Some algebraic geometry problems arising in the field of mechanism theory

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Abstract

Mechanism theory has always been a favorite field of study for mathematicians. But although many powerful results are available, still various open problems remain, especially when one deals with closed-loop mechanisms. This is basically due to the complexity of the equations which have to be solved, even for the most simple mechanism.

The introduction of symbolic computation and of new mathematical tools is of great help for solving many mechanism theory problems. We present a survey of problems arising in this field which are illustrated by examples, some of them being unsolved at this time.

1 Introduction

Mechanism theory deals with *kinematics chains* i.e. rigid bodies (called *links*) connected by *joints*. These joints enable motion between the links and are characterized by the type of motion they allow. The main joints allow only one motion, either a rotation around a given axis (*rotary joint*) or a translation along one given axis (*prismatic joint*). More complex joints can be constructed with these basic joints, e.g. the ball-and-socket joint enabling every rotation around a point. Note that a finite set of parameters defines the status of the joint. For example for a rotary joint the rotation angle fully defines the joint. The independent parameters of the joints will be called the *articular coordinates* of the mechanism.

Among all the joints in a mechanism some may be actuated (i.e. a motor enables to change the value of the parameter of the joint) and the other joints are passive. The parameters of the actuated joints are the *input* of the mechanism.

It is usual to distinguish two special links in a mechanism: the base link which is fixed to the ground and the end-effector whose position/orientation is the *output* of the mechanism. The number of parameters which are required to define the position/orientation of the end-effector is called the number of *degrees of freedom (DOF)* of the mechanism. For planar mechanisms this number is a most 3 (two translations and one rotation) and 6 (three translations and three rotations) for spatial mechanisms. The parameters describing the position/orientation of the end-effector are called the *generalized cartesian coordinates* of the mechanism.

In order to control all the n degrees of freedom of the end-effector it is necessary that the mechanism has at least n independent actuated elementary joints. A mechanism with $m > n$ independent actuated joints is called a *redundant* mechanism but we will restrict ourselves to the case where $m = n$.

We introduce now the *degree of connection* of a link as the number of rigid bodies which are connected to the link. At this points we may define two kinds of kinematics chains:

- open-loop kinematic chains: all the links have a degree of connection of two except the base and the end-effector which have a degree of connection of one.
- closed-loop kinematic chains: at least one link has a degree of connection greater or equal to three or the degree of connection of the end-effector is greater than one.

Figure 1 shows examples of open- and closed-loop planar kinematics chains with only rotary joints.

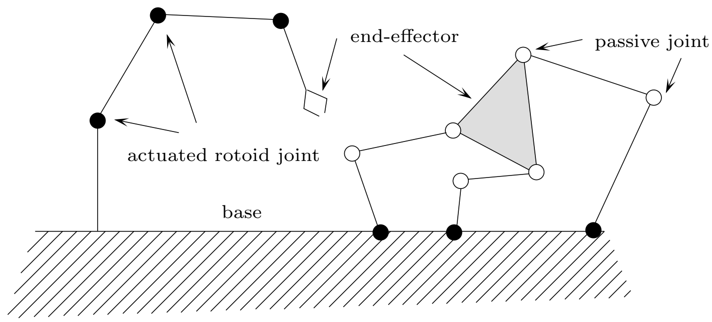

Figure 1: On left an open-loop planar kinematic chain, on right a closed-loop kinematic chain.

2 Classical problems in the field of mechanism theory

When studying a mechanism classical problems arise:

- determining the relation between the articular coordinates of the actuated joints and the generalized cartesian coordinates. The *direct kinematics* consists in determining the generalized cartesian coordinates for a given set of articular coordinates. The *inverse kinematic* problem is finding the opposite relation.
- determining the *workspace* of the end-effector, i.e. the limited region which can be reached by the end-effector owing to constraints on the joints (for example a limited range for a prismatic actuator or no intersection between the links).
- determining the *singular configurations* of the mechanism. In the general case there is a linear relationship between the actuator velocities and the velocities of the end-effector, but this relation is no more valid for some special configurations of the mechanism.
- the *design* or *synthesis* problem: for a given type of mechanism find the geometry of the mechanism such that the trajectory of the end-effector contains a given set of points, called the *precision points*.

