Jacobian, manipulability, condition number and accuracy of parallel robots

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Abstract: Although the concepts of jacobian matrix, manipulability and condition number exist since the very early beginning of robotics their real significance is not always well understood. In this paper we re-visit these concepts for parallel robots as accuracy indices in view of optimal design. We first show that the usual jacobian matrix derived from the input-output velocities equations may not be sufficient to analyze the positioning errors of the platform. We then examine the concept of manipulability and show that its classical interpretation is erroneous. We then consider various common local dexterity indices most of which are based on the condition number of the jacobian matrix. It is emphasized that even for a given robot in a particular pose there are a variety of condition numbers and that their values are not coherent between themselves but also with what we may expect from an accuracy index. Global conditioning indices are then examined. Apart of the problem of being based on the local accuracy indices that are questionable, there is a computational problem in their calculation, that is neglected most of the time. Finally we examine what other indices may be used for optimal design and show that their calculation is most challenging.

1 Introduction

We will use a relatively general definition of parallel robots, focusing on non-redundant mechanisms. A parallel robot is defined as a mechanism having at least two kinematics chains connecting the base to the end-effector, the purpose being to control $n \leq 6$ d.o.f. of the end-effector while the other $6 - n$ d.o.f. have a constant value by using n one d.o.f. actuated joints. Furthermore it will be assumed that when the actuator are locked the mobility of the end-effector is 0 and that the non actuated joints have one d.o.f. Such definition covers the classical 6 d.o.f. robots such as the Gough [17] and Hexa [39] platforms but also robot with less than 6 d.o.f. such as the Delta [9] or 3-UPU [47] robots.

Parallel robots are nowadays leaving academic laboratories and are finding their way in an increasingly larger number of application fields such as telescopes, fine positioning devices, fast packaging, machine-tool, medical application. A key issue for such use is optimal design as performances of parallel robots are very sensitive to their dimensioning. Optimal design methodologies have to rely on kinetostatic performance indices and accuracy is clearly a key-issue for many applications [19]. It has also been a key-issue for serial robots and consequently this problem has been extensively studied and various accuracy indices have been defined. The results have been in general directly transposed to parallel robots. The purpose of this paper is to review how well these indices are appropriate for parallel robots. For that purpose it is necessary to examine the concept of jacobian and inverse jacobian matrices that are essential when examining the positioning accuracy of the end-effector.

2 Jacobian and inverse Jacobian matrix

Let $\mathbf{X_a}$ denotes the generalized coordinates of the end-effector composed of parameters describing the desired n d.o.f. of the end-effector while $\mathbf{X}$ denotes all the generalized coordinates of the end-effector i.e. a set of parameters that allow to describe completely its pose (translation and orientation). We will impose no constraints on the choice of the parameters in $\mathbf{X}$ (e.g. for a Gough robot with a planar platform the pose may be represented by the 9 coordinates of 3 particular non-aligned points on the end-effector).

The twist $\mathbf{W}$ of the end effector is composed of its translational and angular velocities $\mathbf{V}, \mathbf{\Omega}$ and the *restricted twist* $\mathbf{W_a}$ is defined as the restriction of $\mathbf{W}$ to the n desired d.o.f. of the robot. It is well known that for robot having at least 2 rotational d.o.f. $\mathbf{W}$ is not the time-derivative of $\mathbf{X}$ as there is no representation of the orientation whose derivatives corresponds to the angular velocities. However there exists usually matrices H, K such that

$$\mathbf{W} = \mathrm{H}\dot{\mathbf{X}} \qquad \dot{\mathbf{X}} = \mathrm{K}\mathbf{W} \tag{1}$$

The internal geometry of the robot may be described by a set of parameters that describe the motion of some or all of the joints, including the passive, non actuated, one. These parameters are regrouped in the joint variables vector $\mathbf{\Theta}$.

The usual definition of the jacobian matrix $\mathrm{J_k}$ involves a joint variable vector $\mathbf{\Theta_a}$ that is restricted to the actuated joints and is based on the linear relation between the actuated joint velocities $\dot{\mathbf{\Theta}}_\mathbf{a}$ and the restricted twist $\mathbf{W_a}$:

$$\mathbf{W_a} = \mathrm{J_k}\dot{\mathbf{\Theta}}_\mathbf{a} \tag{2}$$

As in this paper we consider only non-redundant robots the matrix $\mathrm{J_k}$ is square and we will call it the *kinematic jacobian*. A feature of parallel robots is that it is usually easy to establish an analytical form for $\mathrm{J_k^{-1}}$ while it is often impossible (or at least very difficult) to obtain $\mathrm{J_k}$. To calculate the inverse kinematic jacobian we may use a velocity analysis but as mentioned by Gosselin [15] it is also possible to use the kinematic closure-loop equations whose general form is:

$$\mathbf{E}(\mathbf{X_a}, \mathbf{\Theta_a}) = \mathbf{0} \tag{3}$$

As we have assumed that the robot is non redundant and its mobility is 0 when the actuators are locked there must be exactly n such equations. By differentiating the system we get

$$\frac{\partial \mathbf{E}}{\partial \mathbf{\Theta_a}}\dot{\mathbf{\Theta}}_\mathbf{a} + \frac{\partial \mathbf{E}}{\partial \mathbf{X_a}}\dot{\mathbf{X}}_\mathbf{a} = \mathrm{U_k}\dot{\mathbf{\Theta}}_\mathbf{a} + \mathrm{V_a}\dot{\mathbf{X}}_\mathbf{a} = \mathbf{0} \tag{4}$$

Using the restriction of equation (1) to $\mathbf{W_a}$ and provided that $\mathrm{U_k}$ is not singular, we get

$$\mathrm{J_k^{-1}} = -\mathrm{U_k^{-1}}\mathrm{V_a}\mathrm{K_a}\mathbf{W_a} \tag{5}$$

The singularity problem of parallel robots has been initially discussed by Gosselin and Angeles [16] and they distinguish *serial singularity (or type 1)* obtained when $\mathrm{U_k}$ is singular and *parallel singularity (or type 2)* when $\mathrm{V_a}$ is singular. According to this definition in a parallel singularity the end-effector may move while the actuators are locked, a typical case of positioning errors as the measurements of the actuated joints variables do not change although there is a displacement of the end-effector. Hence accuracy analysis is also connected to singularity analysis.

