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ON THE SEPARABILITY OF THE SOLUTIONS OF THE DIRECT KINEMATICS OF A SPECIAL CLASS OF PLANAR 3-RPR PARALLEL MANIPULATOR

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ABSTRACT

Parallel manipulators have in general multiple solutions for the forward kinematics. In practice however from the control viewpoint only the current pose of the manipulator is of interest. We consider here a special class of planar parallel manipulator and explain how the solution corresponding to the current pose may be distinguished using a singularity criteria.

INTRODUCTION

We consider a special class of planar 3-RPR parallel manipulator as described in figure 1. The end-effector $B_1B_2B_3$

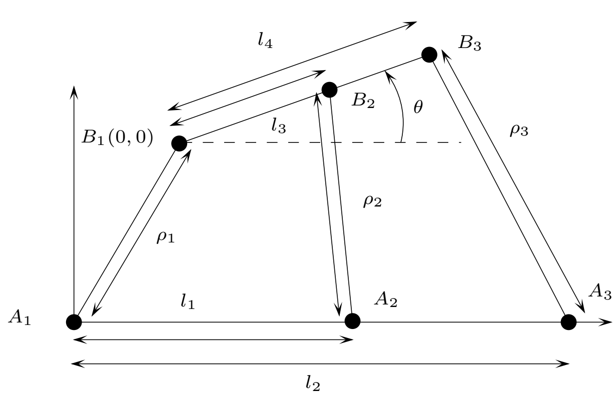

Figure 1. A special class of 3-RPR planar parallel manipulator

is connected to the ground through 3 legs with revolute joints at each extremity. A linear actuator enable to change the leg lengths, which in turn enable to control the position and orientation of the end-effector.

We define a reference frame $A_1, (x, y)$ and a mobile frame attached to the end-effector $B_1B_2B_3$ as $B_1, (x_r, y_r)$. The position and of the end-effector is defined by the coordinates (x, y) of B_1 in the reference frame and its orientation by the angle θ (assumed to lie in the range $[-\pi, \pi]$) between the x-axis of the reference frame and the line going through B_1, B_3. The lengths of the leg A_iB_i will be denoted by ρ_i.

DIRECT KINEMATICS

The problem of the direct kinematics is to determine the position and orientation of the end-effector being given the 3 leg lengths. For a general 3-RPR robot it is well known that this problem may have up to 6 solutions which may be obtained by solving a 6-th order polynomial (?; ?; ?). However for the special class we are considering this problem may be further simplified (?). First we may note that if a set (x_1, y_1, θ_1) is solution of the problem then the set $(x_1, -y_1, -\theta_1)$ is also a solution of the problem: pair of such solution will be denoted *symmetrical solutions*. Furthermore let us consider the equations of the inverse kinematics:

$$x^2 + y^2 = \rho_1^2 \quad (1)$$

$$l_1{}^2 - 2l_3\cos(\theta)l_1 + l_3{}^2 + (2l_3\cos(\theta) - 2l_1)\,x + 2yl_3\sin(\theta) + y^2 + x^2 = \rho_2^2 \quad (2)$$

$$l_2{}^2 - 2l_4\cos(\theta)l_2 + l_4{}^2 + (2l_4\cos(\theta) - 2l_2)\,x + 2yl_4\sin(\theta) + y^2 + x^2 = \rho_3^2 \quad (3)$$

By subtracting equation (1) to equations (2),(3) we get a linear system in x, y. Solving this system leads to the value of x, y as function of θ. Reporting these values in equation (1) leads to a 3-rd order polynomial in the unknown $\cos\theta$. Each solution of this equation leads to 2 solutions for the direct kinematics. Furthermore if we assume that the leg lengths correspond to a real configuration of the robot, then it may be shown that there will always be 4 solutions to the direct kinematics.

It must be noted that the above procedure cannot be used if $l_2l_3 - l_1l_4 = 0$. Indeed under this assumption the equations of the linear system are not independent. In that case equation (2-1) is used to determine the value of y as function of x and θ which is substituted in (3-1) which leads to an equation in $\cos\theta$ only. Solving this equation and reporting the value of $y, \cos\theta$ in (1) leads to a second order polynomial in the unknown x.

