Assembly-modes and minimal Polynomial formulation of the direct kinematics of parallel manipulators

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Abstract

In this paper we will address the problem of parallel manipulator's direct kinematics (i.e. find the position and orientation of the mobile plate as a function of the articular coordinates) and the corollary problem of their assembly-modes.

We consider a 6 d.o.f. manipulator in the case where the mobile plate is a triangle and the links lengths are time-varying. Using geometrical considerations we show that an upper-bound of the maximum number of assembly-modes, and thus the maximum number of solutions of the direct kinematics problem, is 16. We present then the direct kinematics problem as the solution of a sixteenth order polynomial in one variable. Using a numerical procedure we show that this polynomial may have 16 real roots and thus we exhibit one example for which the maximum number of assembly mode is reached.

The same study is done for the famous Stewart platform, for which an upper-bound of the maximum number of assembly-modes is 16, the direct kinematics problem is solution of a twentieth order polynomial and we have found up to eight assembly-modes for a given configuration.

1 Introduction

The direct kinematics problem of a manipulator can be stated in the following manner : the articular coordinates being known is it possible to find the generalized coordinates of the effector? The inverse kinematics problem for parallel manipulator is easily solved : in general each articular coordinate can be expressed as a non-linear function of the generalized coordinates [1], [2]. Thus we have to solve a system of non-linear equations (in general there is 6 equations) to determine the solution of the direct kinematics problem.

Solving a system of non-linear equations is a difficult task and a numerical resolution is tedious. Furthermore we have no a-priori information about the uniqueness of the solution (except in the case of singular configurations [3]).

Nanua and Waldron [4] have initiated a new approach to this problem. They reduce the resolution of the system of non-linear equations to the one of a polynomial in one variable. The number of assembly-modes of the manipulator (i.e. the number of way one can assemble the manipulator with fixed articular coordinates) is clearly related to the degree of this polynomial: it cannot be greater than this degree.

In the case of the manipulator called the TSSM [5], [6] (Triangular Symmetric Simplified Manipulator, see figure 1) these authors show that the direct kinematics problem may be expressed as solution of a polynomial in one variable, which degree is 24. Charentus and Renaud [7][8] have studied the same manipulator, in the case where the mobile plate is an equilateral triangle. They have shown that the degree of the polynomial can be reduced to 16. Hunt [9] has proposed a conjecture for the same manipulator which states that the number of assembly-modes cannot be greater than 16. In a first part we will calculate the polynomial for the TSSM in the general case and prove the conjecture of Hunt. This is done by showing that the TSSM is similar to another mechanism called the *equivalent mechanism* of the TSSM for which we can establish a polynomial and an upper-bound of the maximum number of assembly mode (called the *UBAM* for brief in the following sections). We will present then a configuration for which there is 16 assembly-modes.

2 The TSSM

2.1 Equivalent mechanism

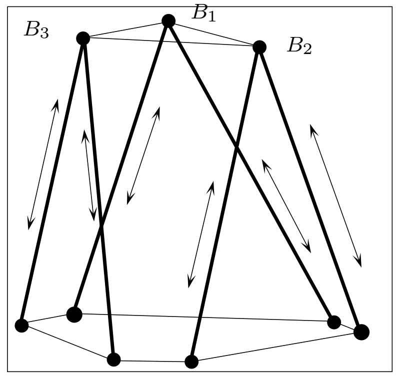

Figure 1: The TSSM parallel manipulator

The TSSM (figure 1) is a 6 d.o.f. parallel manipulator in which a mobile plate is connected to a fixed base through 6 articulated links, each link being connected both at the base and the mobile plate through ball-and-socket and universal joints. By controlling the links lengths we are able to control the position and orientation of the mobile plate [5], [10], [11], [12]. In the following part we will use the notation defined in figure 2. For fixed links lengths the articulation points B_1, B_3, B_5 of the mobile plate are able to describe circles centered in O_{12}, O_{34}, O_{56} whose radius are r_{12}, r_{34}, r_{56}. The characteristics of these circles can be determined using only the knowledge of the links lengths. Thus the TSSM is equivalent to a mechanism constituted of three links articulated on revolute joints and connected to the mobile plate (figure 3) for which the articular coordinates are the angles p_{12}, p_{34}, p_{56}. This mechanism is called the *equivalent mechanism* of the TSSM.