But we may also define other jacobian matrices. We may include parameters in $\mathbf{\Theta}$ that describe the motion of passive joints, which are usually numerous in parallel robots. Hence $\mathbf{\Theta}$ may be defined as a N-dimensional vector $(\mathbf{\Theta_a}, \mathbf{\Theta_p})$ where $\mathbf{\Theta_p}$ correspond to the parameters of the passive joints and N being the total number of joints, actuated or not, of the robot.

Although some robot are designed on purpose to have $n < 6$ controllable d.o.f. while the other d.o.f. are supposed to have a constant value, still the end-effector is a 6 d.o.f. rigid body and positioning errors on *all* d.o.f. should be examined when looking for an optimal design. It is thus interesting to determine an inverse jacobian that involves the full twist $\mathbf{W}$ of the end-effector.

To determine this inverse jacobian matrix we write the kinematics closure-loop equations as

$$\mathbf{G}(\mathbf{\Theta}, \mathbf{X}) = \mathbf{0} \tag{6}$$

The total number of unknowns in these equations is $N + n$. As we have assumed that the robot mobility is 0 when the n actuators are locked the system of equation $\mathbf{G}$ must have a finite number of solutions in that configuration i.e. the number of equations in $\mathbf{G}$ must be equal to the number of unknowns, which is N.

By differentiating equation (6) we get:

$$\frac{\partial \mathbf{G}}{\partial \mathbf{\Theta}}\dot{\mathbf{\Theta}} + \frac{\partial \mathbf{G}}{\partial \mathbf{X}}\dot{\mathbf{X}} = \mathrm{A}\dot{\mathbf{\Theta}}_{\boldsymbol{a}} + \mathrm{B}\dot{\mathbf{\Theta}}_{\boldsymbol{p}} + \mathrm{C}\dot{\mathbf{X}} = \mathbf{0} \tag{7}$$

where A is a $N \times n$ matrix, B is $N \times (N - n)$ while matrix C is $N \times 6$. Zlatanov [58] has derived a similar expression with the difference that he is using the restricted twist $\mathbf{W_a}$ with the assumption that $\mathbf{W_a}$ may be obtained as $\mathbf{W_a} = \mathrm{T}\mathbf{\Theta}$. He proposes to write equation (7) as:

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{CK} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{\Theta}}_{\boldsymbol{a}} \\ \dot{\mathbf{\Theta}}_{\boldsymbol{p}} \\ \mathbf{W} \end{pmatrix} = \mathrm{L} \begin{pmatrix} \dot{\mathbf{\Theta}}_{\boldsymbol{a}} \\ \dot{\mathbf{\Theta}}_{\boldsymbol{p}} \\ \mathbf{W} \end{pmatrix} = \mathbf{0} \tag{8}$$

where L is $N \times (N + 6)$. This may also be written as

$$-\begin{pmatrix} \mathrm{A} & \mathrm{B} \end{pmatrix} \dot{\mathbf{\Theta}} = \mathrm{D}\dot{\mathbf{\Theta}} = \mathrm{CK}\mathbf{W} \tag{9}$$

where D is $N \times N$. Provided that D is not singular we may now derive an inverse jacobian such that

$$\dot{\mathbf{\Theta}} = -\mathrm{D}^{-1}\mathrm{CK}\mathbf{W} = \mathrm{J}^{-1}\mathbf{W} \tag{10}$$

where J^{-1} is $N \times 6$.

In most cases however a velocity analysis allows to eliminate the passive joint velocities $\dot{\mathbf{\Theta}}_{\boldsymbol{p}}$ in (10) to obtain a simpler inverse jacobian matrix through a relation that relates only $\dot{\mathbf{\Theta}}_{\boldsymbol{a}}$ to $\mathbf{W}$:

$$\begin{pmatrix} \dot{\mathbf{\Theta}}_{\boldsymbol{a}} \\ \mathbf{O} \end{pmatrix} = \mathrm{J}_{\mathrm{fk}}^{-1}\mathbf{W} \tag{11}$$

where $\mathrm{J}_{\mathrm{fk}}^{-1}$ is $N \times 6$ and will be called the *full inverse kinematics jacobian*. The usual inverse jacobian matrix is the restriction of $\mathrm{J}_{\mathrm{fk}}^{-1}$ to $\mathbf{W_a}$. An important property of the inverse jacobian J^{-1} of (10) is that it has the same rank than the full inverse kinematic jacobian $\mathrm{J}_{\mathrm{fk}}^{-1}$ whatever is the pose of the end effector.

Zlatanov determine 6 various cases, that he calls *singularity*, in which equation (8) has not a generic behavior. As we are focusing on the motion of the end-effector we will just mention the cases that may involve such motion:

- *redundant output (RO)*: there are $\mathbf{W} \neq \mathbf{0}, \mathbf{\Theta}_{\boldsymbol{p}}$ such that (8) is satisfied with $\mathbf{\Theta}_{\boldsymbol{a}} = \mathbf{0}$. In other words the end-effector is moving while the actuators are locked (which the usual definition of a singularity) and the full inverse kinematic jacobian will not have full rank

- *increased instantaneous mobility (IIM)*: when the rank of L is lower than N

It must be noted that equation(10) describes an intrinsic property of the robot. We may change the pose parameters vector $\mathbf{X}$ (e.g. for a Gough robot with a planar platform by choosing as elements of $\mathbf{X}$ the 9 coordinates of 3 particular non-aligned points on the end-effector) but will get the same equation.

