Direct Geometrico-static Problem of Under-Constrained Cable-Driven Parallel Robots with Three Cables

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***Abstract*— This paper studies the kinematics and statics of under-constrained cable-driven parallel robots with three cables. A major challenge in the study of these robots is the intrinsic coupling between kinematics and statics, which must be dealt with simultaneously. This paper provides a general procedure that solves, in analytical form, the direct geometrico-static problem, which consists in determining the platform posture and the cable tensions once the cable lengths are assigned. The problem is proven to have up to 156 complex solutions.**

I. Introduction

Cable-driven parallel robots (CDPRs) employ cables in place of rigid-body extensible legs in order to control the end-effector posture. A CDPR is referred to as *fully-constrained* if the posture of the end-effector is completely determined when actuators are locked and, thus, all cable lengths are assigned [1]. The minimum number of cables that are necessary to fully control the output motion is equal to the number f of degrees of freedom (dofs) that the end-effector possesses with respect to the base. However, since cables may only exert tensile axial forces, a redundancy of control actions is usually necessary in order to guarantee that no cable becomes slack and, thus, full control is preserved for a generic loading condition [1], [2]. A CDPR is defined, instead, as *under-constrained* if the end-effector preserves some freedoms once actuators are locked and cable lengths are fixed [1]. Typically, this occurs when the end-effector is controlled by a number of cables smaller than f. The employ of CDPRs with a limited number of cables is justified in a number of applications (such as, for instance, rescue, service or rehabilitation operations), in which a limitation of dexterity is acceptable in order to decrease complexity, cost, set-up time, likelihood of cable interference, etc. It must also be observed that a theoretically fully-constrained CDPR operates, in considerable parts of its geometric workspace, as an under-constrained robot, namely when a full restraint may not be achieved because it would require a negative tension in one or more cables.

The above considerations motivate a careful study of under-constrained CDPRs. However, while fully-constrained robots have been extensively investigated [1]–[15], little attention has been dedicated to under-constrained ones [16]–[23]. When a fully-constrained CDPR operates in the portion of its workspace in which the required set of output wrenches is guaranteed to be applicable with purely tensile cable forces [10]–[13], [15], the posture of the end-effector is determined, in a purely geometrical way, by assigning cable lengths. Conversely, for an under-constrained CDPR, when the actuators are locked and the cable lengths are assigned, the end-effector is still movable, so that the actual configuration is determined by the applied forces. As a consequence, the end-effector posture depends on both cable lengths and equilibrium equations, and kinematics and statics (or dynamics) must be dealt with simultaneously. Moreover, as the pose depends on the applied load, it may change due to external disturbances, so that it is important to investigate equilibrium stability.

In [22], [23], a general methodology was proposed for the kinematic, static and stability analysis of general under-constrained nn-CDPRs, namely parallel robots in which a fixed base and a mobile platform are connected to each other by n cables, with $n \leq 5$ and the anchor points on the base and the platform being distinct. In particular, a procedure was provided aimed at effectively solving, in analytical form, the inverse and direct position problems, namely, at finding the overall robot configuration and cable tensions when, respectively, either n platform posture coordinates or the n cable lengths are given, under the assumptions that a constant force is applied on the platform, cables are inextensible and massless, and interference problems are disregarded.

In a companion paper [24], the aforementioned methodology is applied to the inverse geometrico-static problem (IGP) of the general 33-CDPR. In this paper, the direct geometrico-static problem (DGP) of the 33-CDPR is, instead, tackled. The challenge consists in determining the platform posture and the cable tensions once the cable lengths are assigned. Section II presents the robot model. Sections III and IV formulate the geometrical and statical equations that govern the problem, whereas Section V provides the detailed procedures that solve it in analytical form. In Section VII, the main achievements of the paper are discussed.