SINGULARITY

The inverse jacobian matrix J^{-1} of the robot is constituted of three rows J_i^{-1} with:

$$J_i^{-1} = (\frac{A_iB_i}{\rho_i} \quad B_1B_i \times \frac{A_iB_i}{\rho_i})$$

We may also define the semi-inverse jacobian matrix J_s^{-1} which is obtained by multiplying each row J_i of J^{-1} by the leg length ρ_i. We have:

$$|J^{-1}| = \frac{|J_s^{-1}|}{\rho_1\rho_2\rho_3}$$

and consequently $|J^{-1}|, |J_s^{-1}|$ have same sign and vanish at the same point. As $|J_s^{-1}|$ has a slightly less complex formulation than $|J^{-1}|$ we will mostly use J_s^{-1}. Let us now examine the singularity condition for this robot, which is obtained by equating the determinant of the inverse jacobian matrix to zero :

$$\begin{aligned}
&\left(l_3\ (\sin(\theta))^2\, l_4\, l_2 - l_4\ (\sin(\theta))^2\, l_3\, l_1\right) x \\
&+(-l_3\ \cos(\theta) l_4\ \sin(\theta) l_2 + l_1\, l_4\ \sin(\theta) l_2 \\
&+l_4\ \cos(\theta) l_3\ \sin(\theta) l_1 - l_2\, l_3\ \sin(\theta) l_1) y \\
&+(l_1\, l_4\ \cos(\theta) - l_2\, l_3\ \cos(\theta)) y^2 \\
&+(-l_1\, l_4\ \sin(\theta) + l_2\, l_3\ \sin(\theta)) xy = 0
\end{aligned}$$

For a given θ this equation define an hyperbola (**?**) whose asymptotes are the horizontal line defined by:

$$y = \frac{l_3l_4\sin\theta(l_1 - l_2)}{l_2l_3 - l_1l_4} \tag{4}$$

and the line with slope $\tan\theta$ defined by:

$$y = \tan\theta\ x - \frac{\tan\theta l_2l_1(l_3 - l_4)}{l_2l_3 - l_1l_4} \tag{5}$$

Two special cases may occur:

- if $\theta = 0$ the hyperbola is reduced to the line $y = 0$
- if $l_2l_3 - l_1l_4 = 0$ the hyperbola is reduced to the line $y = \sin\theta l_3\ x/(l_3\cos\theta - l_1)$ while the determinant is

$$\sin(\theta)\frac{l_1l_4}{l_3}(xl_3\sin\theta(l_4 - l_3) + y(l_3\cos\theta(l_3 - l_4) + l_1(l_4 - l_3))$$

Let $D(x, y, \theta)$ be the determinant of the inverse jacobian matrix: it must be noted that $D(x, y, \theta) = D(x, -y, -\theta)$. As a consequence the value of the determinant for the symmetrical solutions of the direct kinematics will be identical.

SEPARABILITY OF THE SOLUTIONS

Finding all the solutions of the direct kinematics does not exactly provide an answer to the practical problem: indeed for control purposes we need to determine only the *current pose* of the end-effector. In order to determine this pose we may use the following informations:

- the end-effector cannot cross a singularity
- we know the initial assembly mode of the robot i.e. the pose x_i, y_i, θ_i of the end-effector when the robot was initially assembled
- we know a neighborhood in which the current pose should be located. Indeed we may assume that the pose of the end-effector has to be determined at each sampling time of the controller. Being given the maximal velocity v_x, v_y, ω of the end-effector, the sampling time Δt of the controller and the pose x_0, y_0, θ_0 at the previous sampling time we know that the pose of the end-effector should be located in the ball $x_0 \pm v_x\Delta t$, $y_0 + v_y\Delta t$, $\theta_0 + \omega\Delta t$.

Although it is well known that different solutions of the direct kinematics may be joined by a singularity-free trajectory (**?**; **?**; **?**) we want to investigate if we may separate the different solutions of the direct kinematics and determine which of them may be the current pose under the above assumptions.