We may further extend equation (7) to take into account the geometrical parameters $\mathcal{P}$ of the robot (e.g. the location of the anchor points of the legs in a Gough platform). For that purpose the kinematics equations will be written as $\mathbf{G}(\mathcal{P}, \mathbf{\Theta}, \mathbf{X}) = 0$ and the matrix of the partial derivatives of $\mathbf{G}$ with respect to $\mathcal{P}$ will allow one to quantify the influence of the errors on $\mathcal{P}$ (due, for example, to manufacturing tolerances) on the positioning errors of the end-effector. We will not address this issue although this influence may be important for parallel robots [10, 35, 30, 33, 45, 50, 51] but may partly be decreased by using calibration.

As may be seen there is not a single inverse jacobian matrix but a multiplicity of them. It is also important to note that elements of the inverse jacobian matrix involving the full twist $\mathbf{W}$ of the end-effector will usually not be homogeneous in terms of units. Hence many properties of this matrix such as its determinant, trace, ... will not be invariant under a change of units (see [42] for a discussion on the invariance of dexterity indices). This will also be the case of the inverse kinematic jacobian for robot involving both translation and rotational d.o.f. for the end-effector. Finally it must be noted that by duality the inverse kinematic jacobian is also involved, through its transpose, in the static analysis of parallel robots i.e. the relation between the forces/torques in the joints and wrench on the platform. Hence accuracy indices will be deeply connected to force transmission indices [8, 26].

In this paper we are interested in the possible motion of the end-effector that cannot be detected under measurement of the parameters in $\mathbf{\Theta_a}$. This may occur in two different cases:

- unmeasured motion of the active joints: this corresponds to a limitation $\mathbf{\Delta\Theta_a}$ in the accuracy of the measurement
- singularity classified by Zlatanov as redundant output (RO) or increased instantaneous mobility (IIM)

To study both cases it is necessary to use the full inverse kinematic jacobian and not only the inverse kinematic jacobian as emphasized in the example presented in the next section.

2.1 Example: the $3-U\underline{P}\,U$ robot

Tsai [47] has proposed the $3-U\underline{P}\,U$ robot as a 3 d.o.f. translation robot (figure 1). Each leg of this robot is constituted, starting from the base, by a U joint followed by an extensible leg, whose length may be modified by a prismatic actuator, and which is terminated by another U joint *whose axis are the same than the U joint on the base.* This constraint allows theoretically to obtain only translation for the end-effector. This example will allow us to establish a methodology for determining the full inverse kinematic jacobian and it will also enable to show the importance of this matrix on a practical example. Indeed such a robot was designed at Seoul National University (SNU) and was exhibiting a strange behavior in the poses where the legs have an identical length: although the prismatic actuators were locked, the end-effector was exhibiting significant orientation motion. Two possible reasons were considered for explaining this phenomena:

- *the location of the end-effector was very sensitive to manufacturing tolerances.* Indeed this robot has only *in theory* 3 translational d.o.f., provided a perfect alignment of the U axis on the base and platform. In practice such alignment cannot be realized, which may lead to rotational d.o.f. The effect of manufacturing

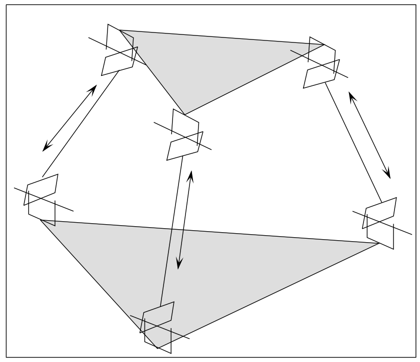

Figure 1: The $3-U\underline{P}U$ robot

tolerances on the positioning errors has been studied by Han and Parenti-Castelli [18, 34] and although it was shown that the robot was indeed very sensitive, the large rotational motion observed on the SNU prototype cannot be explained only by this sensitivity

- *the robot was in a singular pose*: this was a-priori discarded as the determinant of the inverse kinematic jacobian was not 0 in any of the incriminated poses

This phenomena was first explained by Bonev and Zlatanov [4] as a *constraint singularity* and later by DiGregorio, Joshi and Wolf [11, 24, 53] and it has now become a classical example of singularity that cannot be explained only by looking at the input-output velocities equations i.e. by using only the inverse kinematic jacobian.

To start with, we will determine the analytical form of the full inverse kinematic jacobian. We will denote by B_1, B_2, B_3 the center of the U joints on the platform and by $\mathbf{V}, \mathbf{\Omega}$ the translational and angular velocities of the end-effector. The velocity $\mathbf{V_{B_i}}$ of point B_i is

$$\mathbf{V_{B_i}} = \mathbf{V} + \mathbf{B_iC} \times \mathbf{\Omega} \tag{12}$$

Let us define $\mathbf{n_i}$ as the unit vector of the leg i, ρ_i the length of this leg and $\dot{\rho}_i$ the prismatic actuator velocity. We may compute the dot product of the right and left terms of the previous equation:

$$\mathbf{V_{B_i}}.\mathbf{n_i} = \dot{\rho}_i \mathbf{n_i} = \mathbf{V}.\mathbf{n_i} + (\mathbf{B_iC} \times \mathbf{\Omega}).\mathbf{n_i} = \mathbf{V}.\mathbf{n_i} + (\mathbf{CB_i} \times \mathbf{n_i}).\mathbf{\Omega} \tag{13}$$

