Singular configurations of parallel manipulators and Grassmann Geometry

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Abstract

Parallel manipulators have a specific mechanical architecture where all the links are connected both at the base and at the gripper of the robot. By changing the lengths of these links we are able to control the position [1] of the gripper. In general for a given set of links lengths there is only one position for the gripper. But it may be suspected that in some

[1] In this paper position means position and orientation

case more than one solution may be found for the position of the gripper : the robot is in a singular configuration. To determine these singular configurations the classical method is to find the roots of the determinant of the jacobian matrix. In our case the jacobian matrix is complex and it seems not to be possible to find these roots. We propose here a new method based on Grassmann line-geometry. If we consider the set of lines of P^3, it constitutes a linear variety of rank 6. We show that a singular configuration is obtained when the variety spanned by the lines associated to the robot links has a rank less than 6. An important feature of the varieties of this geometry is that they can be described by simple **geometric** rules. Thus to find the singular configurations of parallel manipulators we have to find the configuration where the robot matches these rules.

Such an analysis is performed on a special parallel manipulator and we show that we find all the well-known singular configurations but also some new one.

1 Parallel manipulator

1.1 Introduction

We deal here with the study of parallel manipulator like the model presented in Figure 2.

Basically it consists in two plates connected by 6 articulated links. In the following sections the smaller plate will be called the *mobile* and the larger (which is in general fixed) will be called the *base*. In each articulated link there is one linear actuator.

Manipulators of this type have been designed or studied for a long time. The first one, to the author's knowledge, was designed for testing tyres (see Mc Gough in Stewart paper [18]). But the main use of this mechanical architecture consists in the flight simulator (see for example Stewart [18], Watson [22], Baret [1]). The first design as a manipulator system has been done by Mac Callion in 1979 for an assembly workstation [11] but Minsky [15] has presented in the early 70's some design related to various mechanical architectures. Some other researchers have also addressed this problem: Reboulet [17], Inoue [10], Tanaka [19], Fichter [7], Mohamed [16], Yang [23], Zamanov [24]. This kind of manipulator has a great positionning ability and is very convenient for force-feedback control (see [12]). A prototype of a parallel manipulator is currently under development at INRIA (figure 2). The links articulations are universal joints on the fixed plate and ball-and-socket joints at the mobile plate. The linear actuators are electric rams and the lengths variations are measured through linear potentiometers. The motion ranges of this prototype are:

x	y	z	ψ	θ	ϕ
± 6 cm	±6cm	0-2cm	±15°	±55°	±55°

Figure 1: notation

Figure 2: INRIA parallel manipulator prototype

where ψ, θ, ϕ are the Euler's angles. The measured accuracy is about 1/100 mm without load and 6/100 with a load of 6 kg (improvment of the accuracy seems to be possible). The height of the prototype is about 51 cm, its weight about 11kg. The nominal vertical load vary from 18 daN to 90 daN according to its position.

1.2 Notation

We introduce the absolute frame R with origin C and a relative frame R_b fixed to the mobile with origin O (see Figure 1). The rotation matrix relating a vector in R_b to the same vector in R will be denoted by M.

The centers of the articulations on the base for link i will be denoted A_i and those on the mobile B_i. The length of link i will be noted ρ_i , and the unit vector of this link $\mathbf{n_i}$. The coordinates of A_i in frame R are (xa_i, ya_i, za_i), the coordinates of B_i in frame R_b are (x_i, y_i, z_i) and the coordinates of O, the origin of the relative frame, (x_o, y_o, z_o). We use the Euler's angles ψ, θ, ϕ to characterize the orientation of the mobile.

For the sake of simplicity the subscript i is omitted whenever it is possible and vectors will be noted in **bold** character. A vector with coordinates expressed in the relative frame will be denoted by the subscript $_r$. We will restrict our study to one particular case of parallel manipulators.

We will consider the case where sets of articulation points of both the base and the mobile each lie in a plane and are symmetric along one axe (see Figure 3). The articulation points on the mobile are located only in three different positions. The mobile is homothetic to the base and is rotated at 180 degrees for the connection of the links. In this case, without loss of generality, we will define R such that $za_i = 0$ and R_b such that $z_i = 0$. Each symmetry axis will be used as an axis of its associated frame R, R_b. We exclude the case where three or more articulation points are colinear. We will call this architecture the triangular simplified symmetric manipulator (TSSM). We denote by C_{ij} the intersection point of lines A_kA_i and A_jA_l and by P_{12}, P_{34}, P_{56} the planes defined by $A_1A_2B_1$, $A_3A_4B_3$, $A_5A_6B_5$. We may remark that