II. Geometrico-static model

A mobile platform is connected to a fixed base by three cables and is acted upon by a constant force $Q\mathcal{L}_e$ applied on a point G, e.g. the platform weight acting through its center of mass (Fig. 1). Q is the magnitude of the force, whereas $\mathcal{L}_e$ is the six-dimensional vector grouping the normalized Plücker coordinates of its line of action. $Oxyz$ is a Cartesian

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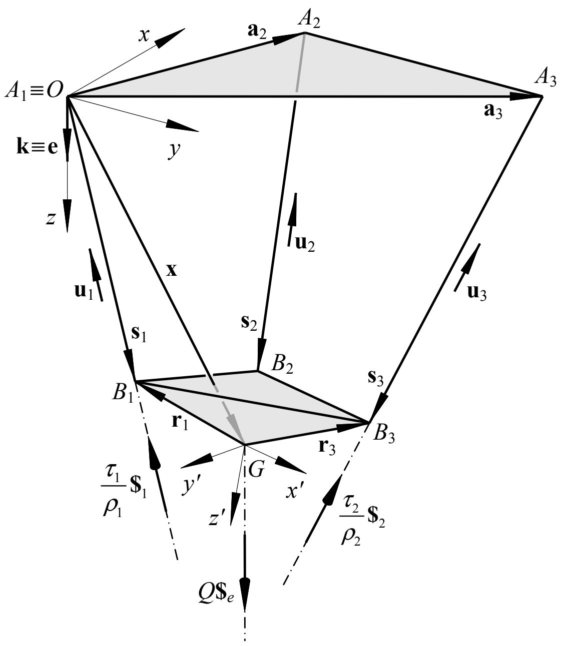

Fig. 1. Geometric model of a cable-driven parallel robot with three cables.

coordinate frame fixed to the base, with **i**, **j** and **k** being unit vectors along the coordinate axes, whereas $Gx'y'z'$ is a Cartesian frame attached to the end-effector. The platform posture is described by $\mathbf{X} = (\mathbf{x}; \boldsymbol{\Phi})$, where $\mathbf{x} = G - O$ and $\boldsymbol{\Phi}$ groups the variables parameterizing the platform orientation with respect to $Oxyz$. If Rodrigues parameters are adopted, i.e. $\boldsymbol{\Phi} = [e_1; e_2; e_3]$, the rotation matrix $\mathbf{R}(\boldsymbol{\Phi})$ between the mobile and the fixed frame is given by

$$\mathbf{R} = \mathbf{I}_3 + 2\frac{\tilde{\boldsymbol{\Phi}} + \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}}{1 + e_1^2 + e_2^2 + e_3^2}, \tag{1}$$

where $\tilde{\boldsymbol{\Phi}}$ denotes the skew-symmetric matrix expressing the operator $\boldsymbol{\Phi}\times$. For the generic ith cable, A_i and B_i are, respectively, the anchor points on the base and the platform, ρ_i is the cable length, $\mathbf{a}_i = A_i - O$, $\mathbf{r}_i = B_i - G$, $\mathbf{s}_i = B_i - A_i$ and $\mathbf{u}_i = (A_i - B_i)/\rho_i = -\mathbf{s}_i/\rho_i$. For apparent reasons, ρ_i is assumed to be strictly positive, so that $\mathbf{s}_i \neq \mathbf{0}$. If $\mathbf{b}_i$ is the projection of $B_i - G$ on $Gx'y'z'$, then $\mathbf{r}_i = \mathbf{R}(\boldsymbol{\Phi})\,\mathbf{b}_i$. $(\tau_i/\rho_i)\,\mathcal{L}_i$ is the force exerted by the ith cable on the platform, with τ_i being the cable tension and $\mathcal{L}_i/\rho_i$ the normalized Plücker vector of the cable line. Without loss of generality, O is chosen to coincide with A_1 (so that $\mathbf{a}_1 = \mathbf{0}$) and $\mathbf{k} = \mathbf{e}$, with $\mathbf{e}$ being a unit vector directed as $\mathcal{L}_e$. For the sake of brevity, the components of **x** in $Oxyz$ are denoted as x, y and z. Finally, vector components along the coordinate axes are denoted by right subscripts reporting the axes names.

III. Geometrical Constraints

When cable lengths are assigned, the set $\mathcal{C}$ of the theoretical restraints imposed by the cables on the platform comprises 3 relations in **X**, i.e.

$$|\mathbf{s}_i| = \sqrt{\mathbf{s}_i \cdot \mathbf{s}_i} = \rho_i, \quad i = 1 \ldots 3, \tag{2}$$

where

$$\mathbf{s}_i = \mathbf{x} + \mathbf{R}\mathbf{b}_i - \mathbf{a}_i. \tag{3}$$

By subtracting the first one from the second and the third one, and by clearing the denominator $1 + e_1^2 + e_2^2 + e_3^2$, the following relations are obtained:

$$\begin{aligned} q_1 := H_{200}x^2 + H_{020}y^2 + H_{002}z^2 + H_{100}x \\ + H_{010}y + H_{001}z + H_{000} = 0, \end{aligned} \tag{4a}$$

$$q_2 := I_{100}x + I_{010}y + I_{001}z + I_{000} = 0, \tag{4b}$$

$$q_3 := K_{100}x + K_{010}y + K_{001}z + K_{000} = 0, \tag{4c}$$

where all coefficients H_{kmn}, I_{kmn} and K_{kmn} are quadratic functions of e_1, e_2 and e_3.