2.2 Minimal degree of the TSSM polynomial

Hunt [9] has conjectured that an UBAM of the TSSM is 16 by using the following method: if we dismantle one of the link of the equivalent mechanism

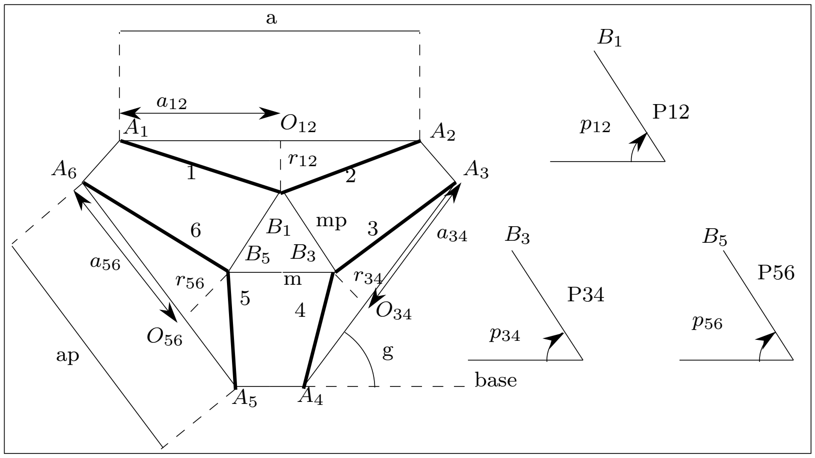

Figure 2: Notation (the TSSM is represented in top view)

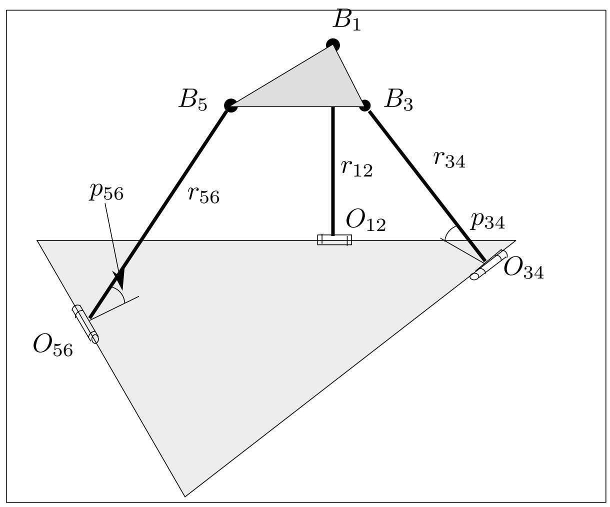

Figure 3: Equivalent mechanism of the TSSM

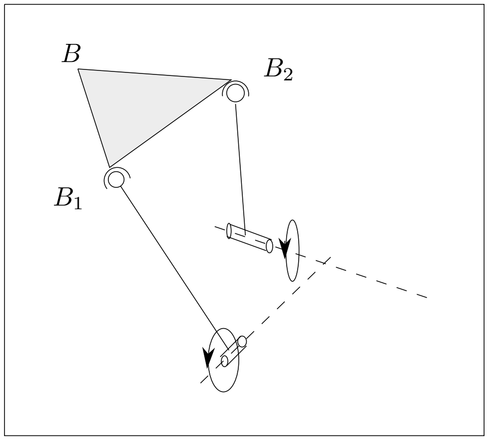

Figure 4: The RSSR mechanism obtained when one link of the equivalent mechanism of the TSSM is dismantled

of the TSSM we get a RSSR mechanism (figure 4). It is known [13] that point B of this mechanism describes a sixteenth order surface, the *RSSR spin surface*. In order to find the possible configurations of mobile plate we have to intersect this surface with the circle described by the extremity of the dismantled link. A sixteenth order surface is intersected by a circle in no more than 32 points. Working on the conjecture that the RSSR spin-surface contain the imaginary spherical circle eight times Hunt deduces that at least 16 points are imaginary, and therefore there is at most 16 assembly-modes for the TSSM.

Thus to demonstrate this conjecture we have to determine the circularity of the RSSR spin-surface.