$$C_{23} \in P_{12} \qquad C_{23} \in P_{34}$$

$$C_{45} \in P_{56} \qquad C_{45} \in P_{34}$$

$$C_{61} \in P_{12} \qquad C_{61} \in P_{56}$$

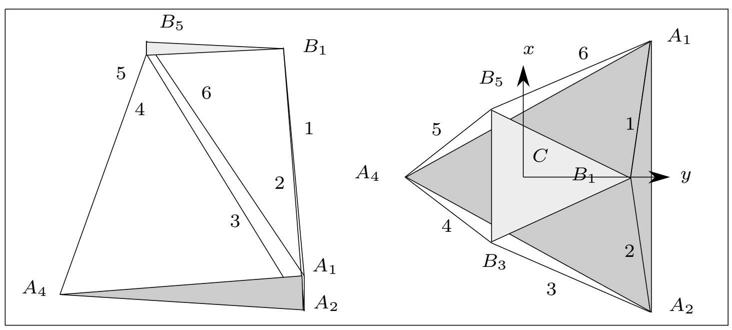

Figure 3: Triangular simplified symmetric manipulator TSSM (face, top, side and perspective view)

1.3 Singular configurations and the jacobian matrix

Let us calculate the fundamental relations relating the links lengths to the position of the mobile. We have :

$$\mathbf{AB} = \rho\mathbf{n} \quad \mathbf{AB} = \mathbf{AC} + \mathbf{CO} + \mathbf{OB} \qquad \mathbf{OB} = M\mathbf{OB_r} \tag{1}$$

where $\mathbf{OB_r}$ means the coordinates of the articulation points with respect to the frame R_b. $\mathbf{n}$ being a unit vector :

$$\rho = ||\mathbf{AC} + \mathbf{CO} + M\mathbf{OB_r}|| = ||\mathbf{U}|| \tag{2}$$

If the position of the mobile is given we are able to calculate the components of $\mathbf{U}$ and thus the length of the segment. At the opposite we have to solve a system of six non-linear equations of type 2 to get the position of the mobile from the links lengths. At this time no theoretical solution of this system has been established. From the rank theorem we know that the solution is unique if the rank of the jacobian matrix J of this system is equal to 6 with:

$$J = ((\frac{\partial \rho}{\partial \mathbf{q}})) \tag{3}$$

where $\mathbf{q}$ is the position parameters vector. Note that this matrix is in fact the inverse jacobian (in a robotics sense) of the manipulator. The symbolic computation of the determinant of J is rather tedious (see [13] for the formulation of this determinant). Mac Callion [11] used a numerical deflation method to find all the roots of the determinant. Mac Callion has found up to nine roots to this determinant, all outside the range of the links lengths, and Hunt has shown that there can be up to 16 roots [9]. Bricard [3] has shown that the resolution of the above system is equivalent to solve a complex trigonometric equation.

Hunt [9] describes a singular configuration (Figure 4). In this case all the segments intersect one line (line B_3B_5). We will see later that a simple mechanical analysis explains why this is a singular configuration.

Fichter [7] describes another singular configuration which is obtained when one rotates the mobile plate around the z axis with an angle of $\pm\frac{\pi}{2}$. This configuration was obtained by noticing that in this case two lines of the determinant were constant. But outside this two particular configurations no systematic method was proposed to find *all* the singular configurations of a parallel manipulator. Let us investigate now a geometrical approach.

2 Plücker coordinates of lines, rigidity and geometry

It is well known that a line can be described by its Plücker coordinates. Let us introduce briefly these coordinates. We consider two points on a line, say M_1 and M_2, and a reference frame R_0 whose origin is O (see Figure 5). Let us consider now the two three dimensional vectors $\mathbf{S}$ and $\mathbf{M}$ defined by :

$$\mathbf{S} = \mathbf{M_1M_2}$$

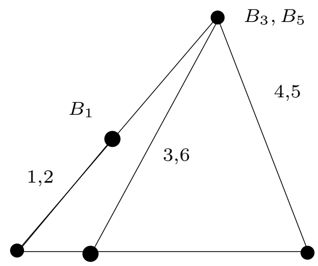

Figure 4: Hunt's singular configuration for the TSSM

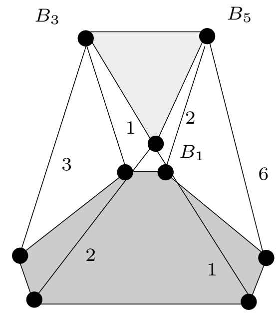

Figure 5: Plücker coordinates

$$\mathbf{M} = \mathbf{OM_1} \wedge \mathbf{OM_2} = \mathbf{OM_2} \wedge \mathbf{S} = \mathbf{OM_1} \wedge \mathbf{S}$$

If we assemble these vectors to form a six-dimensional vector we get the vector $\mathbf{U}$ of the Plücker-coordinates of this line.

$$\mathbf{U} = [S_x, S_y, S_z, M_x, M_y, M_z]$$

It is useful to introduce the normalized vector $\mathbf{U}'$ defined by:

$$\mathbf{U}' = \frac{\mathbf{U}}{||\mathbf{S}||} = \left[S'_x, S'_y, S'_z, M'_x, M'_y, M'_z\right]$$