The above problems involve solving algebraic equations or inequalities which are deduced from the geometry of the mechanism. All of them are of great practical interest: the kinematics problems are to be solved to control the mechanism, the workspace and synthesis problems are useful for the design of robots and the singularities are essential as a mechanism can suffer a breakdown when crossing a singular configuration..

3 Some results

Very elegant results has been established in the past, especially for planar mechanisms. I will mention a result established by Freudenstein which has shown that it is always possible to design a planar mechanism such that the trajectory of the end-effector (or at least a part of it) corresponds to a given algebraic curve [1].

3.1 The four-bar mechanism

Another well-studied mechanisms is the so-called *four-bar mechanism* (figure 2). Basically the end-effector is a

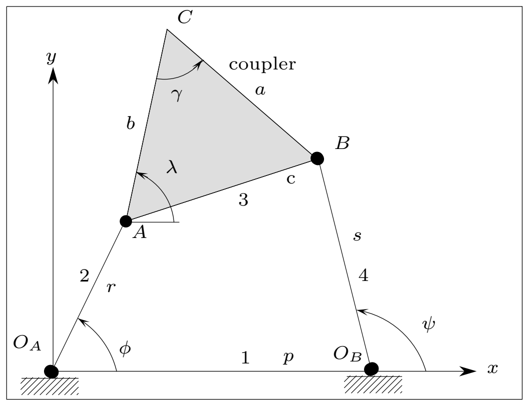

Figure 2: The four-bar mechanism.

triangle with two vertices connected by two segments to the base. At the extremities of the segments rotary joints enable a rotation. Note that both the base and the end-effector have a degree of connection of two and consequently, the four-bar mechanism is a closed-loop mechanism. If the joint at O_A is actuated then point C of the end-effector will describe a curve (the *coupler curve*) which happens to be a sextic [5] which has at most four double points. This sextic has a full circularity i.e. it intersects the imaginary circle three times at the circular imaginary points. Although this mechanism is one of the most simple planar mechanism the equation of the sextic is rather complex. For example the synthesis problem with three or four precision points has been solved [4] but it is only recently that a solution to the nine precision points synthesis has been proposed [16]. Solving this synthesis problem is equivalent to solve a system of four fourth-degree polynomials in four unknowns. The solution proposed by Wampler uses the numerical method called *continuation* which has been successfully used for solving kinematics problems [15].

3.2 The 6R manipulator

As for spatial kinematic chains, a classical problem is the resolution of the inverse kinematic of the 6R robot. Most industrial robots are open-loop kinematic chains with only rotary joints. A 6-DOF robot will have 6 such joints, all actuated. The articular coordinates are the 6 rotation angles θ_i of the joints and the generalized cartesian coordinates are the 3 coordinates of the center of the end-effector together with the three angles describing the orientation of the end-effector. If the θ_i angles are known it is easy to calculate the generalized cartesian coordinates but the opposite problem is far more complex and it is only recently that the final result has been established. First it has been shown that by combining the algebraic equations from the direct kinematic it is possible to obtain a 16th-order polynomial in one unknown [12] which can be solved numerically and in some cases leads to 16 real roots. Reducing a system of algebraic equations to a polynomial in one unknown is a common practice in kinematics. This method has however drawbacks as a complex polynomial is obtained from an initial set of usually simple equations. Consequently, some numerical problems may arise when computing the coefficient of the final polynomial. Another drawback is that no-closed form solution will be obtained. An alternative has been proposed by Wampler which used the continuation method on the original set of equations to get all the solutions [14].