The coordinates of the articulation points can be expressed as a function of the three unknown angles p_{12}, p_{34}, p_{56}. Thus we are able to write three equations relating the known distance between the articulation points to the three unknowns p_{12}, p_{34}, p_{56}. We have

$$a_{12} = \frac{\rho_1^2 + a^2 - \rho_2^2}{2a} \qquad a = 2xa_2 \tag{1}$$

$$x_{O_{12}} = -xa_2 + a_{12} = \frac{\rho_1^2 - \rho_2^2}{4xa_2} \quad y_{O_{12}} = ya_2 \quad r_{12}^2 = \rho_1^2 - a_{12}^2 \tag{2}$$

If n_{ij} denote the unit vector between O_{ij} and the corresponding articulation points we get :

$$n_{12} = -\cos(p_{12})\mathbf{j} + \sin(p_{12})\mathbf{k} \tag{3}$$

In the same way:

$$a_{34} = \frac{\rho_3^2 + ap^2 - \rho_4^2}{2ap} \tag{4}$$

$$x_{O_{34}} = xa_3 - a_{34}\cos(g) \quad y_{O_{34}} = ya_3 - a_{34}\sin(g) \quad r_{34}^2 = \rho_3^2 - a_{34}^2 \tag{5}$$

$$n_{34} = -\cos(p_{34})\sin(g)\mathbf{i} + \cos(p_{34})\cos(g)\mathbf{j} + \sin(p_{34})\mathbf{k} \tag{6}$$

and

$$a_{56} = \frac{\rho_6^2 + ap^2 - \rho_5^2}{2ap} \tag{7}$$

$$x_{O_{56}} = xa_6 + a_{56}\cos(g) \quad y_{O_{56}} = ya_6 - a_{56}\sin(g) \quad r_{56}^2 = \rho_6^2 - a_{56}^2 \tag{8}$$

$$n_{56} = \cos(p_{56})\sin(g)\mathbf{i} + \cos(p_{56})\cos(g)\mathbf{j} + \sin(p_{56})\mathbf{k} \tag{9}$$

We may write then

$$\mathbf{OB_1} = \mathbf{OO_{12}} + r_{12}\mathbf{n_{12}} \tag{10}$$

$$\mathbf{OB_3} = \mathbf{OO_{34}} + r_{34}\mathbf{n_{34}} \tag{11}$$

$$\mathbf{OB_5} = \mathbf{OO_{56}} + r_{56}\mathbf{n_{56}} \tag{12}$$

We are able to express the norm of the vectors $\mathbf{B_1B_3}$, $\mathbf{B_1B}$, $\mathbf{B_3B}$ i.e. the distances between the articulation points of the mobile whose values are mp and m, m. This yields to the following three equations :

$$||B_1B_3||^2 - mp^2 = 0 \quad ||B_1B||^2 - m^2 = 0 \quad ||B_3B||^2 - m^2 = 0 \tag{13}$$

We get three equations which can be written as :

$$E_1\cos(p_{12}) + E_2\sin(p_{12}) + E_3 = 0 \tag{14}$$

$$F_1\cos(p_{34}) + F_2\sin(p_{34}) + F_3 = 0 \tag{15}$$

$$K_{11}\sin(p_{34})\sin(p_{12}) + (K_{21}\cos(p_{34}) + K_{22})\cos(p_{12}) + K_{32}\cos(p_{34}) + K_{33} = 0 \tag{16}$$

where the E_i, F_j coefficients does not depend upon the angles but only upon the three coordinates of B. Equations 14,16 are linear in term of $\sin(p_{12}), \cos(p_{12})$. We solve this linear system and write the equation $\cos(p_{12})^2 + \sin(p_{12})^2 = 1$ which yields:

$$(N_1 - N_2)\cos(p_{34})^2 + N_3 \sin(p_{34})\cos(p_{34})$$
$$+N_4 \sin(p_{34}) + N_5 \cos(p_{34}) + N_6 + N_2 = 0 \tag{17}$$