Special case $l_2l_3 - l_1l_4 = 0$

We have seen in the direct kinematics section that the solutions of the direct kinematics can be obtained by first determining the unique value of $\cos\theta$ which is only a function of the leg lengths and then solving a second order polynomial in x:

$$Ux^2 + Vx + W = 0 \tag{6}$$

where U, V, W are only functions of the leg lengths. Being given the real roots x_1, x_2 of this polynomial and the value θ_1 of θ we may then compute the value of y. The solutions of the direct kinematics are therefore $x_1, y(x_1), \theta_1$, $x_1, -y(x_1), -\theta_1$, $x_2, y(x_2), \theta_1$, $x_2, -y(x_2), -\theta_1$. We may also plug in the value of $y, \cos\theta$ in the determinant of the inverse jacobian matrix and get its value as function of x and the leg lengths. This leads to a linear expression in x:

$$|J^{-1}| = Ax + B \tag{7}$$

with $A = U/l_1$, $B = V/(2l_1)$. As a consequence the determinant will be 0 for $x = -V/(2U)$ and the line $x = -V/(2U)$ split the $x - y$ plane into two half-plane in which the determinant of the inverse jacobian has opposite sign. Let Δ be the discriminant of equation (6). The roots of this equation may therefore be written as:

$$x_1 = \frac{-V - \sqrt{\Delta}}{2U} \quad x_2 = \frac{-V + \sqrt{\Delta}}{2U} \tag{8}$$

Consequently all the solutions of the direct kinematics which have x_1 as value for x will have a determinant of opposite sign from the determinant of the solutions which have x_2 of value for x. Being given the sign of the determinant of the inverse jacobian matrix for the initial assembly mode we may determine the only two possible solutions of the direct kinematics which have the same sign for the determinant. Furthermore we notice that the determinant has $\sin\theta$ as factor: consequently there is no singularity-free trajectory between two poses that have opposite value for the orientation angle. As the two solutions that have the same sign for the determinant have also opposite value for θ, then they are always separated by a singularity surface.

In summary we are able to find the current pose by using the following algorithm:

- compute the 4 solutions of the direct kinematics
- retain the 2 solutions whose determinant of the inverse jacobian matrix has the same sign than for the initial assembly mode x_i, y_i, θ_i
- among these two solutions the current pose is the one which has the same sign for $\sin\theta$ than $\sin\theta_0$.

GENERAL CASE

We will restrict our study to the case where $l_2l_3 - l_1l_4 > 0$, as the result for the other case can be established in the same way.

For a given θ the singularity hyperbola $\mathcal{H}(\theta)$ split the $x - y$ plane into two regions $\mathcal{H}_-, \mathcal{H}_+$, with opposite sign for $|J^{-1}|$ as shown in figure 2.

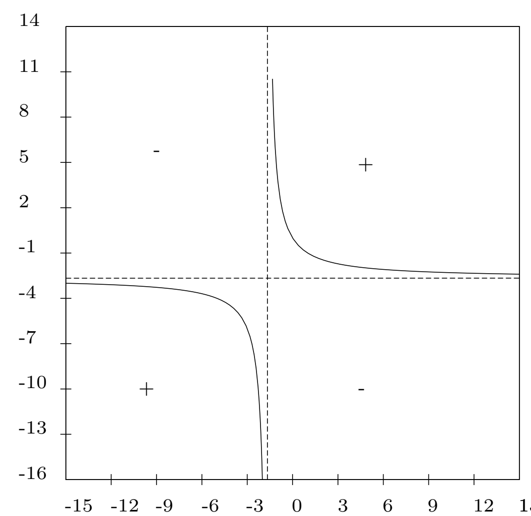

Figure 2. Singularity hyperbola and the sign of $|J^{-1}|$ (for $l_1 = 1, l_2 = 5, l_3 = 1, l_4 = 2$ and $\theta = \pi/2$

Singularity-free trajectory between symmetrical solutions

We have seen that symmetrical solutions $X_1(x_1, y_1, \theta_1), X_2$ have the same x value, opposite y and θ, and same sign for $|J^{-1}|$. It must also be noted that we have:

$$\theta = 0 \quad \Rightarrow |J^{-1}| = -y^2(l_2l_3 - l_1l_4)$$
$$\theta = \pi \quad \Rightarrow |J^{-1}| = y^2(l_2l_3 - l_1l_4)$$

Consequently a singularity-free trajectory from a solution X_1 to its symmetrical X_2 must be such that at some point the trajectory angle must go through $\theta = 0$ if $|J^{-1}(X_1)| < 0$ or $\theta = \pi$ if $|J^{-1}(X_1)| > 0$. Now it is easy to find a singularity-free trajectory between two poses in a symmetrical solutions:

- if $|J^{-1}(X_1)| < 0$: without changing the orientation of the end-effector moves the end-effector from (x_1, y_1) to the intersection point P of the asymptotes of the hyperbola $\mathcal{H}(\theta_1)$, then from P to $X_2 = (x_1, -y_1)$. Then, without moving the origin of the end-effector, if $\theta_1 > 0$ rotate counter-clockwise to reach $-\theta_1$ or if $\theta_1 < 0$ rotate clockwise to reach $-\theta_1$ (figure 3). The translation motion from X_1 to X_2 ensure that the end-effector at X_2 lie in the same component $\mathcal{H}_x$ than X_1. Then the rotation keep the sign of the determinant identical.

- if $|J^{-1}(X_1)| > 0$: let D be the line going through the intersection point P of the asymptotes with a slope $\tan(\theta/2)$ and $d(\theta)$ being the distance between P and the hyperbola. Without changing the orientation of the end-effector go from (x_1, y_1) to the closest point M on D such that the distance between P and M is $d(\theta) + \epsilon$, ϵ being an arbitrary positive constant. Then we will combine a rotation motion of the end-effector with a translation motion which ensure that the origin of the end-effector remains on D and at a distance $d(\theta) + \epsilon$ from P. If $\theta_1 > 0$ the rotation motion is clockwise, goes through π and end at $-\theta_1$. If $\theta_1 < 0$ the rotation motion is counter-clockwise. As soon as we have have reached the orientation $-\theta_1$ we move M to X_2 (figure 4). This motion ensure that the end-effector remains in the same $\mathcal{H}_x$ than X_1.

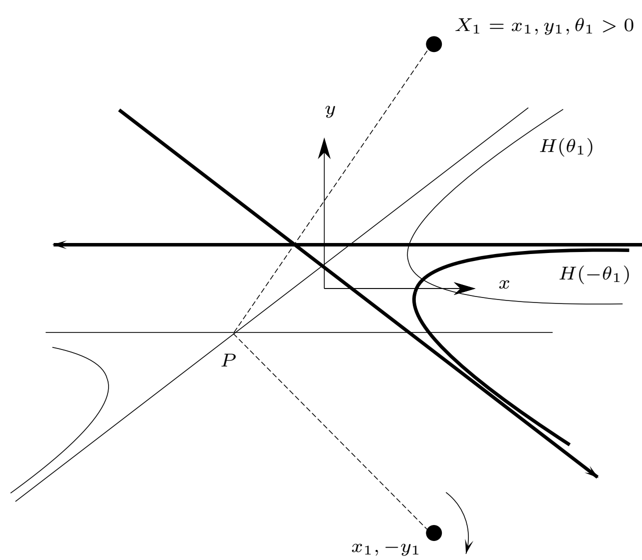

Figure 3. A singularity-free trajectory between two symmetrical solutions X_1, X_2 if $|J^{-1}(X_1)| < 0$. Without changing the orientation of the end-effector we move its origin from X_1 to P, the intersection point of the asymptotes of the hyperbola $\mathcal{H}(\theta_1)$, then from P to X_2. We rotate then counter-clockwise which transform the hyperbola $\mathcal{H}(\theta_\infty)$ into the hyperbola $\mathcal{H}(-\theta_1)$, $|J^{-1}(X_2)|$ remaining negative during this transformation.

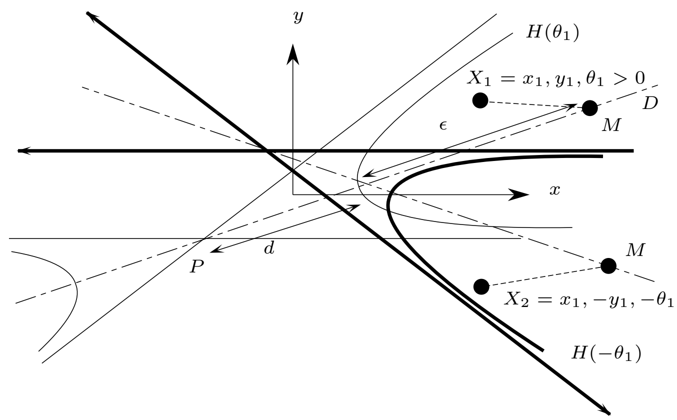

Figure 4. A singularity-free trajectory between two symmetrical solutions X_1, X_2 if $|J^{-1}(X_1)| > 0$. Without changing the orientation of the end-effector we move its origin from X_1 to the closest point M on the line D (which goes through P and has a slope $\tan(\theta_1/2)$) which is at a distance $d(\theta_1) + \epsilon$ from P. Then we rotate clockwise the end-effector while maintaining M on D and at a distance $d(\theta) + \epsilon$ from P until we reach the angle $-\theta_1$. Then we move M to X_2. $|J^{-1}(X_2)|$ remains positive during this transformation.