It may be seen that the first three components of this vector are the components of the unit vector $\mathbf{n_i}$ of the line. The last three components are given by :

$$\mathbf{OM} \wedge \mathbf{n_i}$$

M being any point of the line. Let us consider now the matrix P defined by:

$$P = ((U'_1, U'_2, ...U'_6))$$

where U'_i is the coordinate vector of line i. If we denote by $\mathcal{T}$ the generalized force vector we get:

$$\mathcal{T} = P\mathbf{f} \tag{4}$$

From this relation it is easy to show that we have:

$$J^T = P \tag{5}$$

Equation 4 is a linear system of equations in term of the articular forces. If the system is rigid this means that whatever the generalized forces are, there exists one set of articular forces such that the system is in an equilibrium state (and in our case the solution will be unique). This will be true if the matrix P is of full rank which is equivalent to say that the Plücker vectors are linearly independent. Thus *a singular configuration of a parallel manipulator corresponds to a configuration where it is not rigid*. One main result for the rigidity of polyhedron was obtained by Cauchy [4]. He shows that a *convex* polyhedron with invariable faces is always rigid. But in our case a parallel manipulator may be not convex. However from this result we may say that all the singular configurations we will find must be such that the parallel manipulator is not convex.

As a matter of example let us consider Hunt's singular configuration. We notice that the torque around the axis B_3B_5 exerted by the segments on the mobile is always equal to zero (remember that every lines intersect B_3B_5). Thus if we apply an external force on the mobile such that the resulting torque around the axis B_3B_5 is not equal to zero, the mobile cannot be in an equilibrium state: it is not rigid.

Let us assume now that the Plücker vectors belong to a vector space V_6 and we consider the one-dimensional subspaces of V_6 as points of a projective P_5. Then every line g in P_3 corresponds to exactly one point $\mathbf{G}$ in P_5.

It is well known that point $\mathbf{G}$ belongs to a quadric Q_p (see [5], [21], [2]). Indeed we have for every line of P_3 :

$$S_xM_x + S_yM_y + S_zM_z = 0$$

This equation defines the quadric Q_p which is called the *Grassmannian* or the *Plücker quadric*. At this point we have defined a one-to-one relation between the set of lines in the real P_3 and the quadric Q_p in P_5. The rank of this mapping is 6 (there is at most 6 independent Plücker vectors).

Let us consider now the various sub-spaces of P_5 (or more precisely their intersection with Q_p). We get various varieties which rank ranges from 0 to 6. As a matter of example a point in P_5 (rank=1) corresponds to a line in P_3. As for Q_p (which represents the set of line of P_3) it is defined through 6 linearly independent Plücker vectors and is therefore of rank 6.

Let us come back to the rigidity of parallel manipulators. We have seen that this manipulator is rigid (and therefore not in a singular configuration) if and only if the 6 lines are linearly independent. Therefore **a parallel manipulator will be in a singular configuration if, and only if, there is a subset spanned by n of its lines which has a rank less than n.** At this point the problem is far from solved because we are not able to find the generalized coordinates of the mobile for which there is a linear dependency between the n Plücker vectors. But these dependencies can be described by geometric rules.

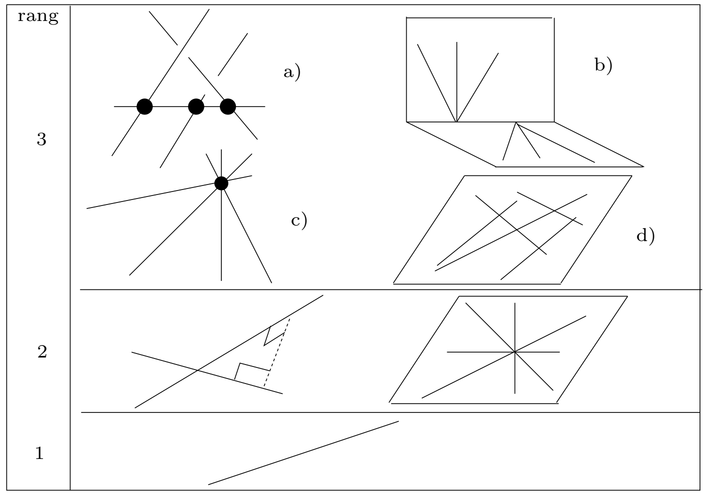

Figure 6: Grassmann varieties of rank 1,2,3

3 Grassmann Geometry

The varieties of lines has been studied by H. Grassmann (1809-1877). The purpose of this study was to find geometric characterization of each varieties. We will introduce now the various results which can be found in [6] or, with more mathematical justifications, in [21].