Then $\sin(p_{34})$ is determined using equation 15. If we put this value in equation 17 and write $\sin(p_{34})^2 + \cos(p_{34})^2 = 1$ we get two equations:

$$I_1 \cos(p_{34})^2 + I_2 \cos(p_{34}) + I_3 = 0 \tag{18}$$
$$H_1 \cos(p_{34})^2 + H_2 \cos(p_{34}) + H_3 = 0 \tag{19}$$

where the coefficients of I_i, H_j are function only of the coordinates of B. The orders of these coefficients are 4, 4, 4, 2, 3, 4. The equations 18, 19 yield to :

$$\begin{vmatrix} |I_1 H_2| & |I_1 H_3| \\ |I_1 H_3| & |I_2 H_3| \end{vmatrix} = 0 \tag{20}$$

where
$$|I_i H_j| = I_i H_j - I_j H_i \tag{21}$$

Using this method we get a sixteenth order polynomial. Its higher degree term is:
$$F_{21}^4 (Y^2 + X^2 + Z^2)^8 (N_{13} - N_{21})^2 \tag{22}$$

If $(N_{13} - N_{21})$ is equal to zero the mobile plate is reduced to a line. As for F_{21} it cannot be equal to zero. Therefore the circularity of the RSSR spin-surface is 8 and the conjecture of Hunt is verified. Thus there is at most 16 assembly-modes for the TSSM.

2.3 Determination of the polynomial

The principle of the determination of the polynomial is similar to the one used for the determination of the circularity of the RSSR spin surface.

We are able to express the norm of the vectors $\mathbf{B_1B_3}$, $\mathbf{B_1B_5}$, $\mathbf{B_3B_5}$ i.e. the distances between the articulation points of the mobile whose values are mp, mp et m as a function of the angles p_{12}, p_{34}, p_{56}. This yields to the following three equations :

$$||B_1B_3||^2 - mp^2 = 0 \quad ||B_1B_5||^2 - mp^2 = 0 \quad ||B_3B_5||^2 - m^2 = 0 \tag{23}$$

It is possible to show [14] that these equations may be written as :

$$K_{11}\sin(p_{34})\sin(p_{12}) + (K_{21}\cos(p_{34}) + K_{22})\cos(p_{12}) + K_{32}\cos(p_{34}) + K_{33} = 0 \tag{24}$$

$$L_{11}\sin(p_{56})\sin(p_{12}) + (L_{21}\cos(p_{56}) + L_{22})\cos(p_{12}) + L_{32}\cos(p_{56}) + L_{33} = 0 \tag{25}$$

$$M_{11}\sin(p_{34})\sin(p_{56}) + (M_{21}\cos(p_{34}) + M_{22})\cos(p_{56}) + M_{32}\cos(p_{34}) + M_{33} = 0 \tag{26}$$

where the coefficients K, L, M does not depend upon the angles p_{12}, p_{34}, p_{56}. We may notice that for a given set p_{12}, p_{34}, p_{56}, solution of these equations, the set $-p_{12}, -p_{34}, -p_{56}$ is also a solution. This mean simply that for a given position of the mobile plate the symmetrical position with respect to the base has the same links lengths.

Noticing that equations 24, 25 are linear in term of $\sin(p_{12}), \cos(p_{12})$ we solve this linear system and write the equation $\cos(p_{12})^2+\sin(p_{12})^2 = 1$ which has the following form:

$$(N_1 - N_2)\cos(p_{56})^2 + N_5\sin(p_{56}) + (N_3\sin(p_{56}) + N_4)\cos(p_{56}) + N_2 + N_6 = 0 \tag{27}$$

From equation 26 we get the value of $\sin(p_{56})$:

$$\sin(p_{56}) = -\frac{(M_{21}\cos(p_{34})+M_{22})\cos(p_{56})+M_{32}\cos(p_{34})+M_{33}}{M_{11}\sin(p_{34})} \tag{28}$$

From now on the process is similar to the one proposed by Nanua and Waldron. Charentus and Renaud [7][8] (in the case where the mobile plate is equilateral) have noticed that the coefficients N_3, N_5 can be written as :

$$N_3 = N'_3\sin(p_{34}) \qquad N_5 = N'_5\sin(p_{34})$$

The term $\sin(p_{34})$ being present in the denominator of $\sin(p_{56})$ we get a simplification when we use the value of $\sin(p_{56})$ in equation 27.