3.3 Parallel robots

3.3.1 Planar mechanisms

The most difficult problems in the field of mechanism theory appear when dealing with closed-loop mechanism. However new results have been obtained recently. These advances will be illustrated by examples dealing with a special kind of robot: the *parallel manipulator*. Figure 3 shows an example of planar parallel manipulator, called the $3-RPR$ manipulator, as the end-effector is linked to the the ground by three identical chains consisting of a rotary joint (R) connected to the ground, followed by an active prismatic joint (P) which is connected to the platform by a rotary joint (R). By changing the length of the three prismatic actuators it is possible to control

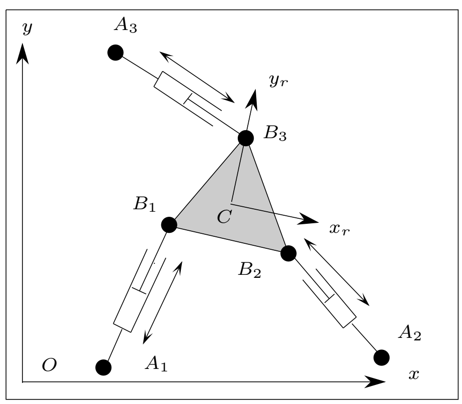

Figure 3: The $3-RPR$ parallel manipulator.The end-effector is the gray triangle.

the position and orientation of the end-effector. Consequently, this closed-loop mechanism is a three DOF robot. Consider the inverse kinematics problem for this mechanism, i.e. find the link lengths (the distance between the points A_i, B_i) for a given position/orientation of the end-effector. As the end-effector location is known it is easy to determine the coordinates of the B_i points in the reference frame. By construction the location of the A_i are known. Consequently, computing the norm of the vector A_iB_i is straightforward. Solving the direct kinematic is far more complex and basically the problem can be reduced to solving a set of three algebraic equations in three unknowns. First of all the solution may be not unique. The maximum number of solutions can be established in the following manner. Suppose we disconnect the mechanism at one joint (say B_3). We get therefore two mechanisms: a four-bar mechanism ($A_1B_1B_3^1B_2A_2$) and a circular mechanism ($A_3B_3^2$). A solution of the direct kinematic is obtained when the points B_3^1, B_3^2 are at the same location. Consequently, the solutions are the *real* intersection points of the coupler curve of the four-bar mechanism (i.e. a sextic) and a circle. Using Bezout's theorem we deduce that there is no more than 12 intersection points. But remind that the circularity of the coupler curve is 3 and the circularity of a circle is 1. Consequently, among the 12 intersection points, six are the circular imaginary points and therefore the number of real intersection points is no more than 6.

The system of equations of the inverse kinematic can be combined to obtain a 6th-order polynomial in one of the unknown [3] and examples with 6 real solutions have been found. Figure 4 shows an example of such a configuration: the coupler curve intersects a circle in 6 different points. A particular example illustrates the

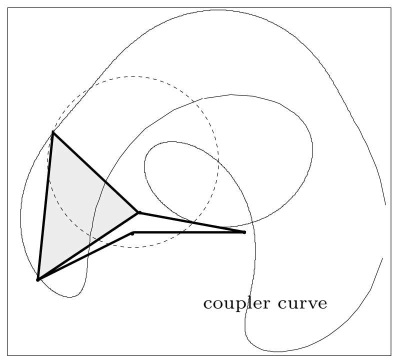

Figure 4: The coupler curve (in thin line) intersects a circle (in dashed line) in 6 different locations.

influence of algebraic geometry in kinematics. Assume that the (A_1, A_2, A_3), (B_1, B_2, B_3) points lie on two lines. For this special case the degree of the direct kinematics polynomial is reduced to 3 but each root leads to two positions for the end-effector, thus there is also 6 potential solutions. But Sturm's method has enabled us to show that the polynomial has at most 2 real solutions and consequently the direct kinematic has at most 4 solutions [2].