Separability between non-symmetrical solutions

We have seen that the forward kinematics may be solved by deriving a 3rd order polynomial P_k in $X = \cos\theta$. Using the expression of x, y as function of θ we may transform the singularity condition into a fourth order polynomial P_s in X. For the polynomial P_k we have

$$P_k(-1) > 0 \quad P_k(1) > 0 \tag{9}$$

Hence this polynomial may have 0 or 2 real roots in the interval $[-1, 1]$. As we assume that their is at least one assembly mode in the interval we deduce that we have two real roots $X_1, X_2 \geq X_1$ in this interval. Furthermore as the leading term of P_k is

$$U = -8 l_3 l_1 l_4 l_2 (l_1 - l_2)(l_3 - l_4)$$

we may deduce from equation (9) that the third root (which is clearly real) will be lower than -1 if $U > 0$ and greater than 1 if $U < 0$. Indeed if $U > 0$, then $P_k(+\infty) \to +\infty$: if the third root was greater than 1 and as $P_k(1) > 0$, then we must have two real roots in $[1, +\infty]$, and a total of 4 real roots for a third order polynomial, which is impossible. A similar reasoning can be made for $U < 0$.

We will now use Sturm theorem to determine the number of real root of P_s in the interval $[-1, 1]$. Sturm sequences for a n-th order polynomial $P(X)$ is defined as a sequence of functions $f_0, f_1, \ldots f_n$ such that $f0 = P$, $f_1 = P'$ where P' is the derivative of P. The remaining function are defined by the sequence:

$$f_{i+1} = -rem(f_{i-1}, f_i)$$

Thus f_{i+1} is the opposite of the remainder of the division of f_{i-1} by f_i. If the polynomial is of order n we will get $n+1$ f functions, the last function f_n being a constant i.e has a constant sign. We define as $N(x)$ the number of change of sign between two successive f_i considered at $X = x$. Sturm theorem states that the number of real roots of P in the interval $[a, b]$, counted with their order of multiplicity, is $N(a) - N(b)$. For the polynomial P_s we have:

$$\begin{array}{lll} f_0(-1) > 0 & f_1(-1) < 0 & f_2(-1) < 0 \\ f_0(1) < 0 & f_1(1) < 0 & f_2(1) > 0 \end{array}$$

and thus $N(-1) \geq 1$ and $N(1) \geq 1$. Remember that the last function f_4 in the Sturm sequence has a constant sign. So the only possible sign combination for the last elements f_3, f_4 of the sequence are (+,+),(-,+),(+,-),(-,-). Consider, for example, that the sign sequence for $f_3(-1)$ and $f_4(-1)$ is (+,+). Then

$$\begin{array}{l} \text{if} \quad f_3(-1) = + \quad f_4(-1) = + \Rightarrow N(-1) = 2 \\ \text{then} \quad f_3(1) = - \quad f_4(1) = + \Rightarrow N(1) = 3 \\ \text{or} \quad f_3(1) = + \quad f_4(1) = + \Rightarrow N(1) = 1 \end{array}$$

Thus there is one real root in the range [-1,1]. If we plug these other possible sign combinations in the sequence we will always find the same result. Therefore P_s has only one real root X_3 in the interval [-1,1]. We will now determine the location of X_3 with respect to the interval $[X_1, X_2]$. Indeed if we may prove that X_3 always lie inside this interval, then the non-symmetrical solutions will be separated by a singularity. We will first show that it is not possible to find a sequence of leg lengths such that the initial X_3 is outside $[X_1, X_2]$, then become coincident with either X_1 or X_2 and then lie inside $[X_1, X_2]$. This is simply done by reminding that the inverse jacobian is the mathematical jacobian of the kinematics equations; when its determinant vanished there is a multiple root for P_k and $X_1 = X_2$. Hence for a leg lengths sequence that does not lead to a singularity the root X_3 has always a fixed position with respect to the interval $[X_1, X_2]$: if for a fixed value of the leg lengths X_3 is inside the interval, then for any leg lengths it will remain inside the interval.