Let us begin with the linear varieties of rank 0 through 3 (Figure 6). We have first the empty set of rank 0. Then the *point* (rank=1), which is a line in the 3D space. The *lines* (rank=2) are either a pair of skew lines in R^3 or a flat pencil of lines: those lying in a plane and passing through some point on that plane.

The *planes* (rank=3) are of four types:

- all lines in a plane (3d)
- all lines through a point (3c)
- the union of two flat pencils having a line in common but lying in distincts planes and with distinct centers (3b)
- a regulus (3a)

Let us define the regulus. Take three skew lines in space and consider the set of lines which intersect these three lines : this set of lines build a surface which is an hyperboloid of one sheet (a quadric surface, Figure 7) and is called

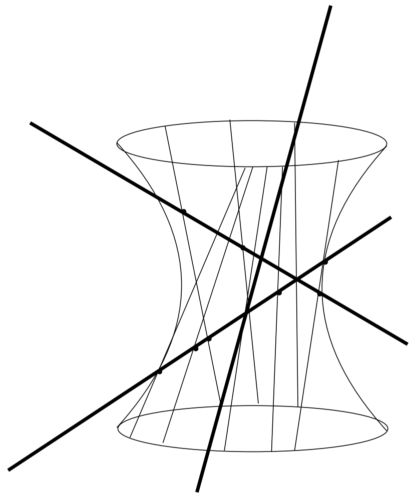

Figure 7: hyperboloid of one sheet

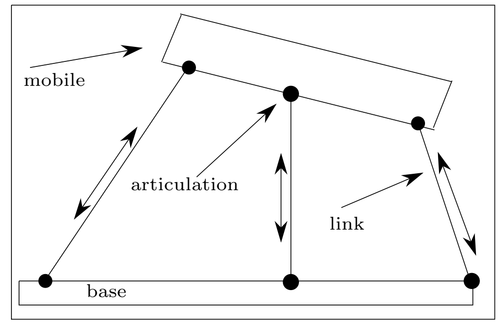

Figure 8: 2D parallel manipulator

a **regulus**. Each line belonging to the regulus is called a *generator* of the regulus.

It is shown in [8],[21] that this surface is *doubly ruled*. This means that there exist two reguli (a regulus and its "complementary" regulus) which generate the same surface or that each point on the surface is on more than one line.

Therefore there are two families of straight lines on the hyperboloid and each family covers the surface completely. A line on this surface is dependent on the lines of either the regulus or the complementary regulus. An interesting property is that a line of one family intersects all the lines of the other family and that any two lines of the same family are mutually skew (see [20] for the hairy details).

Let us describe now the linear varieties of higher rank of the Grassmann geometry (Figure 9).Linear varieties of dimension 4 are called linear *congruences* and are of four types:

- a linear spread generated by four skew lines i.e. no line meet the regulus generated by the three others lines in a proper point (*elliptic congruence*, 4a)
- all the lines concurrent with two skew lines (*hyperbolic congruence*, 4b)
- a one-parameter family of flat pencil, having one line in common and forming a variety (*parabolic congruence*, 4c)

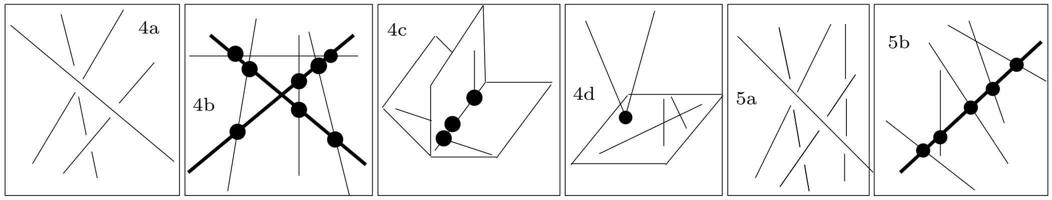

Figure 9: Grassmann varieties of rank 4,5

- all the lines in a plane or passing through one point in that plane (*degenerate congruence*, 4d)

Linear varieties of dimension 5 are called linear *complexes* and are of two types:

- *non singular* (or *general*): generated by five independent skew lines (5a)
- *singular* (or *special*): all the lines meeting one given line (5b)

The geometric characterization of a general linear complex is that through any point of the space there is one and only one flat pencil of line such that all the lines which belong to the pencil belong also to the complex. In other words all the lines of a linear complex which are coplanar intersect one point.

We will give now a simple example for the use of this geometrical description of linear varieties of lines.

4 A basic example: the 2D parallel manipulator

Let us consider a basic example: a 2D parallel manipulator (Figure 8). In this case we have three segments and these three lines must constitute a variety of Grassmann of rank 3. Thus we will consider any subset of 1,2,3 segments and determine the condition for which any such subset has a rank less than 1,2,3.

The case of 1 and 2 segments are rather trivial: for one line we have only to verify that this line exists and for two lines that the lines are distincts. This is clearly the case if we except the configuration where the base and the mobile are collinear.