3.3.2 Spatial mechanisms

The concept of parallel robots can be extended to build 6-DOF robots. Figure 5 shows an example of such a manipulator. The end-effector is a triangular platform which is connected to the ground by 6 legs. At each extremity of one leg there is a ball-and-socket joint and in each leg a prismatic actuator enables to change the leg length. Similarly to the planar robot by changing the leg lengths we can control the position/orientation of the platform. The inverse kinematic is straightforward and leads to a set of 6 algebraic equations, but the direct kinematic is much more complex. It is however possible to show that there will be at most 16 solutions [8]. The

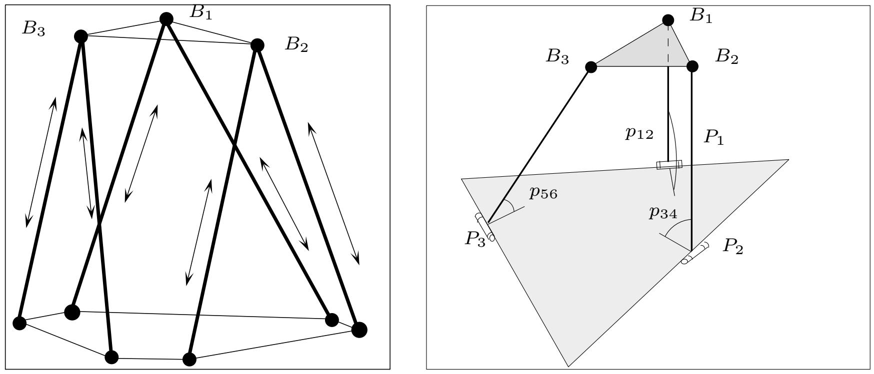

Figure 5: On left a 6-DOF parallel robot.

basic idea of the demonstration is first to notice that each of the B_i is connected to two legs and consequently is only able to move on a circle. The mechanism can therefore be replaced by a new mechanism with only three legs, attached to the platform by a ball-and socket joint and to the base by a rotary joint (figure 5, on right). Then we disconnect one of the leg at one of the B_i points (say B_3) and get two mechanisms: a circular mechanism and a mechanism called the $RSSR$ mechanism. Using one of Cayley's theorems [5] it is possible to show that the coupler surface of the $RSSR$ (i.e. the surface described by B_3) is of order 16 with a circularity of 8. As the solutions of the direct kinematics are the intersection points of the coupler surface and the circle there cannot be more than 16 solutions. This claim has been verified as it is possible to deduce from the inverse kinematic equations a 16th-order polynomial in one variable. This polynomial may have 16 real solutions [8].

3.3.3 Singular configurations

We have seen that for parallel manipulator determining the leg lengths ρ when the cartesian coordinates $\mathbf{X}$ are known is straightforward. Let

$$\rho = F(\mathbf{X})$$

From this relation it is easy to deduce the linear relation between the articular velocities $\dot{\rho}$ and the cartesian velocities $\dot{\mathbf{X}}$.

$$\dot{\rho} = J^{-1}(\mathbf{X})\dot{\mathbf{X}}$$

where $J^{-1} = ((\partial F/\partial X))$ is a 6x6 matrix called the *inverse jacobian matrix* of the robot. For a given $\dot{\rho}$ there will be in general an unique $\dot{\mathbf{X}}$ except if J^{-1} is singular. The configurations where J^{-1} is singular are called the *singular configurations* of the robot: the velocity of the platform may be non zero although the articular coordinates do not change and consequently the robot cannot be controlled. The determinant of the inverse jacobian matrix is also used for determining the forces in the legs τ for given forces and torques $\mathbf{F}$ acting on the platform as

$$\tau = J^{-T}\mathbf{F}$$

A consequence is that the articular forces are obtained by dividing some quantity by the determinant of the inverse jacobian matrix. Consequently the forces increase as the robot comes close to a singular configuration which may cause a breakdown. For finding this configurations we may try to compute the determinant of J^{-1}. But the expression of this determinant is huge and finding its roots is a difficult task. Another approach is to notice that the rows of J^{-1} are the Plücker vectors of the lines associated to the legs. If J^{-1} is singular then we have a linear dependency between the lines, which can occur only for some special geometric configurations of the lines as stated by Grassmann geometry. The problem is thus reduced to find the cartesian coordinates such that these special configurations occur and this leads to rather simple conditions [10].