Thus equation 27 is written as :

$$I_1\cos(p_{56})^2 + I_2\cos(p_{56}) + I_3 = 0 \tag{29}$$

Using the equation $\sin(p_{56})^2 + \cos(p_{56})^2 = 1$ we get from equation 28:

$$H_1\cos(p_{56})^2 + H_2\cos(p_{56}) + H_3 = 0 \tag{30}$$

The I_i, H_j coefficients are second order polynomial in $\cos(p_{34})$ only. The equations 29, 30 yield to :

$$\begin{vmatrix} |I_1H_2| & |I_1H_3| \\ |I_1H_3| & |I_2H_3| \end{vmatrix} = 0 \tag{31}$$

where

$$|I_iH_j| = I_iH_j - I_jH_i \tag{32}$$

These terms are fourth order polynomials in $\cos(p_{34})$. Therefore equation 31 is a eighth order polynomial in $\cos(p_{34})$. If we define $x = tan(\frac{p_{34}}{2})$ we have $\cos(p_{34}) = \frac{1-x^2}{1+x^2}$, and we get from equation 31 a sixteenth order polynomial in x. In fact the odd power of x in this polynomial have zero as coefficient. Thus we have to solve only a eighth order polynomial (this means that if p_{34} is solution of the polynomial $-p_{34}$ is also a solution, as known from the beginning). From the determination of p_{34} it is easy to determine the value of p_{12}, p_{56} by following the process described by Nanua.

2.4 Example

We present here an example of a TSSM with 16 assembly-modes (i.e. the polynomial has 16 real roots). The positions of the articulation points are given in Table 1. The nominal position is: $x_0 = y_0 = 0, z_0 = 20, \psi =$

Table 1: Positions of the articulation points on the base and the mobile plate for the TSSM with 16 assembly-modes

base			mobile		
x_a	y_a	z_a	x	y	z
9.7	9.1	0.0	0.0	7.3	0.0
12.76	3.9	0.0	4.822	-5.480722	0.0
3.0	-13.0	0.0	4.822	-5.480722	0.0
-3.0	-13.0	0.0	-4.822	-5.480722	0.0
-12.76	3.9	0.0	-4.822	-5.480722	0.0
-9.7	9.1	0.0	0.0	7.3	0.0

$-10, \theta = -5, \phi = 10$ where ψ, θ, ϕ are the Euler's angles in degree. The equivalent configurations for which the mobile plate is over the base are given in Table 2.

We show in figure 5 the eight positions of the mobile plate for which the mobile plate is over the base (the 8 others configurations are the symmetrics with respect to the base of the drawn configurations).

Table 2: Position and orientation of the 16 assembly-modes of the TSSM

x_0	y_0	z_0	ψ	θ	ϕ
0.1099	-6.8071	15.1572	178.79	104.2473	-179.39757
0.0	0.0	20.0	170.000	4.3	-170.0
2.8029	-4.6660	12.7407	55.3895	89.1782	136.1996
1.3617	4.9038	17.3824	-106.3317	149.9318	58.9676
0.1606	5.3765	17.1868	-170.3808	164.0139	7.9545
-0.3525	-3.8663	11.9183	-12.5596	45.1107	-168.3013
-1.4134	4.8262	17.4299	102.6405	147.3844	-61.9768
-2.3355	-4.4679	12.5479	-50.8490	79.0396	-137.3533