IV. Statical Constraints

The platform equilibrium may be written as

$$\underbrace{\begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_e \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} (\tau_1/\rho_1) \\ (\tau_2/\rho_2) \\ (\tau_3/\rho_3) \\ Q \end{bmatrix} = \mathbf{0}, \tag{5}$$

with

$$\tau_i \geq 0, \quad i = 1 \ldots 3. \tag{6}$$

Equations (5) amount to 6 scalar relations involving 9 variables, namely $\mathbf{x}$, $\boldsymbol{\Phi}$ and τ_i, $i = 1 \ldots 3$. Following [22], cable tensions may be eliminated by observing that Eq. (5) holds only if

$$\operatorname{rank}(\mathbf{M}) \leq 3, \tag{7}$$

namely if $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_e$ are linearly dependent. This is a purely geometrical condition, since **M** is a 6×4 matrix only depending on the platform posture. By setting all 4×4 minors of **M** equal to zero, a set of 15 scalar relations that do not contain cable tensions may be obtained[1].

If O is chosen as the reduction pole of moments, $\mathcal{L}_i$ and $\mathcal{L}_e$ may be respectively expressed, in axis coordinates, as $-[\mathbf{s}_i;\ \mathbf{a}_i \times \mathbf{s}_i]$ and $[\mathbf{e};\ \mathbf{x} \times \mathbf{e}]$, so that **M** becomes

$$\mathbf{M}(O) = \begin{bmatrix} -\mathbf{s}_1 & -\mathbf{s}_2 & -\mathbf{s}_3 & \mathbf{e} \\ \mathbf{0} & -\mathbf{a}_2 \times \mathbf{s}_2 & -\mathbf{a}_3 \times \mathbf{s}_3 & \mathbf{x} \times \mathbf{e} \end{bmatrix}. \tag{8}$$

The equations[2]

$$p_1 := \det \mathbf{M}_{1236}(O) = 0, \tag{9a}$$

$$p_2 := \det \mathbf{M}_{1235}(O) = 0, \tag{9b}$$

$$p_3 := \det \mathbf{M}_{1234}(O) = 0, \tag{9c}$$

[1]In very special conditions, Eq. (7) is fulfilled because $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ become linearly dependent. In this case, equilibrium is possible only if $\operatorname{rank}(\mathbf{M}) \leq 2$, for, in any case, the external wrench must belong to the subspace generated by cable lines. Cases like the ones mentioned here, however, are sufficiently unlikely to occur not to be, in practice, of particular concern.

[2]The notation $\mathbf{M}_{hij,klm}$ denotes the block matrix obtained from rows h, i and j, and columns k, l and m. When all columns of **M** are used, the corresponding subscripts are omitted.

comprise the lowest-degree polynomials in $\mathbf{X}$ among all minors of $\mathbf{M}(O)$. They are of degree 4 in the Rodrigues parameters and degree 2 in the components of $\mathbf{x}$, thus being of degree 6 in $\mathbf{X}$. All other minors have degree ranging from 7 to 9 in $\mathbf{X}$.

An additional sextic relation in $\mathbf{X}$ emerges by setting $\det \mathbf{M}_{j456}(O) = 0$ for $j = 1 \ldots 3$, so that

$$\mathbf{s}_1 \cdot \det \mathbf{M}_{456,234}(O) = \mathbf{0}, \tag{10}$$

and thus, since $\mathbf{s}_1 \neq 0$,

$$p_4 := \det \mathbf{M}_{456,234}(O) = 0. \tag{11}$$

Equation (11) is, indeed, of degree 4 in $\mathbf{\Phi}$, degree 2 in $\mathbf{x}$ and degree 6 in $\mathbf{X}$.