We have now to study what is occurring when crossing a singularity and in the neighbor of a singularity. Let θ_1 such that $P_s(cos\theta_1) = 0$. In the neighborhood of the singularity we have:

$$\begin{aligned} P_s(\cos(\theta_1 + \epsilon)) &= P_s((cos\theta_1)) + \frac{dP_s}{d(\cos\theta)}(\cos(\theta_1))\ \epsilon + \ldots \\ &= \frac{dP_s}{d(\cos\theta)}(\cos(\theta_1))\ \epsilon + \ldots \end{aligned}$$

Hence if the derivative D of P_s at θ_1 does not vanish, then the determinant of the inverse jacobian for the two roots X_1, X_2 will have opposite sign and hence X_3 must lie in the interval $[X_1, X_2]$. Assume now that $D = 0$: this means that P_s has a double root in the interval [-1,1] which is clearly in contradiction with the result of Sturm theorem.

Hence we have shown that in the neighborhood of a singularity X_3 lie in the range $[X_1, X_2]$ and as we have already shown that X_3 is always in the same position with respect to this range we may conclude that X_3 is always in the range. In summary the non-symmetrical solutions of the forward kinematics are always separated by a singularity.

Separability

Being given the sign of the determinant of the inverse jacobian matrix at the initial assembly mode we may now determine the 2 solutions that have same sign for the determinant, these solutions being symmetrical. In a real-time context we may also have the latest known position $X_l = (x_l, y_l, \theta_l)$ of the platform and the elapsed time Δt between the time at which the end-effector was located at X_l and the current time. Being given the maximal velocity v_x, v_y, ω of the manipulator we are able to determine the range in which should lie the current posture $X_c = (x, y, \theta)$:

$$x \in [x_l \pm v_x \Delta t] \quad y \in [y_l \pm v_y \Delta t] \quad \theta \in [\theta_l \pm \omega \Delta t]$$

If one solution is outside this range then we have determined the current posture. Otherwise there is no singularity-based way to distinguish between the solutions and additional information is needed.

EXAMPLE

In this example we have chosen:

$$l_1 = 1 \quad l_2 = 5 \quad l_3 = 1 \quad l_4 = 2$$

and

$$\rho_1 = 3.5 \quad \rho_2 = 2 \quad \rho_3 = 4$$

Figure 5 presents the 4 possible solutions together with their value for J_s^{-1}. It may be seen that the 4 solutions verify

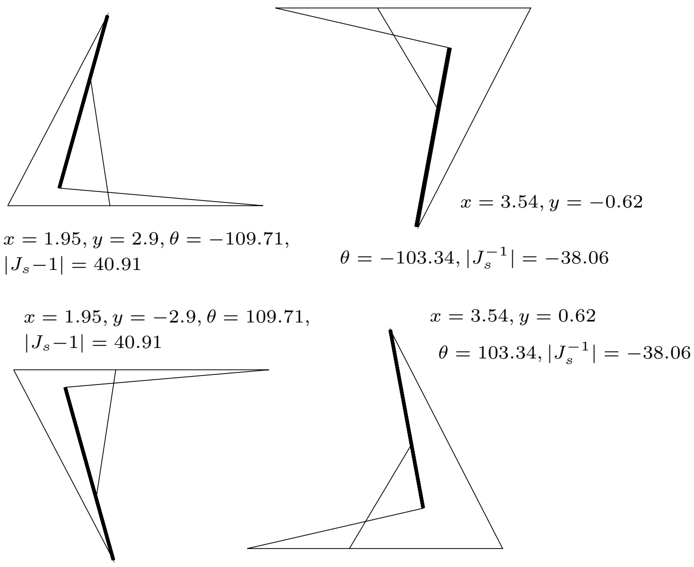

Figure 5. Four solutions for the forward kinematics

the theorems presented in the previous sections.

CONCLUSION

It is now well known that sorting the forward kinematics solution using only the singularity condition is not possible. Still we have shown that, in a special case, singularity may be useful to determine the current pose of the robot among all the possible solutions of the forward kinematics.