$

$$\textit{Subject To} : \text{AllDifferent}\,(X)$$

$$X_{i \in N} \in N$$

Reduced cost vector C^1

$$\textit{Minimize} : \sum_{i \in N} C^1_{iX_i}$$

$$\textit{Subject To} : \sum_{i \in N} (X_i \in \beta_{X_i}) = k$$

$$X_{i \in N} \in N$$

First bound from a *primal-dual* Algorithm

+

Second bound from counting the *k* smallest reduced cost in *ß*

Discrepancy-Based Additive Bounding Linear Model

$$x_{ij} = 1 \Leftrightarrow X_i = j \quad \forall i, j \in N$$

$$\sum_{j \in N} x_{ij} = 1 \quad \forall i \in N$$

$$x \in \{0,1\}^N$$

Mapping between linear model and CP model

$$Minimize: \sum_{i \in N} c_{ij} x_{ij}$$

$$Subject\ To: \sum_{i \in N} x_{ij} = 1 \quad \forall j \in N$$

$$\sum_{j \in N} x_{ij} = 1 \quad \forall i \in N$$

$$\sum_{i \in N} \sum_{j \in \beta_{X_i}} x_{ij} = k$$

$$\text{AnyLinearSideConstraint}\,(X)$$

$$x \in \{0,1\}^N$$

Linear Programming model

Discrepancy-Based Additive Bounding
a second (general) bound

$$Minimize \ : \sum_{i \in N} c^1_{ij} x_{ij}$$

$$Subject \ To : \sum_{j \in \beta_{X_i}} x_{ij} \leq 1 \quad \forall i \in N$$

$$\sum_{i \in N} \sum_{j \in \beta_{X_i}} x_{ij} = k$$

$$x \in \{0,1\}^N$$

To solve:

- Identify the minimum reduced cost associated with every CP variable
- Sum the *k* minimum reduced costs selected

Discrepancy-Based Additive Bounding
Generality of the method

- The technique has been applied in conjunction with the AP but it is obviously independent on it.

- More precisely, the framework can be applied with any (combinatorial) relaxation providing the reduced cost vector or, even more likely, with a sequence of relaxations.

- “Combinatorial relaxation” means that a special purpose algorithm is used to solve the relaxation, thus the framework does not affect the structure of the relaxation itself.

Discrepancy-Based Additive Bounding
Further exploiting the structure: AllDifferent

$$Minimize:\ \sum_{i \in N} c_{ij} x_{ij}$$

$$Subject\ To: \sum_{i \in N} x_{ij} \boxed{\le 1} \quad \forall j \in N$$

$$\sum_{j \in N} x_{ij} \boxed{\le 1} \quad \forall i \in N$$

$$\sum_{\boxed{i,j \in N}} x_{ij} = k$$

$$x \in \{0,1\}^N$$

use of a second bound

it is known as the *k*-Assignment problem and can be solved in polynomial time

it is also a relaxation of the previous model (since $c^1_{ij} = 0$)

Incorporate more information than just the discrepancy constraint

Defining benchmarks (1)

Lower Assignment Problem (LAP)

$$Minimize: \sum_{i \in N} C_{iX_i}$$

$$Subject\ To: \text{AllDifferent}(X)$$

$$\sum_{i \in N} C_{iX_i} \geq L$$

$$X_{i \in N} \in N$$

Assignment Structure

L constraint abstracts the *side constraints* and arbitrarily makes the AP bound poor.

Didactic but NP-hard and very hard to solve

Defining benchmarks (2)

Asymmetric Traveling Salesman Problem (ATSP)

$$Minimize: \sum_{i \in N} C_{iX_i}$$

$$Subject\ To: \mathrm{AllDifferent}(X)$$

$$\mathrm{SubTourElimination}(X)$$

$$X_{i \in N} \in N$$

Defining benchmarks (3)

Resource Constraint Assignment Problem (RCAP)

$$Minimize: \sum_{i \in N} C_{iX_i}$$

$$Subject\ To: \text{AllDifferent}(X)$$

$$\sum_{i \in N} R_{iX_i} \leq M$$

$$X_{i \in N} \in N$$

Assignment Structure

classical knapsack constraint modeling a resource.

NP-hard to solve

Experimental Results

- For all benchmarks we generated 60 test problems using structured cost matrices of the DIMACS ATSP instances.
- Size of the problem was set to N = 25.

- For LAP, *L* is 110% the AP relaxation value.
- For RCAP, first solve AP using *R* as cost vector, then *M* is set 4 times this value.

- Ilog Solver 5.2. on a Intel 1.5 GHz Centrino laptop.
- Time limit (TL) for each run is 3,600 CPU seconds.
- Variable selection based on first fail (min. domain size).
- Value selection based on minimum reduced cost.

Aggregated experimental results (1)

%reduction	Counting		*K* assignment	
Problem	Time	BT	Time	BT
LAP	55%	57%	56%	58%
ATSP	31%	39%	39%	49%
RCAP	25%	23%	30%	23%

average *k*			
Problem	normal DBS	counting	*K*-assignment
LAP	24.75	4.38	4.35
ATSP	26.00	5.68	5.63
RCAP	23.91	5.16	5.10

Aggregated experimental results (2)

- Taking **search** into account in the **bounding procedures** seems to be particularly effective.
- This is a general (and easy) approach which can be widely used when an efficient algorithm for a relaxation provides reduced costs.
- Further exploiting a problem structure can improve the behavior of discrepancy-based additive bounding.

k-discrepancy: Full Integration via LP

- Fully integrate the *Discrepancy Constraint* with the lower bound
  - Use the linear relaxation as a lower bound
  - Not as efficient since an LP is solved at each node
  - Maximum use of discrepancy information

$$\mathit{Minimize}: \sum_{i \in N} c_{ij} x_{ij}$$

$$\mathit{Subect\ To}: \sum_{i \in N} x_{ij} = 1 \quad \forall j \in N$$

$$\sum_{j \in N} x_{ij} = 1 \quad \forall i \in N$$

$$\sum_{i \in N} \sum_{j \in \beta_{X_i}} x_{ij} = k$$

$$\text{AnyLinearSideConstraint}\,(X)$$

$$\boxed{x \in (0,1)^N}$$

k-discrepancy: Partial Integration via Lagrangean Relaxation

Solve the Linear Relaxation

Once per discrepancy

Lagrangean Relaxation of the Discrepancy Constraint with optimum multiplier

Solve the Modified Assignment Problem

During Search

- Partially integrate the *Discrepancy Constraint* via Lagrangean Relaxation
  - Only some information on discrepancy is used
  - At each node an assignment problem is solved instead of a linear program

Drawback:
the Lagrangean multiplier used is not anymore optimal during search

Additional experimental results

- More integrated is the *Discrepancy Constraint*, less discrepancy level is needed to prove optimality, i.e. in terms of the value of the bound we have:

  Additive Bounding < Lagr. Relaxation < Linear Relaxation

- LP and Lagrangean relaxations are less effective both in terms of computing time and number of backtracks (they also solve less problems in TL):
  - LPs are more time consuming
  - Lagrangean multiplier deteriorates
  - Cost-based propagation in pure AP case is more effective

Conclusion

- The AP is extensively used as a relaxation for different purposes.

- On the other hand, the techniques shown in this talk do not depend on the AP.

- The key issue is the use of a combinatorial relaxation, i.e., a relaxation which models a linear program but can be solved with a special purpose technique.

- This is often the case with graph theory models!!!

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