For the purpose of this paper, it is worth deriving as many independent lowest-degree equations in $\mathbf{X}$ as possible. Further sextics may be obtained as follows. Let $\mathbf{M}$ be written by choosing a generic P as the reduction pole of moments, namely as

$$\mathbf{M}(P) = \begin{bmatrix} \cdots & \mathbf{s}_i & \cdots & \mathbf{e} \\ \cdots & (B_i - P) \times \mathbf{s}_i & \cdots & (G - P) \times \mathbf{e} \end{bmatrix}. \tag{12}$$

When $P \equiv B_i$ or $P \equiv A_i$, $i = 1 \ldots 3$, the moment vector in the ith column vanishes, so that setting $\det \mathbf{M}_{j456}(B_i) = 0$ or $\det \mathbf{M}_{j456}(A_i) = 0$ for $j = 1 \ldots 3$ yields, respectively,

$$\mathbf{s}_i \cdot \det \mathbf{M}_{456,km4}(B_i) = \mathbf{0}, \tag{13}$$

or

$$\mathbf{s}_i \cdot \det \mathbf{M}_{456,km4}(A_i) = \mathbf{0}, \tag{14}$$

with $k, m \in \{1, 2, 3\} - \{i\}$. This way, the following equations may be obtained:

$$p_5 := \det \mathbf{M}_{456,234}(B_1) = 0, \tag{15a}$$
$$p_6 := \det \mathbf{M}_{456,134}(B_2) = 0, \tag{15b}$$
$$p_7 := \det \mathbf{M}_{456,134}(A_2) = 0, \tag{15c}$$
$$p_8 := \det \mathbf{M}_{456,124}(B_3) = 0, \tag{15d}$$
$$p_9 := \det \mathbf{M}_{456,124}(A_3) = 0. \tag{15e}$$

Analogously, by setting $P \equiv G$, one obtains

$$p_{10} := \det \mathbf{M}_{456,123}(G) = 0, \tag{16}$$

All polynomials p_j, with $j = 5 \ldots 10$, have degree 4 in the Rodrigues parameters and degree 2 in the components of $\mathbf{x}$. These are the only linearly independent sextics in $\mathbf{X}$ that may be derived from the minors of $\mathbf{M}$ by varying the moment pole.

V. DGP

Solving the DGP of the 33-CDPR requires solving, simultaneously, both the equations emerging from the geometrical constraints and those inferred from static equilibrium.

The 3 point-to-point distance relations in Eq. (4) represent the typical constraints governing the forward kinematics of parallel manipulators equipped with telescoping legs connected to the base and the platform by ball-and-socket joints. In particular, the DGP of the general Gough-Stewart manipulator depends on six equations of this sort, one of which is equivalent to Eq. (4a) and five more to Eqs. (4b)-(4c). This problem is known to be very difficult and it has attracted the interest of researchers for several years [25], [26]. The DGP of the 33-CDPR appears to be a even more complex task, since, in this case, three equations analogous to Eqs. (4b)-(4c), namely of degree 3 in $\mathbf{X}$, are replaced by relationships that are, at least, of degree 6 in $\mathbf{X}$. If the platform posture is described by Study homogeneous coordinates[3] $\mathbf{X}_S := (g_0, g_1, g_2, g_3, e_0, e_1, e_2, e_3)$, with

$$q_0 := e_0 g_0 + e_1 g_1 + e_2 g_2 + e_3 g_3 = 0, \tag{17}$$

the relations in Eqs. (4) become quadratic in $\mathbf{X}_S$ [26], but the polynomials in Eqs. (9), (11), (15) and (16) remain of degree 6. The task does not appear to be significantly simplified. In the following, the number of complex solutions that the problem admits is determined by a hybrid approach based on Groebner bases and Sylvester's dialytic method. Results are confirmed by homotopy continuation.

Let $\langle J \rangle$ be the ideal generated by the polynomial set $J = \{q_1, q_2, q_3, p_1, \ldots, p_{10}\}$. q_1, q_2 and q_3 have, respectively, degree 4, 3 and 3 in the elements of $\mathbf{X}$, whereas all other generators of $\langle J \rangle$ have degree 6 in the same variables. In order to ease numeric computation via a computer algebra system, namely the $GroebnerPackage$ provided within the mathematical software $Maple13^{TM}$, all geometric parameters of the 33-CDPR are assumed to be rational. Accordingly, $\langle J \rangle \subset \mathbb{Q}[\mathbf{X}]$, where $\mathbb{Q}[\mathbf{X}]$ is the set of all polynomials in $\mathbf{X}$ with coefficients in $\mathbb{Q}$. All Groebner bases are computed with respect to graded reverse lexicographic monomial orders (grevlex, in brief)[4].