3.4 Workspace of parallel manipulators

Consider a 6-DOF parallel manipulator. The position/orientation of its end-effector is therefore described by a set of 6 parameters: the coordinates of its center and three angles describing its orientation. In practice there is always a limitation on the leg lengths and mechanical limits on the passive joints. Furthermore some positions of the end-effector cannot be reached as they will lead to an intersection between some legs. Consequently the position which can be reached by the center of the end-effector, the *workspace* of the robot, is a limited region of the physical space. But as we have to deal with a region in a 6-parameters spaces there is no human readable representation of the full workspace. However it is usually admitted to represent a restriction of the full workspace by assigning some fixed values to some of the parameters. For example it is usual to fix the orientation of the end-effector and one of the coordinates of the center of the end-effector (for example the x, y coordinates of the center remain free). The reduced problem is now to compute the border of planar cross-sections of the workspace. This problem can be stated as an algebraic problem as we have to deal with algebraic inequalities in the unknowns. For example the leg lengths can be expressed as polynomials in x, y which have to verify the limitation on the lengths. But the geometrical meaning of these polynomials is used to simplify the problem and leads to compute the intersection of simple geometric objects: circles for the constraints on the leg length, polygons for the mechanical limits on the joints and conics when dealing with the intersection of the legs [8]. Figure 6 shows an example of such cross-sections.

4 Open problems

4.1 Intersection of two coupler curves

The *maximal workspace* of a robot is the region which can be reached by the center of the end-effector with at least one orientation. Consider a planar $3-RPR$ parallel robot with some limitations on the leg lengths and assume that the center of the end-effector is one of its vertice (say B_3). It is easy to show that a location of B_3 will belong to the border of the maximal workspace if at least one of the leg length has an extremal value. Thus the border of the maximal workspace is the intersection of the trajectory obtained when one or two actuators have a maximal value. For only one actuator the extremal location of B_3 are circles and when two have an extremal values the trajectory of B_3 is the coupler curve of the four bar mechanism $A_1B_1B_3B_2A_2$. Therefore to compute the border we have to calculate the intersection of the coupler curves of the four bar mechanisms whose link lengths are the combination of the extremal leg lengths.

Innocenti has shown that in general two coupler curves have at most 18 intersection points. He has then exhibited a 18x18 matrix whose determinant is a polynomial of 18th order and enables to compute the solution

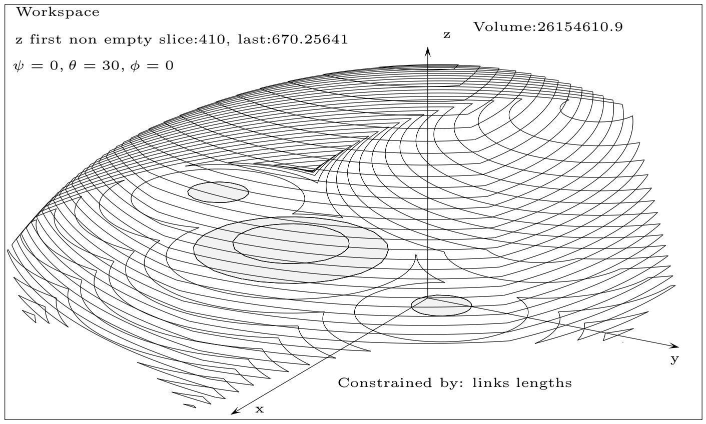

Figure 6: A 3D view of the workspace of a parallel manipulator for a fixed orientation of the end-effector.

of the intersection problem [6]. Unfortunately Innocenti was not able to obtain this polynomial although the matrix is spare. Instead he takes 19 particular values for the unknowns, compute numerically the determinant and get a linear system of 19 equations in the coefficients of the polynomial. But this method is not robust and is computer intensive. For the maximal workspace problem the four bar mechanisms have the same connection points to the ground and the same coupler but differ by the lengths of their links (figure 7). In that particular case it can be shown that there cannot be more than 12 intersection points but finding an algorithm for computing the intersection is still an open problem.