Now let us define $\mathbf{u_i}, \mathbf{v_i}$ the unit vectors of the two joint axis of the U joint at B_i, these vectors being the same for the base and platform. The angular velocity of the leg $\boldsymbol{\omega_l}$ with respect to the base and the angular velocity of the platform $\boldsymbol{\omega_p}$ with respect to the leg are

$$\boldsymbol{\omega_l} = \dot{\theta}^i_A \mathbf{u_i} + \dot{\alpha}^i_A \mathbf{v_i} \qquad \boldsymbol{\omega_p} = \dot{\theta}^i_B \mathbf{u_i} + \dot{\alpha}^i_B \mathbf{v_i}$$

The angular velocity of the platform is obtained as

$$\mathbf{\Omega} = \boldsymbol{\omega_l} + \boldsymbol{\omega_p} = (\dot{\theta}^i_A + \dot{\theta}^i_B)\mathbf{u_i} + (\dot{\alpha}^i_A + \dot{\alpha}^i_B)\mathbf{v_i}$$

Now define $\mathbf{s}_i = \mathbf{u_i} \times \mathbf{v_i}$ and compute the dot product of the right and left terms of the previous equation by $\mathbf{s_i}$. We get the constraint equation:

$$\mathbf{s_i}.\mathbf{\Omega} = 0 \tag{14}$$

that states that the end-effector cannot rotate around a line going through B_i, whose unit vector is $\mathbf{s_i}$. Combining equations (13, 14) we get the full velocities equations involving the twist $\mathbf{W}$ as

$$\begin{pmatrix} \dot{\rho}_i \\ \mathbf{0} \end{pmatrix} = \mathrm{J}_{\mathrm{fk}}^{-1}\mathbf{W} = \begin{pmatrix} \mathbf{n_i} & (\mathbf{CB_i} \times \mathbf{n_i}) \\ \mathbf{0} & \mathbf{s_i} \end{pmatrix} \mathbf{W} \tag{15}$$

which establish the full inverse kinematic jacobian. The inverse kinematic jacobian may be extracted from $\mathrm{J}_{\mathrm{fk}}^{-1}$ as the 3×3 matrix whose rows are the $\mathbf{n_i}$ vectors.

Important information may be gained from this matrix. First of all the rows of the matrix are Plücker vectors that describe lines in space. The 3 first rows are the Plücker vectors of the line associated to the legs of the robot while the 3 last rows correspond to lines at infinity. Hence the singularity analysis approach based on line geometry may be applied although the robot has less than 6 d.o.f. As a matter of fact it appears that for all known parallel robots, the rows of the full inverse kinematics jacobian will always be Plücker vector, although to the best of my knowledge no formal proof has been given that this will always be the case. As far as accuracy analysis is concerned it is important to consider the lower part of $\mathrm{J}_{\mathrm{fk}}^{-1}$. They represent the lines at infinity that are lying in any plane perpendicular to $\mathbf{s_i}$. According to line geometry the three lines will be dependent if the vectors $\mathbf{s_i}$ are lying in the same plane or are parallel i.e. if $\mathbf{s_1}.(\mathbf{s_2} \times \mathbf{s_3}) = 0$. In that case the platform with locked actuators will exhibit orientation motion that may be infinitesimal or finite according to the geometry of the robot [4]. It happens that the design of the SNU robot was such that, with actuators of the same length, the axis of the U joints were all horizontal, leading to parallel vertical $\mathbf{s_i}$ and therefore to a rotational singularity.

There are many different types of 3-UPU robots (for example it is possible to exhibit a 3-UPU that has only rotational motion) and Zlatanov [57] has presented an extensive analysis of the singularity of these robots.

This example has shown that accuracy analysis cannot be decoupled from singularity analysis and that is it always necessary to consider the full inverse kinematic jacobian.

3 Manipulability

Now that we are on a safe ground as far as the inverse jacobian is concerned. let us consider another classical concept: the kinematic *manipulability* whose purpose is to help quantify the manipulator's velocity transmission capabilities or, equivalently, the dexterity of the robot.

For a given robot it is realistic to assume that the joint measurement errors are bounded and consequently so will be the positioning errors. The norm of the bound may be chosen arbitrary as (11) is linear, so that a simple scaling will allow to determine the positioning errors from the errors obtained for a given bound on the joint measurement errors. A value of 1 for this bound is usually chosen so that:

$$||\mathbf{\Delta\Theta}|| \leq 1 \tag{16}$$

which leads to

$$\mathbf{\Delta X}^T \mathrm{J}^{-\mathrm{T}} \mathrm{J}^{-1} \mathbf{\Delta X} \leq 1 \tag{17}$$

If the Euclidean norm is used (16) represents a circle in the joint errors space. This circle is mapped through matrix $J^{-T}J^{-1}$ into an ellipse in the generalized coordinates error space. More generally, the mapping transform the hyper-sphere in the joint errors space into an ellipsoid, usually called the *manipulability ellipsoid*, in the generalized coordinates error space. A classical geometrical interpretation of this relation, that may be found in many text books, is presented for the 2D case in figure 2. It is usually claimed that the size and shape of

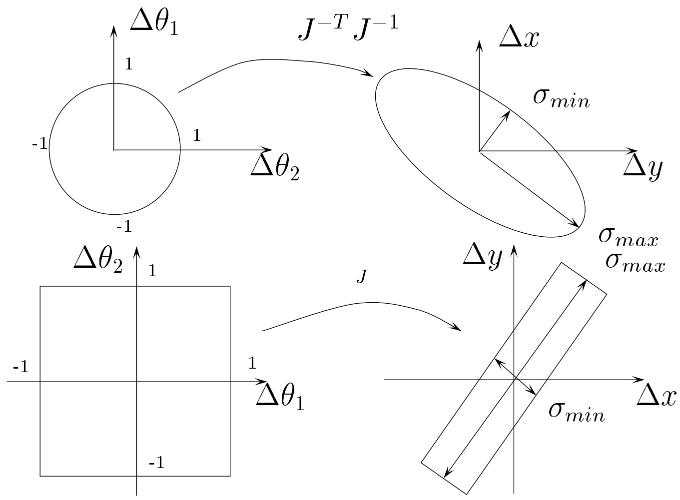

Figure 2: The mapping between the joint errors space and the generalized coordinates error space induced by $J^{-T}J$ according to the norm: on top the Euclidean norm and on bottom the infinity norm.