In general, a Groebner basis $G[J]$ of $\langle J \rangle$ with respect to $\mathrm{grevlex}(z, y, x, e_1, e_2, e_3)$ may be computed in a fairly expedited way. A key factor for the efficiency of such a computation is the abundance of generators available in $\langle J \rangle$, which significantly simplifies and speeds up calculation. $G[J]$ comprises 137 polynomials, namely 2 of degree 3 in $\mathbf{X}$, 41 of degree 4 in $\mathbf{X}$ and 94 of degree 5 in $\mathbf{X}$.

Once $G[J]$ is known, the number of complex roots in the variety V of $\langle J \rangle$ may be evaluated by the command $PolynomialIdeals[NumberOfSolutions]$ [27], [28]. In this case, the returned number is 156. In order to actually solve J, and thus eliminate unknowns, Groebner bases with respect to some elimination monomial orders are, however, needed. If $\mathbf{X}_l$ is a list of l variables in $\mathbf{X}$ and $\mathbf{X}\backslash\mathbf{X}_l$ is the (ordered) relative complement of $\mathbf{X}_l$ in $\mathbf{X}$, a monomial order $>_l$ on $\mathbb{Q}[\mathbf{X}]$ is of l-elimination type provided that any monomial involving a variable in $\mathbf{X}_l$ is greater than any

[3]Study coordinates are advantageously employed to solve the DGP of the Gough-Stewart manipulator.

[4]The lexicographic monomial order is particularly suitable to solve systems of polynomial equations, for it provides polynomial sets whose variables may be eliminated successively. However, the Grobner bases that it provides tend to be very large and thus, even for problems of moderate complexity, they have little chance to be actually computable. Conversely, the graded reverse lexicographic order produces bases that are endowed with no particular structure suitable for elimination purposes, but it ordinarily provides for more efficient calculations.

TABLE I

COMPUTATION TIME TO OBTAIN GROEBNER BASES OF THE ELIMINATION IDEALS OF $\langle J\rangle$ FOR THE EXAMPLE REPORTED IN TABLE II

l	J_l	$T_{G[J_l]}$ [min]	$T_{\langle J\rangle\cap\mathbb{Q}[e_3]}$ [min]
0	$\langle J\rangle$	1.3	1919
1	$\langle J\rangle\cap\mathbb{Q}[y,x,e_1,e_2,e_3]$	19	2159
2	$\langle J\rangle\cap\mathbb{Q}[x,e_1,e_2,e_3]$	42 (27)	579
3	$\langle J\rangle\cap\mathbb{Q}[e_1,e_2,e_3]$	49 (24)	33
4	$\langle J\rangle\cap\mathbb{Q}[e_2,e_3]$	160 (80)	11
5	$\langle J\rangle\cap\mathbb{Q}[e_3]$	$\dots$	–

monomial in $\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$. If $G_{>l}[J]$ is a Groebner basis of $\langle J\rangle$ with respect to $>_l$, then $G_{>l}[J]\cap\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$ is a basis of the lth elimination ideal $\langle J_l\rangle := \langle J\rangle\cap\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$ [29]. The l-elimination monomial order implemented in $Maple$ is a product order that induces $\mathrm{grevlex}$ orders on both $\mathbb{Q}[\mathbf{X}_l]$ and $\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$. In this perspective, the FGLM algorithm [30], which converts a Groebner basis from one monomial order to another, may be called upon to compute elimination ideals of type $\langle J\rangle\cap\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$, starting from $G[J]$. By this approach, a least-degree univariate polynomial in one of the original variables may be (theoretically) obtained.