We will consider now the whole system of three bars. By reference to Figure 6 we can see that the only possibility for a system of three coplanar bars to be a 2-rank Grassmann variety is obtained when the three lines cross the same point (Figure 10). In particular if the mobile and the base are homothetic we get a rather disturbing singular configuration when the base

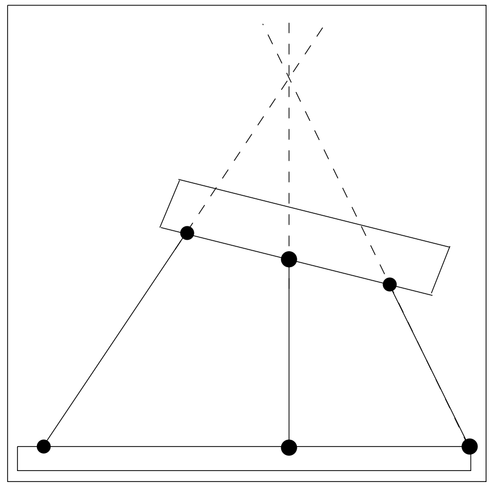

Figure 10: Singular configuration for the 2D parallel manipulator

Figure 11: A 2D parallel manipulator without singular configuration

and the mobile are parallel, whatever is their relative position. This is easy to verify by building a paper model.

Another design is straightforward to avoid the above singular configuration (Figure 11).

We can see here that practically, whatever is the position of the mobile the three segments cannot cross the same point.

5 Study of the TSSM

We will deal now with the case of the TSSM (Figure 3).

Let us make a preliminary remark: for the TSSM we may have at most 2 coplanar lines. Indeed we notice that there are at most two segments with collinear articulation points on the base. Therefore we have not to consider the degeneracy of subset where more than two lines must be coplanar. This will be the case for:

- the subset of two lines
- the subset of three lines,
- the subset of four lines in configuration 3d, 3b
- the subset of five lines in configuration 4c

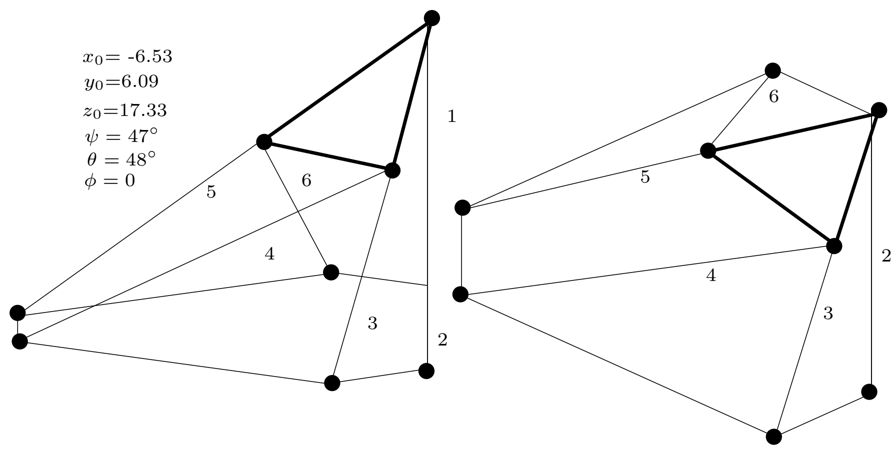

Figure 12: Perspective and top view of a singular configuration of type 3c

5.1 Subset of 4 bars

5.1.1 Type 3c

(Four lines cross the same point).

Among a set of four bars two have a common articulation point on the mobile. Thus this common point must be the common point to the four lines. We will assume that the two lines with a common point are 1,2. Lines 3,4 and 5,6 have a common point different from B_1 and thus cannot have another one. Thus the only sets to be considered are (1,2,3,5),(1,2,3,6), (1,2,4,5) and (1,2,4,6). The demonstration of the following result will be the same in each case and we will study only the case of the set (1,2,3,5).

If 3 crosses B_1 then 3 is collinear to the edge B_1B_3. In the same manner if 5 crosses B_1 then 5 is collinear to the edge B_1B_5 and thus 3 and 5 are coplanar. Thus we get a singular configuration if lines 3,5 are coplanar and intersect the articulation point B_1. Figure 12 shows such a case.

5.1.2 Type 3a

The problem is to find four lines which are on the same regulus. An hyperboloid of one sheet has two regulus $\Re_1$ and $\Re_2$ and we denote by (1) the family of lines which are spanned by $\Re_1$ and (2) the family of lines spanned by $\Re_2$. Remember that each line of (1) has an intersection point with every lines of (2) and none with the other lines of (1).