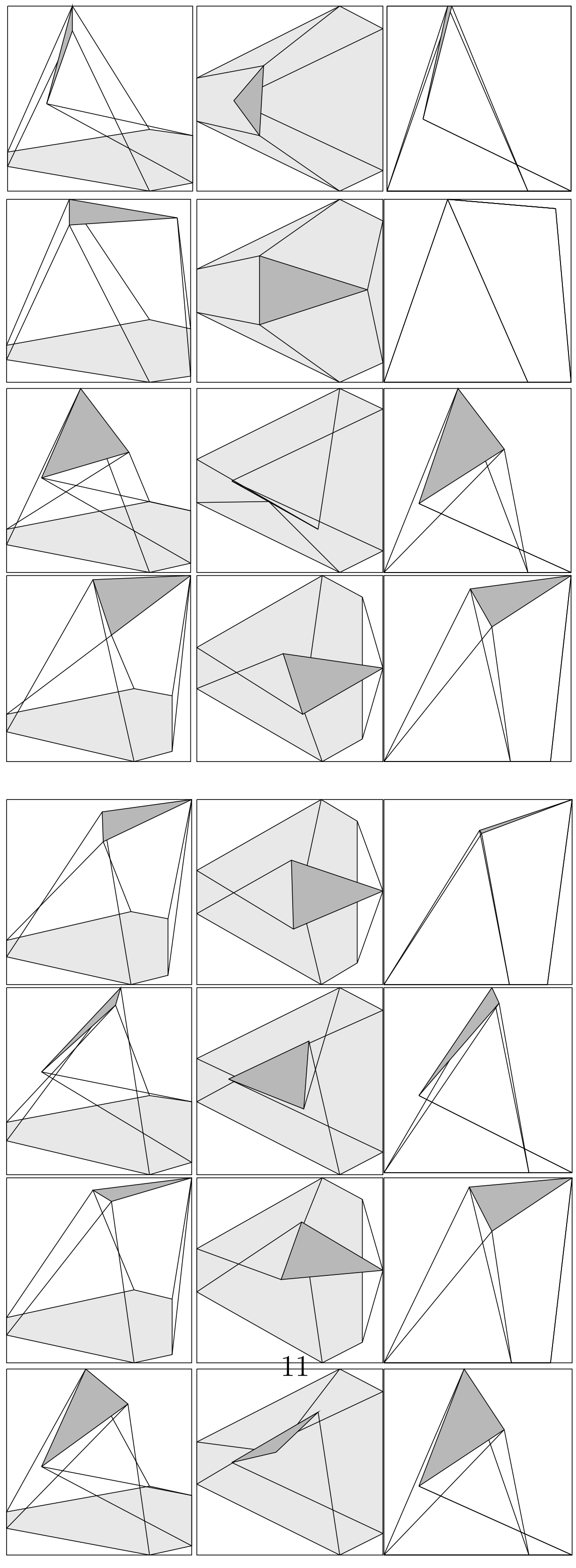

Figure 5: 8 over-the-base assembly-modes of the TSSM (perspective, top and side view)

3 Stewart Platform

This famous manipulator [15] is presented in figure 6. In this mechanism

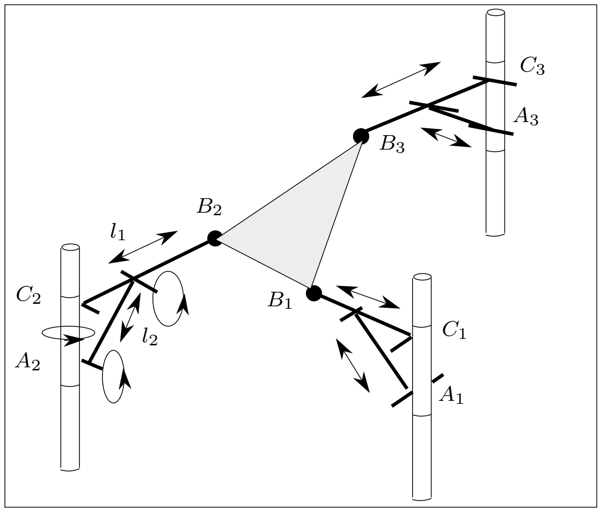

Figure 6: Stewart Platform

two rams are articulated through revolute joints on a beam which can rotate along a vertical axis. The other extremity of ram 1 is connected to the mobile plate and ram 2 enables to change the orientation of ram 1. For fixed lengths of the 6 rams the articulation points of the mobile plate can only rotate along a vertical axis around the beam center. Thus the equivalent mechanism of the Stewart Platform is described by figure 7. Using a similar method as for the TSSM we have been able to show that an UBAM is 16. But we may find only a direct kinematics polynomial of degree 20 and a numerical study enables to find only a configuration with 8 assembly-modes described in figure 8.

4 Conclusion

The direct kinematics problem is one of the most challenging problem of parallel manipulators. In a first part we show that there can be at most