Another method to compute a least-degree univariate polynomial of $\langle J\rangle$ emerges from the following observation. Let $N_{G[J_l]}$ be the number of generators in $G[J_l]$, with $G[J_l]$ being the Groebner basis of $\langle J_l\rangle$ with respect to $\mathrm{grevlex}\,(\mathbf{X}\backslash\mathbf{X}_l)$. Furthermore, let w be the last variable in $\mathbf{X}\backslash\mathbf{X}_l$. It is not difficult to verify that $G[J_l]$ comprises a number of monomials in $\mathbf{X}\backslash\mathbf{X}_l-\{w\}$ which is exactly equal to $N_{G[J_l]}$. For example, the Groebner basis $G[J_3]$ of $\langle J\rangle\cap\mathbb{Q}[e_1,e_2,e_3]$ with respect to $\mathrm{grevlex}(e_1,e_2,e_3)$ comprises 45 polynomials (9 of degree 8 in $\mathbf{\Phi}$ and 36 of degree 9 in $\mathbf{\Phi}$), including 45 monomials in e_1 and e_2 (of degree ranging from 0 to 8), whereas the Groebner basis $G[J_4]$ of $\langle J\rangle\cap\mathbb{Q}[e_2,e_3]$ with respect to $\mathrm{grevlex}(e_2,e_3)$ contains 18 polynomials (15 of degree 17 in $\{e_2,e_3\}$ and 3 of degree 18 in $\{e_2,e_3\}$), including 18 monomials in e_2 (of degree ranging from 0 to 17). It follows that, if w is assigned the role of 'hidden' variable, the resultant in w of J may be obtained from $G[J_l]$ by Sylvester's dialytic method. Indeed, by writing the generators of $G[J_l]$ in the form

$$\mathbf{T}\,(w)\,\mathbf{E}_w=\mathbf{0}, \tag{18}$$

where $\mathbf{T}(w)$ is a $N_{G[J_l]}\times N_{G[J_l]}$ matrix that only depends on w and $\mathbf{E}_w$ is a $N_{G[J_l]}$ column vector comprising all monomials in $G[J_l]$ with variables in $\mathbf{X}\backslash\mathbf{X}_l-\{w\}$, the sought-for resultant is

$$\det\mathbf{T}(w)=\sum_{h=0}^{156}L_h w^h=0, \tag{19}$$

with the coefficients L_h only depending on the input data, namely the robot geometry and the cable lengths. The degree of $\det\mathbf{T}(w)$ is confirmed to be 156.

Table I reports, for the exemplifying 33-CDPR whose dimensions are given in Table II, the CPU time required to compute $\mathrm{grevlex}$ Groebner bases for the elimination ideals of $\langle J\rangle$, with $l=0\dots5$, on a PC with a 2.67GHz Intel Xeon processor and 4GB of RAM. In particular, the third column reports the CPU time $T_{G[J_l]}$ required to get $G[J_l]$ both by computing $\langle J\rangle\cap\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$ and, in parentheses, by computing $\langle J_{l-1}\rangle\cap\mathbb{Q}[\mathbf{X}\backslash\mathbf{X}_l]$. The elimination task proves to be, in general, computationally very expensive and time consuming[5]. In particular, the 'deeper' the elimination process goes (i.e. the smaller the number of variables in $\mathbf{X}\backslash\mathbf{X}_l$ is), the longer is the time necessary to perform the computation and, above all, the larger is the amount of memory that is required. The latter issue is particularly critical. Indeed, for the example at hand, the last elimination ideal cannot be computed on the given PC, due to excessive memory usage[6]. The fourth column reports the CPU time $T_{\langle J\rangle\cap\mathbb{Q}[e_3]}$ required to calculate $\langle J\rangle\cap\mathbb{Q}[e_3]$ by applying Sylvester's dialytic method on $G[J_l]$, for $l=0\dots4$. In this case, computation time depends on the dimension of $\mathbf{T}\,(w)$ and, thus, it normally decreases with the number of variables in $\mathbf{X}\backslash\mathbf{X}_l$. Memory requirements are modest and the algorithm is ordinarily successful. It emerges from the above consideration that a hybrid approach, which eliminates a subset of variables by the FGLM algorithm and further applies Sylvester's method on the Groebner basis of the corresponding elimination ideal, provides an effective strategy to compute a least-degree univariate polynomial in $\langle J\rangle$.