Let us suppose that line 1 belongs to the family (1) spanned by the regulus. Line 2 intersects line 1 and thus belongs to the family (2) spanned by the

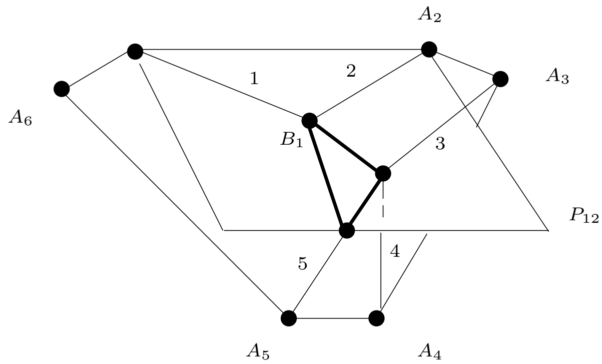

Figure 13: Perspective and top view of a singular configuration of type 4d

complementary regulus. For the same reason lines 3,4 and 5,6 cannot belong to the same family. Therefore four lines cannot belong to the same regulus.

5.2 Set of five bars

5.2.1 Configuration 4d

(five lines in a plane or crossing a point of this plane)

Let us remember that we have at most two coplanar lines. Among a set of 5 lines two pairs are coplanar and have a common point which is their articulation points on the mobile. These two points being different, this imply that the plane to be considered is spanned by a pair of lines and that the common point to the three other lines is the articulation point of the second pair. For example if we consider lines 1,2,3,4,5 we will consider the plane spanned by 1,2 , put the articulation point B_3 common to 3,4 in this plane and look if 5 can intersect B_3. Figure 13 shows such a case.

5.2.2 Configuration 4b

(five lines intersect two skew lines)

Let us consider first lines 1,2,3,4. We have to find two skew lines D_1, D_2 which intersect these four lines. We have four possibilities for a line D which intersects lines 1,2,3,4:

-$D \in P_{12}$ and D intersects B_3
-$D \in P_{34}$ and D intersects B_1
-$D = P_{12} \cap P_{34}$
-D intersects both B_1 and B_3

Figure 14: Two skew lines intersecting 1,2,3,4,5, configuration 4b, first case

5.2.3 $D_1 \in P_{12}$ and D_1 intersects B_3

If D_2 is skew to D_1 then $D_2 \notin P_{12}$ and D_2 does not intersect B_3. Therefore D_2 cannot be either $P_{12} \cap P_{34}$ or B_1B_3. Thus the only remaining case is $D_2 \in P_{34}$ and D_2 intersects B_1. In this case we have :

$$B_1 \in P_{34} \quad B_1 \in P_{12} \Rightarrow B_1 \in P_{12} \cap P_{34}$$

$$B_3 \in P_{34} \quad B_3 \in P_{12} \Rightarrow B_3 \in P_{12} \cap P_{34}$$

Thus C_{23}, B_1, B_3 belong to the same line. Let M_{12} be the intersection point of line 5 with P_{12} and M_{34} the intersection point of line 5 with P_{34}. If M_{12} is different from M_{34} then the lines B_3M_{12} and B_1M_{34} are skew and intersect the lines 1,2,3,4,5. This is then a singular configuration. (Figure 14).

5.2.4 $D_1 \in P_{34}$ and $B_1 \in D_1$

As in the previous part we get:

$$D_2 \in P_{12} \quad B_3 \in D_2$$

which is the case we investigated above.

5.2.5 $D_1 = P_{12} \cap P_{34}$

If D_2 is not coplanar with D_1 then we must have:

$$D_2 = B_1B_3$$

But if line 5 intersects D_2 then line 5 and B_1B_3 are coplanar and thus the mobile and line 5 are coplanar. We must then investigate if line 5 intersects D_1. If we write that line 5 intersects D_1 we get four linear equations in term of the intersection point. A numerical resolution of the three first equations give the position of the intersection point and then we have to verify if the fourth equation is satisfied. This will be the second case of degeneracy of type 4b.

5.2.6 D_1 intersects both B_1 and B_3

In this case D_1 is the edge B_1B_3 of the mobile. If a fifth line (5 or 6) intersects D_1 then both lines 5 and 6 are coplanar with the mobile. Thus we get a configuration where all the lines intersect the line D_1 : this is Hunt's singular configuration.

Figure 15: Perspective and top view of a singular configuration of type 5a ($\psi = \pm\frac{\pi}{2}, \theta = \phi = 0$)

5.3 Set of six bars

5.3.1 Configuration 5a

In this case the variety spanned by the six lines is a general linear complex. We consider the lines D_i belonging to the flat pencils spanned by lines 1-2, 3-4, and 5-6 and lying in the mobile plane. We get a general linear complex if and only if these three lines intersect the same point.

We will consider first the case where we have only rotation around the vertical axis. We have shown [14], with the help of MACSYMA [2] , that we get then a singular configuration if, and only if, we have $\psi = \pm\frac{\pi}{2},\ \theta = \phi = 0$ whatever is the position of the center of the mobile (figure 15). This is Fichter singular configuration.