the ellipsoid are indices of the "amplification" between the joint space errors and the generalized coordinates errors. More precisely the lengths of the principal axis of the ellipsoid, which correspond to the minimal and maximal eigenvalues $\sigma_{min}, \sigma_{max}$ of $J^{-T}J^{-1}$, are considered as an image of the minimum and maximum velocity amplification factor. The closer these lengths are the closer is the manipulability ellipsoid to a circle. In order to evaluate this closeness Yoshikawa [54] introduces a manipulability index for serial robot as

$$m(\mathrm{J}) = \sqrt{|\mathrm{J}\mathrm{J}^T|} \tag{18}$$

which correspond to the product of the half axis lengths of the ellipsoid.

In fact the use of the Euclidean norm in (16) is not realistic: it implies for example that if one of the joint measurement error is 1, then by some mysterious influence all the other joint errors become exactly 0. The appropriate norm is the infinity norm that states that the absolute value of the joint errors are independently bounded by 1. With this norm equation (16) represents a n-dimensional square in the joint errors space that is mapped into what we will call the *manipulability polyhedron*, that includes the manipulability ellipsoid, in the generalized coordinates errors space and figure 2 illustrates this mapping in the 2D case. Using the manipulability ellipsoid it is possible to believe that there are perfect mappings having a constant amplification factor over the whole workspace. This is no more the case with the manipulability polyhedron.

It must be noted that, apart of being more realistic, the mapping with the infinity norm leads to geometrical object that can be more easily manipulated than the ellipsoid. For example assume that one want to determine what are all the possible end-effector velocities that can be obtained in 2 different poses of the end-effector. For that purpose we will have to calculate the intersection of the 2 polyhedra obtained for the 2 poses, a well known problem of computational geometry, that can be much more easily solved than computing the intersection of 2 ellipsoids. Finally note that the manipulability index of Yoshikawa characterizes the volume of the manipulability polyhedron.

But major drawbacks of the manipulability concept are that it mixes arbitrary translational and rotational capabilities and that it is usually not invariant with respect to the choice of the units. As a consequence it has been proposed to split the jacobian into its translational and rotational parts and to calculate the manipulability index for each of them. But this not satisfactory for estimating the amplification factor for motion involving both translation and rotation.

4 Condition number

A large dimension along a given axis of the manipulability polyhedron indicates a large amplification error. It is therefore necessary to quantify this amplification factor. Let us consider the linear system:

$$\mathrm{J}^{-1}\Delta\mathbf{X} = \Delta\mathbf{\Theta} \ ,$$

where J^{-1} is a $n \times n$ inverse kinematic jacobian matrix. A possible error amplification factor for this system expresses how a **relative** error in $\mathbf{\Theta}$ gets multiplied and leads to a **relative** error in $\mathbf{X}$. It characterizes in some sense the dexterity of the robot and has been proposed as a performance index. We use a norm such that

$$||\mathrm{J}^{-1}\Delta\mathbf{X}|| \leq ||\mathrm{J}^{-1}||||\Delta\mathbf{X}|| \ ,$$

and obtain

$$\frac{||\Delta\mathbf{X}||}{||\mathbf{X}||} \leq ||\mathrm{J}^{-1}||||\mathrm{J}||\frac{||\Delta\mathbf{\Theta}||}{||\mathbf{\Theta}||} \ ;$$

The error amplification factor, called the *condition number* κ, is therefore defined as

$$\kappa(\mathrm{J}) = \kappa(\mathrm{J}^{-1}) = ||\mathrm{J}^{-1}||||\mathrm{J}|| \ .$$

The condition number is thus dependent on the choice of the matrix norm. The most used norms are:

- the 2-norm defined as the square root of the largest eigenvalue of matrix $\mathrm{J}^{-\mathrm{T}}\mathrm{J}^{-1}$: the condition number of J^{-1} is thus the square root of the ratio between the largest and the smallest eigenvalues of $\mathrm{J}^{-\mathrm{T}}\mathrm{J}^{-1}$,

- the Euclidean (or Frobenius) norm defined for the $m \times n$ matrix A by: $||\mathrm{A}|| = \sqrt{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}|a_{ij}|^2}$ or equivalently as $\sqrt{\mathrm{tr}(\mathrm{A}^{\mathrm{T}}\mathrm{A})}$: if λ_i denotes the eigenvalues of $\mathrm{J}^{-\mathrm{T}}\mathrm{J}^{-1}$, then the condition number is the ratio between $\sum\lambda_i^2$ and $\prod\lambda_i$. Note that sometime is also used a weighted Frobenius norm in which $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ is substituted by $\mathrm{A}^{\mathrm{T}}\mathrm{W}\mathrm{A}$, where W is a weight matrix whose purpose is to "normalize" the components of the matrix

For these two norms, the smallest possible value of the condition number is 1. The inverse of the condition number, which has a value in [0,1], is also often used. Whatever is the choice for the norm, a value of 0 indicates that the inverse jacobian matrix is singular. The necessity of specifying the norm and the change that the choice of the norm induces will be illustrated later on in this section. But we may illustrate it on the simple example of a serial Cartesian X-Y table. A possible matrix norm is the infinity norm defined as the maximum row sum, where the row sum is the sum of the magnitudes of the elements in a given row. If we define the reference frame as the one having the same axis direction than the the two prismatic actuators, then the jacobian matrix of the robot is the identity, whose condition number using the infinity norm is 1. Now let's rotate the reference frame by 45 degrees around the vertical axis: the row of the jacobian matrix will be $(\sqrt{2}/2, -\sqrt{2}/2)$, $(\sqrt{2}/2, \sqrt{2}/2)$ and its condition number using the infinity norm is 2. Hence the claim that every matrix norm is equivalent is not exactly true from a kinematic view point.