4.2 Maximal workspace of a spatial parallel robot

The concept of maximal workspace can be extended to spatial parallel manipulators. It is assumed that the leg lengths are constrained to lie in some given intervals and we want to determine all the possible position of the center of the end-effector which can be reached with at least one orientation. At this time no method has been proposed to solve this problem. The even simpler problem *is a given location of the center of the end-effector inside the maximal workspace?* has no known solution. Basically this problem is to show that it exists three unknowns (the three orientation angles) which satisfy a set of 12 inequalities (the inverse kinematic equations).

4.3 Direct kinematics of a general parallel manipulator

A general parallel manipulator is a mechanism with two bodies connected by 6 legs, whose attachment points are in a general position on the bodies. It has been recently shown that the direct kinematic problem of a general parallel manipulator will have at most 40 solutions (complex and real) [13],[7],[11]. But no practical algorithm is known to determine these solutions. What is the maximum number of real solutions? In the case where the base points and platform points lie in two planes an algorithm is known [9] but no configuration with more than 16 real solutions have been found.

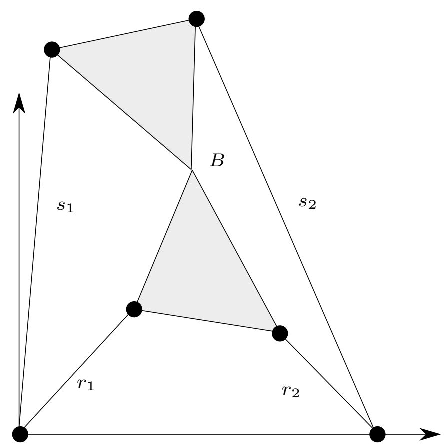

Figure 7: For determining the maximal workspace of a planar parallel manipulator it is necessary to compute the intersection of the coupler curves of two four bar mechanisms.

Another problem can be stated as follows. Consider a parallel manipulator which has been mounted in some initial configuration. Assume now that an algorithm exists for determining all the solutions of the direct kinematic problem. Among the set of solutions is there one unique solution which can be reached from the initial configuration without dismantling the manipulator or crossing a singular configuration?

4.4 Intersection of a cycloid and a circle

Not all problems in mechanism theory are algebraic. Here is an example of open problem which involves a non-algebraic curve. Assume that a planar $3-RPR$ parallel manipulator has to move from an initial configuration M_1 to a final configuration M_2, the intermediate configurations being obtained by linear interpolation. As the center of the platform translates along the segment M_1M_2 the platform will rotate around it and consequently the vertices B_i of the platform will lie on an arc of cycloid. Assume now that we have some restriction on the leg lengths of the robot i.e. the lengths ρ has to belong to the interval $[\rho_{min}, \rho_{max}]$. Consequently each B_i has to lie in an annular region with center A_i and radii ρ_{min}, ρ_{max}. Assume now that we want to verify that the trajectory of the robot is feasible i.e. there is no violation of the leg lengths constraints on the whole trajectory. We have thus to check that the arcs of cycloid lie fully inside the annular regions (figure 8). As we may assume that the initial and final point are valid position we have therefore to check is there is no intersection between the arcs of cycloid and the circles.