In a second part we consider the general case. We have shown [14] that it is also possible to find constraint on the position of the mobile such that we get a singular configuration (see part 6 for a summary of these conditions). In this case we must have either $\psi = \phi$ or $\theta = \pm\frac{\pi}{2}$ and z_0 is solution of a third order polynomial for fixed x_0, y_0. Figures 17, 16 show a configuration in these cases.

5.3.2 Configuration 5b

We have to consider the case where the six segments cross the same line. Let us consider lines 1,2,3,4.

We have four possibilities for line D to intersect 1,2,3,4:
-$D = P_{12} \cap P_{34}$
-D intersects both B_1 and B_3
-$D \in P_{12}$ and D intersects B_3
-$D \in P_{34}$ and D intersects B_1
Let us now consider lines 5,6 in each of these cases.

5.3.3 $D = P_{12} \cap P_{34}$

We may have:
-$B_5 \in P_{12} \cap P_{34}$
-$D = P_{56} \cap P_{12} \cap P_{34}$

[2] MACSYMA is a large symbolic manipulation program developed at MIT

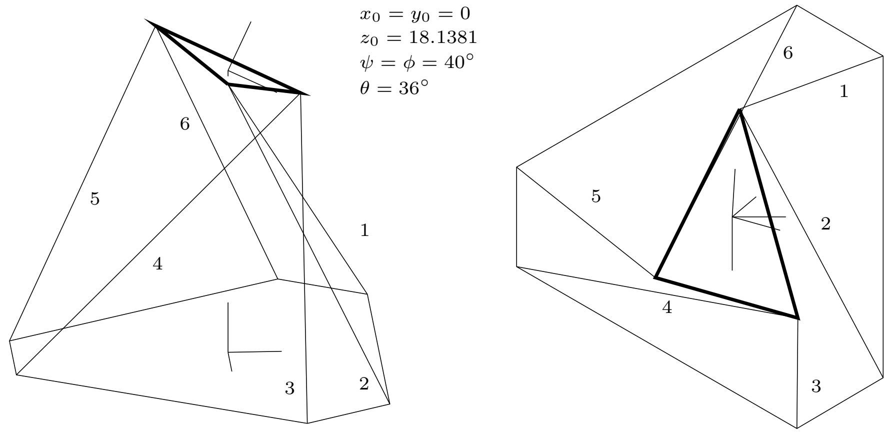

Figure 16: Perspective and top view of a singular configuration of type 5a ($\psi = \phi$)

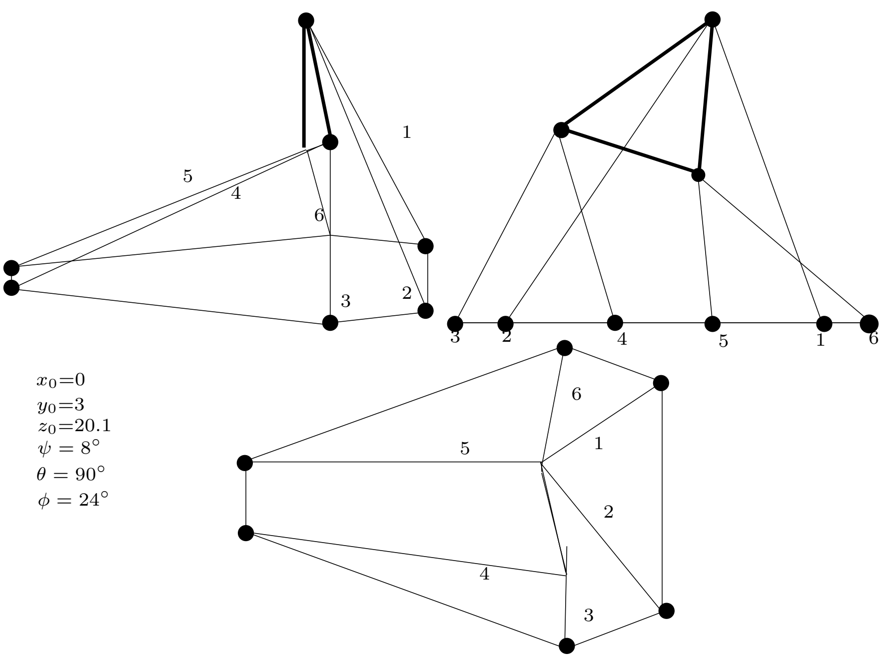

Figure 17: Perspective, side and top view of a singular configuration of type 5a ($\theta = \pm\frac{\pi}{2}$)