The condition number has the main advantage of being a single number for describing the overall kinematic behavior of a robot. It is used as an index to describe

- the accuracy/dexterity of a robot [40, 41, 44]
- the closeness of a pose to a singularity [12, 14, 49]. It is in general not possible to define a mathematical distance to a singularity for robots whose d.o.f. is a mix between translation and orientation: hence the use of the condition number is as valid an index than any other one.
- as a performance criteria for optimal design and robots comparison [3, 6, 20, 22, 21, 31, 43, 48, 56]
- as a criteria to determine the useful workspace of a robot [7]

The definition of the condition number makes clear that we cannot calculate its analytical form as a function of the pose parameters except for very simple robot. But robust linear algebra software allows to calculate it numerically for a given pose.

But for robot having both translation and orientation d.o.f. there is a major drawback of the condition number: the matrix involved in its calculation are not homogeneous in terms of units. Hence the value of the condition number for a given robot and pose will change according to the unit choice, while clearly the kinematic accuracy is constant. To deal with this problem Ma and Angeles [29] and Kim [27] suggested to define a *normalized inverse jacobian matrix* by dividing the rotational elements of the matrix by a length such as the length of the links in a nominal position, or the *natural length* defined as that which minimizes the condition number for a given pose. Still the choice of the length remains arbitrary as it just allows to define a correspondence between a rotation and a translation and as mentioned by Park [36] "this arbitrariness is an unavoidable consequence of the geometry of SE(3)".

The condition number is not an intuitive way to measure the accuracy of a robot. Indeed end-users are more interested by the *maximal* positioning errors than by a *relative* value. However a condition number may be an acceptable performance index if:

1. its value is consistent with the maximal positioning errors
2. its calculation over a given workspace is easier than a similar calculation for the maximal positioning errors

We will examine the later point in the next sections and will focus here on the first point. The simplest way to examine the consistency of a condition number is to consider a given robot and a set of poses, a configuration in which it is easy to calculate both the condition number and the maximal positioning errors. We may then rank the poses according to the maximal positioning errors and compare it with the ranking according to the condition number.

For that purpose we have used one of our Gough platform prototype and we have chosen three representative reference poses. They are defined by the coordinates of the center of the platform and the 3 Euler angles as P_1=$x = y = 0, z$=53 cm, $\psi = 0$, $\theta = 0$, $\phi = 0$ (roughly the pose obtained for the mid-stroke of the actuators), P_2=$x = y = 0, z$=53 cm, $\psi = 30°$, $\theta = 0$, $\phi = 0$ (whose orientation is roughly 1/3 of the total possible rotation around the z axis) and P_3=$x = y = 10, z$=53 cm, $\psi = 0$, $\theta = 0$, $\phi = 0$ (close to the border of the translation workspace for this orientation). We have then computed the absolute value of the maximal positioning error at these poses, given in table 1, obtained as the sum of the absolute value of the elements of the rows of the kinematic jacobian, that has been obtained by a numerical inversion of the inverse jacobian. It can be seen in

Pose	ΔX_x	ΔX_y	ΔX_z	ΔX_{θ_x}	ΔX_{θ_y}	ΔX_{θ_z}
P_1	0.1184	0.1268	0.010087	0.1185	0.1184	0.697
P_2	0.1189	0.1274	0.01266	0.1333	0.1429	0.808
P_3	0.123	0.1309	0.0372	0.15	0.1663	0.7208

Table 1: Maximal positioning errors for the 3 reference poses

this table that the positioning errors are significantly larger for P_2 and P_3 compared to P_1. As for P_3 the errors are usually larger compared to P_2 except for the rotation around z. Hence as far as accuracy is concerned the ordering of the poses from the most to the least accurate is P_1, P_2, P_3 and we expect to obtain a similar ordering for the condition numbers.

For the calculation of the condition numbers we have used both the inverse kinematic jacobian matrix and a normalized inverse jacobian matrix $\mathrm{J_n^{-1}}$ obtained by dividing the orientation components of the $\mathrm{J_k^{-1}}$ by 53, which is roughly the value of the legs lengths at pose P_1. The considered accuracy indices will be

- C_d: the determinant of $\mathrm{J_k^{-1}}$
- C_2, C_2^n: the 2-norm condition number of $\mathrm{J_k^{-1}}$, $\mathrm{J_n^{-1}}$
- C_F, C_F^n: the Frobenius-norm condition number of $\mathrm{J_k^{-1}}$, $\mathrm{J_n^{-1}}$
- C_2^3, C_F^3: the 2-norm and Frobenius norm condition number of the inverse jacobian matrix obtained when the inverse kinematics equations are based on the coordinates of 3 points of the end-effector. The chosen points will be all possible triplets in the set B_i. We will provide a drawing that describes the result for all 20 possible combinations in which we will present the ratio $\kappa(P_2)/\kappa(P_1)$ and $\kappa(P_3)/\kappa(P_1)$.
- M_t, M_o: the manipulability index of the restriction of $\mathrm{J_k}$ to its translation, orientation parts

For all condition numbers we expect to have a value that decreases in the order P_1, P_2, P_3. For the determinant we expect an absolute value that decrease in the same order. The computed condition numbers for this very simple test are presented in table 2 (for C_2^3, C_F^3 we provide the minimal and maximal value obtained for all reference points choice) and the relative values of C_2^3, C_F^3 are presented in figure 3.