5 Conclusion

Algebraic geometry plays an essential role in mechanism theory as most of the equations used to describe the geometric model of a mechanism are algebraic. The main problems that have to be solved are:

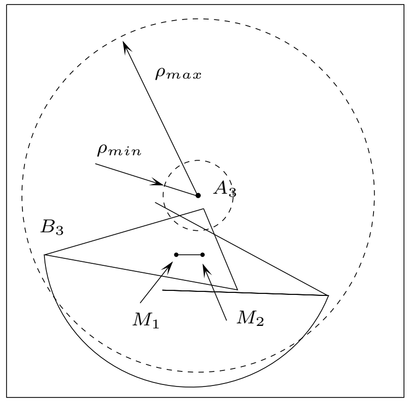

Figure 8: When the platform moves on the segment M_1M_2 point B_3 describe an arc of cycloid (in thin line). If the trajectory is feasible this arc should lie inside an annular region centered in A_3 (in dashed line). Here the trajectory is not feasible.

- find the number of real solutions and the solutions of a system of algebraic equations.
- find if there are some unknowns which satisfy a system of algebraic inequalities (although the values of the unknowns need not to be necessarily determined).
- being given a set of inequalities with n unknowns find the border of the region of the n parameters space for which the system of inequalities is satisfied.

An advantage of mechanism theory is that the algebraic equations have in general a geometric meaning which may often help to solve the problem. But it remains that the equations which are used in this field are often huge, even for the simplest mechanism, and consequently constitute an important challenge for algebraic geometry with a potentially huge field of applications.

References

[1] Freudenstein F. On the variety of motion generated by mechanisms. *Transaction of the ASME*, pages 156–160, February 1962.

[2] Gosselin C. and Merlet J-P. On the direct kinematics of planar parallel manipulators: special architectures and number of solutions. *Mechanism and Machine Theory*, 29(8):1083–1097, November 1994.

[3] Gosselin C., Sefrioui J., and Richard M.J. Solution polynomiale au problème de la cinématique directe des manipulateurs parallèles plans à 3 degrés de liberté. *Mechanism and Machine Theory*, 27(2):107–119, March 1992.

[4] Hartenberg R.S. Denavit J. *Kinematic synthesis of linkages*. McGraw-Hill, New-York, 1964.

[5] Hunt K.H. *Kinematic geometry of mechanisms*. Clarendon Press, Oxford, 1978.

[6] Innocenti C. Analytical determination of the intersection of two coupler-point curves generated by two four-bar linkages. In J. Angeles P. Kovacs, G. Hommel, editor, *Computational Kinematics*, pages 251–262. Kluwer, 1993.

[7] Lazard D. Generalized Stewart Platform: How to compute with rigid motions? In *IMACS Symp. on Symbolic Computation*, pages 85–88, Lille, June, 14-17, 1993.

[8] Merlet J-P. *Les Robots parallèles*. Hermès, Paris, 1990.

[9] Merlet J-P. An algorithm for the forward kinematics of general parallel manipulators. In *ICAR*, pages 1131–1135, Pise, June, 19-22, 1991.

[10] Merlet J-P. Singular configurations of parallel manipulators and Grassmann geometry. *Int. J. of Robotics Research*, 8(5):45–56, October 1989.

[11] Mourrain B. Enumeration problems in Geometry, Robotics and Vision. In *MEGA*, Santander, April, 5-9, 1994.

[12] Raghavan M. and Roth B. Kinematic analysis of the 6R manipulator of general geometry. In *5th Int. Symp. of Robotics Research*, pages 263–270, Tokyo, , 1990.

[13] Ronga F. and Vust T. Stewart platforms without computer? In *Conf. Real Analytic and Algebraic Geometry*, pages 197–212, Trento, 1992.

[14] Wampler C. and Morgan A. Solving the 6R inverse position problem using a generic-case solution methodology. *Mechanism and Machine Theory*, 26(1):91–106, , 1991.

[15] Wampler C.W., Morgan A.P., and Sommese A.J. Numerical continuation methods for solving polynomial systems arising in kinematics. *ASME J. of Mechanical Design*, 112:59–68, March, , 1990.

[16] Wampler C.W., Morgan A.P., and Sommese A.J. Complete solution of the nine-point path synthesis problem for four-bar linkages. *ASME J. of Mechanical Design*, 114:153–159, September - October , 1988.