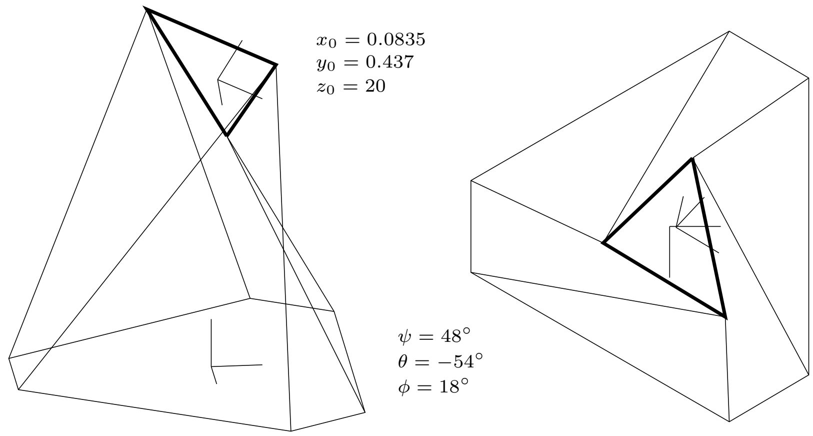

Figure 18: Perspective and top view of a singular configuration of type 5b

In the first case we may deduce from the preliminary remark that line D intersects both B_5 and C_{23}. Figure 18 shows such a singular configuration.

Let us consider now the second case. The three planes must have a line in common. Let us consider the intersection line of plane P_{34}, P_{56}. We know that C_{45} belongs to this line. If the intersection line lies also in the plane P_{12} then C_{45} must also lie in this plane. This is impossible under our assumption and therefore the three planes cannot have a line in common.

5.3.4 D intersects both B_1 and B_3

Thus the line common to the 6 segments is the edge B_1B_3 of the mobile. If lines 5,6 both intersect this edge this means that the edge is coplanar to P_{56}. This is the singular configuration described by Hunt.

5.3.5 $D \in P_{12}$ and D intersects B_3

Thus B_3 belongs to P_{12}. If D also intersects 5,6 we may have two possibilities.

-$B_5 \in P_{12}$

-the intersection line $P_{12} \cap P_{56}$ intersects B_3

In the first case D is the edge B_3B_5 of the mobile and two of the segments are coplanar to the mobile. This is Hunt's singular configuration.

In the second case we may deduce that the intersection line must be the line joining C_{16} and B_3. We have dealt with a similar problem in a previous part ($D = P_{12} \cap P_{34}$ and $B_5 \in D$).

Spanned variety	condition
3c	$\tan\psi = (ya_3 - ya_4)/(xa_3 + xa_4)$ $x_0 = A(\psi, \theta, \phi), y_0 = B(\psi, \theta, \phi), z_0 = C(\psi, \theta, \phi)$
degenerate congruence, 4d	$x_0 = A(\psi, \theta, \phi), y_0 = B(\psi, \theta, \phi), z_0 = C(\psi, \theta, \phi)$
hyperbolic congruence,4b (first case)	$x_0 = A(z_0, \psi, \theta, \phi), y_0 = B(z_0, \psi, \theta, \phi)$
hyperbolic congruence (second case)	$y_0 = A(x_0, z_0, \psi, \theta, \phi)$, $F(x_0, z_0, \psi, \theta, \phi) = 0$
general complex, 5a (first case)	$\theta = \phi = 0 \quad \psi = \pm\frac{\pi}{2}$
general complex (second case)	$\theta = \pm\frac{\pi}{2}$ or $\psi = \phi$ $A(x_0, y_0, \psi, \theta, \phi)z_0^3 + B(x_0, y_0, \psi, \theta, \phi)z_0^2+$ $C(x_0, y_0, \psi, \theta, \phi)z_0 + D(x_0, y_0, \psi, \theta, \phi) = 0$
special complex, 5b	$x_0 = A(z_0, \psi, \theta, \phi), y_0 = B(z_0, \psi, \theta, \phi)$

Table 1: The conditions on the mobile parameters for spanning degenerate varieties

5.3.6 $D \in P_{34}$ and D intersects B_1

This case is similar to the previous one.

6 Summary of the results

The conditions on the position parameters which must be satisfied to get a singular configuration according to the above geometric conditions have been established in [14]. This calculation are beyond the scope of this paper but the table below summarizes the results.

Among these singular configurations we find Hunt's configuration (in this case the manipulator is a special complex) and Fichter's configuration (as a general complex).

7 Conclusion

The study of the singular configurations by the use of Grassmann geometry yields to interesting results and new singular configurations. With this method we have found *every* singular configurations of the TSSM. We have established the constraints on the position parameters which must be satisfied to obtain the various singular configuration. This work has been extended in [14] for various architecture of parallel manipulator. It appears that in the most general case of parallel manipulators it may be difficult to find the singular configurations which satisfy these geometric rules. But we are at the beginning of this approach and we hope that by the use of the powerful theorems which have been established by the researchers in the field of Grassmann geometry we may be able to obtain additional results.

Another extension of this work will be to deduce from the constraints on the *position parameters* the constraints on the *links lengths.* From this point it will be possible to determine, for a given architecture, if the singular configurations are in the working area.

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