From this very simple analysis we may deduce interesting results:

	C_d	C_2	C_2^n	C_F	C_F^n	C_2^3	C_F^3	M_t	M_o
P_1	-29.22	75.14	63.9	152.8	70.2	[9.55,55.47]	[258.8,3204.9]	12.65	0.04266
P_2	-24.64	75.16	73.8	154	80.9	[9.62,43.84]	[218.8,2383.6]	20.451	0.0754
P_3	-23.93	80.65	68.4	158.3	74.7	[10.06,58.95]	[286.5,3618]	13.995	0.0471

Table 2: Accuracy indices at the 3 reference poses

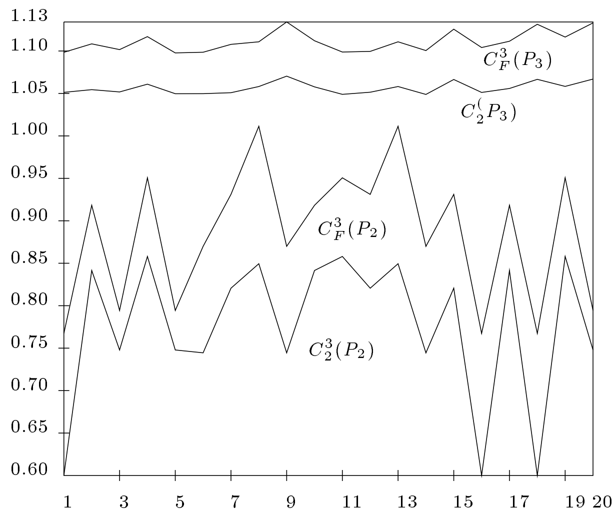

Figure 3: The value of $C_2^3(P_{23})/C_2^3(P_1), C_F^3(P_{23})/C_F^3(P_1)$ according to the choice of the 3 reference points on the platform

- C_d: the value of this index is coherent with the maximal positioning errors
- C_2: it may be seen that the difference is surprisingly very small between P_1, P_2 and significant between P_3, P_2. This is not what we may expect from an accuracy index
- C_2^n: the accuracy ordering between P_2, P_3 is not respected
- C_F: the accuracy ordering is respected but the change between P_1 and P_2 is relatively small
- C_F^n: the accuracy ordering between P_2, P_3 is not respected
- C_2^3, C_F^3: for P_3 the condition number is either very close to the one of P_1 (C_2^3) or always larger. On the contrary for P_2 the condition number is in general significantly smaller than the condition number for P_1 and sometime very close but in all cases smaller than the condition number for P_3. This completely disqualify these condition numbers as accuracy indices

- M_t, M_o: the manipulability indices of P_1, P_3 are close while the one of P_2 is significantly larger. According to the maximal positioning errors this disqualify M_t as an accuracy index while M_o does not reflect exactly the orientation errors

Hence none of these condition numbers exhibits a completely consistent behavior with respect to the positioning errors of this robot. This simple example shows clearly that the concept of condition number has to be carefully considered when addressing optimal design for robot.

5 Isotropy

An *isotropic* pose of a robot is defined as a pose where κ is equal to 1 and a robot which has only isotropic poses in its workspace is coined an *isotropic robot*. Designing a parallel robot that is isotropic in one pose or is isotropic over its full workspace is often considered as a design objective [1, 2, 5, 13, 46, 52, 55]. A trivial example of isotropic robot is a serial Cartesian X-Y robot whose kinematic jacobian matrix is the identity. But this is a surprising denomination as, stricto sensu, isotropy indicates that the performances of a robot should be the same whatever is the motion direction. Now if we assume that all the actuator velocities of a X-Y robot are bounded by 1, then the maximal velocity of the end-effector lie in the range $[1, \sqrt{2}]$: as far as velocity or accuracy are considered such robot is far from isotropy. Still the concept may have some interest: for example any Cartesian robot whose actuator axis are not mutually orthogonal will exhibit a ratio between its maximal velocities over its workspace that will be larger than $\sqrt{2}$. Hence, instead of using the name "isotropic robot" we may consider using the name "maximally regular robot". Looking for a maximally regular robot is thus justified but, except for robot having a small workspace [23], designing a robot to be isotropic only in one pose is less justified.

Note also that for redundant robots the isotropy concept is even less justified. For example Krut [28] exhibits a redundant robot whose kinematic behavior is the same than a serial Cartesian X-Y table, but whose condition number is not 1.

6 Global conditioning indices

The condition number is a local indication for the dexterity of a robot. To evaluate the dexterity of a robot over a given workspace W Gosselin [15] has introduced the *global conditioning index* (GCI) as:

$$\text{GCI} = \frac{\int_W \left(\frac{1}{\kappa} \right) dW}{\int_W dW} .$$

which correspond to the average value of $1/\kappa$. Clearly this concept makes sense for the optimal design of robot for which the extremal and average value of any performance are important design factors. The main problem with the GCI, apart of the validity of the condition number, that has been discussed in a previous section, is its *robust* calculation, i.e. its computation as a number that is reasonably close to its true value. Clearly we cannot expect to obtain a closed-form for the GCI and we must rely on a numerical evaluation. The usual method is to discretize the workspace using a regular grid, compute $1/\kappa_i$ at each node N_i and approximate the GCI as GCI_a, the sum of the $1/\kappa_i$ divided by the number of nodes and by the workspace size. This calculation may be computer intensive as its complexity is exponential with respect to the number of d.o.f. of the robot.