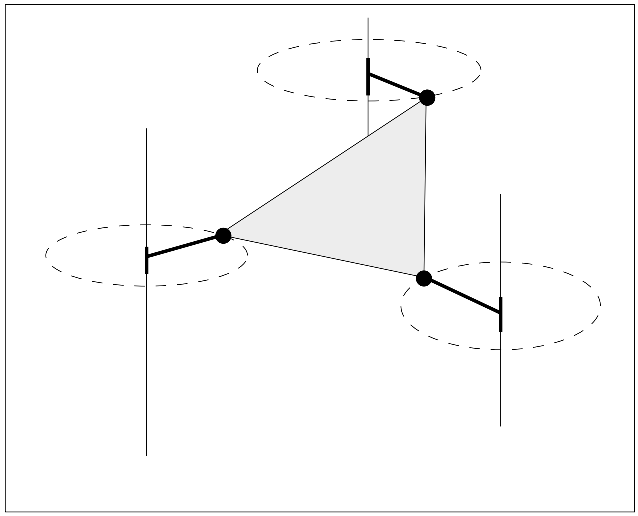

Figure 7: The equivalent mechanism of the Stewart Platform

sixteen different configurations of the manipulator for a given set of links lengths. Then we have demonstrated that in the general case it will not be possible to find an analytical solution to this problem because it is equivalent to solve a polynomial which order is 16 (or 20 for the Stewart Platform). The interest of the polynomial formulation of the direct kinematics problem is that it gives all the possible configurations of the manipulator. As for the numerical efficiency the resolution using the polynomial is very slow compared to others known methods based on an estimate of the solution [16] (we get at least a factor 10 on a Sun workstation). Thus this method can be useful only during the initialization process, where no estimate of the position of the mobile plate is known.

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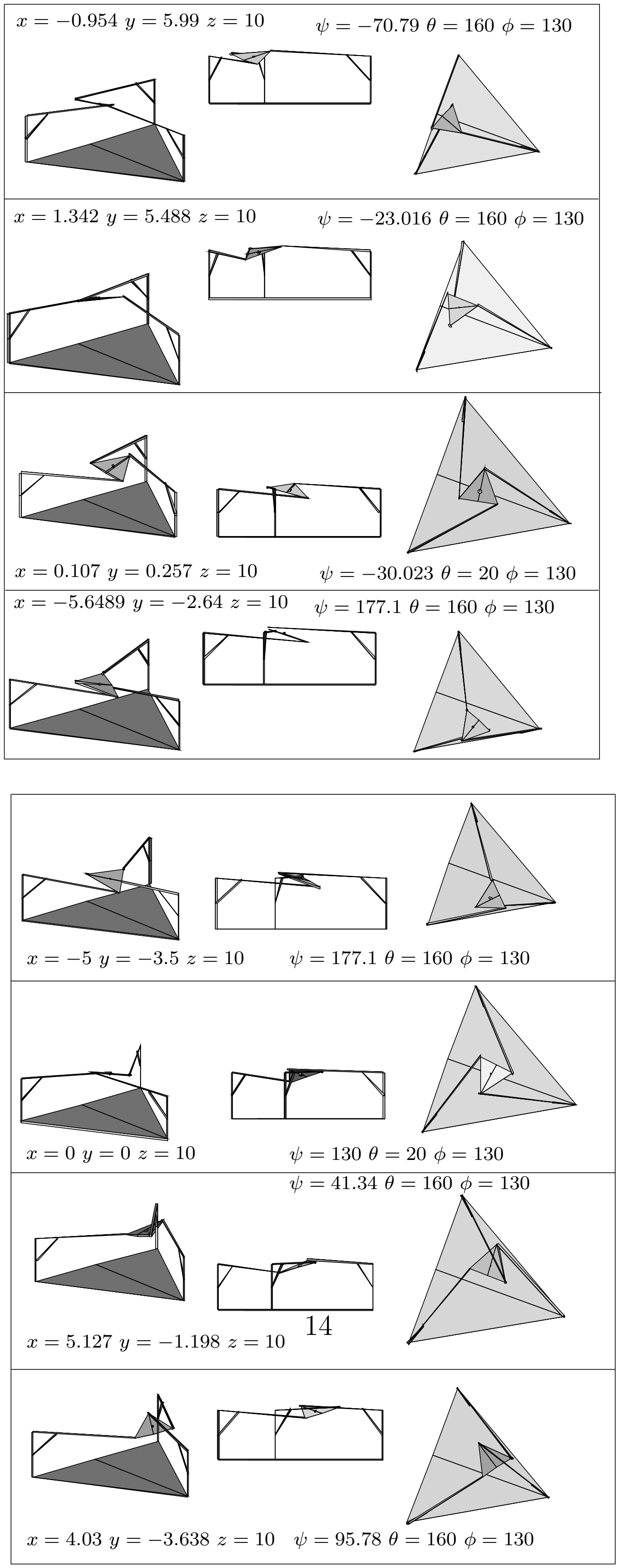

Figure 8: Stewart Platform: the eight assembly-modes (perspective, side and top view)

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