Furthermore this method does not allow to get a bound on $|\text{GCI} - \text{GCI}_a|$. To deal with this error problem it is sometimes assumed that if the result $\text{GCI}_a(m_1)$ with m_1 sampling points is close to the result $\text{GCI}_a(m_2)$ obtained with m_2 points, m_2 being significantly larger than m_1, then $\text{GCI}_a(m_2)$ is a good approximation of the GCI. This assumption will be true only if the condition number is smooth enough, a claim that is difficult to support.

To illustrate this problem consider for example a simple planar serial 2R robot with identical link lengths set to 10. The GCI can be computed very precisely with a numerical integration scheme as it depends only on a single parameter. We then use the discretization method by sampling the parameter using 10, 20, ..., m_1, $m_2 = m_1 + 10$ points and stop the calculation when the relative error between $\text{GCI}_a(m_1), \text{GCI}_a(m_2)$ is lower than 0.5% and approximate the GCI by $\text{GCI}_a(m_2)$. For this example when $m_1 = 50$ the relative error is 0.377% while the relative error on the GCI is still 1.751%, i.e. about 5 times larger. It may be assumed that such error will even be larger for more complex robot.

A better evaluation will probably be obtained by using Monte-Carlo integration (with an error that decreases as $1/\sqrt{n}$ where n is the number of sampling nodes) or quasi-Monte Carlo. In the previous example (which is not favorable for Monte-Carlo method as there is only one parameter) we found out that by using the same stop criteria the relative error on the GCI was reduced to 0.63%. A certified evaluation of the global conditioning index is therefore an open problem but nevertheless the calculation of such index will probably be computer intensive.

Another global conditioning index is the *uniformity of manipulability* defined as the ratio of the minimum and maximum values of the manipulability index over a given workspace [38]. It suffers from the same problems than the GCI.

7 Challenges for accuracy indices

As seen in the previous sections classical dexterity indices are not very adequate for parallel robots. The purpose of this section is to examine what other possible indices may be of interest, especially in view of optimal design.

A first possibility, that is almost always required by end-users, it to determine the maximal value of the positioning errors over the workspace [25, 37]. This is a difficult optimization problem as we are looking for a global optimum and as we do not have an analytic formulation for the objective function. But we must note that for comparison purpose it is not necessary to compute *exactly* the maximal errors as soon as we are able to bound the calculation errors and if the algorithm allows to define upper bounds on this error.

We have presented in a recent paper a a computer intensive method for finding the largest maximal positioning errors, up to an arbitrary accuracy, of a 6 d.o.f. robot [32]. It is a derivation of a more general algorithm that allows to determine an approximation of all the design parameters so that the corresponding robots will have positioning accuracies lower than given thresholds.

We will now outline the principle of this algorithm. If we assume that the pose parameters X_i all lie in a range R_i, then interval analysis allows to compute a range for each element of J_k^{-1}, that will include any possible value of the element for any pose whose parameters lie in the ranges R_i. Hence J_k^{-1} is an interval matrix $\mathcal{J}(R)$ that depends upon the ranges R_i. The maximal positioning errors will always be obtained for extremal values of the joint measurement errors, that may be fixed arbitrary to -1, 1 and we will denote by $\boldsymbol{\Delta\Theta_e}$ any joint errors vector whose element has a value -1 or 1.

A classical problem in interval analysis is to bound the possible solutions $\mathbf{\Delta X}$ of an interval linear system $\mathcal{J}(R)\mathbf{\Delta X} = \boldsymbol{\Delta\Theta_e}$ i.e. to determine a range S_i for each ΔX_i so that for any instance of J_k^{-1} in $\mathcal{J}(R)$ the solution in ΔX_i of the linear system $\mathrm{J}_{\mathrm{k}}^{-1}\mathbf{\Delta X} = \mathbf{\Delta\Theta_e}$ is included in S_i. Usually S_i is an over approximation of the solution set for ΔX_i, which will come closer to this solution set as soon as the width of the ranges R_i decreases. With these elements a branch and bound algorithm allows to compute the maximal value of ΔX_i up to an arbitrary accuracy. The idea is to compute the maximal value of ΔX_i for a given set of ranges R by selecting an arbitrary pose in R (e.g. the pose defined by the mid-point of the ranges in R). This maximal value is used to update a current estimation ΔX_i^M of the maximum of ΔX_i. If S_i is such that the absolute value of its lower and upper bounds are smaller than $\Delta X_i^M + \epsilon$, where ϵ is the accuracy with which we want to calculate the maximal positioning error, then there is no pose in R such that $|\Delta X_i| > \Delta X_i^M$. Otherwise we choose a range in R, bisect it and create two new elements R_1, R_2. All these elements will be submitted to the same process, that will stop when all elements have been processed.

Beware that such algorithm is usually computer intensive and needs to be carefully implemented to be efficient, as drastic differences in the computation time will be obtained according to the implementation.

But if the values of the maximal positioning errors over a workspace are necessary, they are not sufficient to determine an optimal design. Clearly the average values of the maximal positioning errors and even their variance will be needed. Unfortunately there is no known algorithms to compute these accuracy indices and finding such algorithms is one of the greatest challenge of accuracy analysis.

8 Conclusion

Classical local dexterity indices defined for serial robot, such as the condition number or the manipulability index, are not very appropriate for parallel robots. Furthermore we have shown that they do no reflect exactly the positioning accuracy of the robot. Global dexterity indices based on these local indices are therefore questionable and we have also shown that their guaranteed numerical evaluation (i.e. with a bound on the calculation error) is an open problem. In our opinion the most appropriate global accuracy indices are the determination of the maximal positioning errors, their average values and their variance. A real challenge is to design algorithms for calculating these indices. One may take advantage that it is not necessary to calculate these indices *exactly* as soon as it is possible to impose a bound on the calculation error. Indeed for comparison purposes an approximate value with a guaranteed error will be sufficient.

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