For the numeric solutions of the problem to be actually calculated, however, working with polynomials of degree as high as 156 is unpractical and it poses substantial numerical problems. In this perspective, homotopy continuation offers a robust alternative [26]. If no information is *a priori* known about the roots in V, the DGP of the 33-CDPR may be cast, on the basis of the degree of the polynomials contained in J, into the larger family of all polynomial systems made up by 1 quartic, 2 cubics and 3 sextics on $\mathbf{X}\in\mathbb{P}^6$. General members of this family have $4^1 3^2 6^3=7776$ isolated roots. This is, indeed, the number of paths N_{paths} tracked by the homotopy-continuation software used in this paper, namely $Bertini$ [31]. If the platform posture is described by Study coordinates, N_{paths} lowers to $2^4 6^3=3456$, which is still a very high number. N_{paths} significantly improves if solutions are computed starting from three 8-degree polynomials in $\mathbf{\Phi}$ chosen within $G[J_3]$. N_{paths} drops, in this case, to $8^3=512$. Computation converges in a fairly robust way. However, since only a subset of the generators available for the ideal is used (6 out 13 if homotopy continuation is applied to J, and 3 out of 45 if homotopy continuation is applied to $G[J_3]$), the results must be successively sifted in order to retain only those that actually lie in the variety of J. As expected, 156 solutions are finally obtained. If the roots in

[5]Computation time may significantly increase depending on the complexity of the coefficients of the polynomials in J.

[6]In a computation performed on a more powerful workstation, $Maple$ estimated a required memory usage of about 12GB, in order to derive $\langle J_5\rangle$ from $\langle J_4\rangle$.

$\mathbf{\Phi}$ are computed via $G[J_3]$, the problem solutions may be completed by calculating the corresponding roots in $\mathbf{x}$ as follows. x and y may be linearly eliminated from Eqs. (4b) and (4c), so that algebraic functions $x = x(z, \mathbf{\Phi})$ and $y = y(z, \mathbf{\Phi})$ may be derived. By way of them, q_1 and p_1 may be written as quadratic expressions in z, thus allowing z and z^2 to be linearly computed. Back-substitution of z in $x = x(z, \mathbf{\Phi})$ and $y = y(z, \mathbf{\Phi})$ completes the solution. Due to space limitations, only the real solutions of the example reported in Table II are listed in the table.

After an equilibrium configuration is found, it proves *feasible* only if it is stable and therein cable tensions are positive. Cable tensions may be computed by a suitable set of linear independent relations chosen within Eq. (5), whereas stability may be assessed by determining the definiteness of the reduced Hessian matrix $\mathbf{H}_r$ defined in [22]. In Table II, the symbols $>$, $\geq$, $<$, $\leq$ and $<>$ denote, respectively, a positive-definite, a positive-semidefinite, a negative-definite, a negative-semidefinite and an indefinite matrix.

It is worth observing that the procedures described so far are aimed to find an analytic solution to the problem and to ascertain its number of complex roots. Once the latter information is known, more efficient computational techniques may be used to numerically solve practical cases. For example, the complete family of 33-CDPR DGPs lies in a 21-dimensional parameter space, parametrized by the geometric quantities $\mathbf{a}_i$, $\mathbf{b}_i$ and ρ_i, $i = 1 \dots 3$. Accordingly, when the 156 isolated roots of the DGP of a generic 33-CDPR are known, 'parameter' homotopy continuation may be applied to find the solutions for any other member of the family. In this case, only 156 paths need to be tracked and the algorithm may be quite fast [26]. Another possibly very efficient approach to solve the problem relies on techniques based on interval analysis. This method brings about the significant advantage of easily incorporating in the calculation the constraints (6), as well as uncertainties in the parameter values, physical bounds on variables ranges, etc. [32]. Such an approach will be subject of future research.

VI. Equilibrium Configurations with Unloaded Cables

Equation (2) represent a set of *theoretical* constraints. Indeed, since the *actual* constraints imposed by the cables are that

$$|\mathbf{s}_i| \leq \rho_i, \quad i = 1 \dots 3, \tag{20}$$

the number of tensioned cables for which equality relations such as those in Eq. (2) hold is *a priori* unknown. Accordingly, the *overall* solution set must be obtained by solving the DGP for *all possible* constraint sets $\{|\mathbf{s}_j| = \rho_j, j \in \mathcal{W}\}$, with $\mathcal{W} \subseteq \{1, 2, 3\}$ and $\mathrm{card}(\mathcal{W}) \leq 3$, and by retaining, for each corresponding solution set, the solutions for which $|\mathbf{s}_k| < \rho_k$, $k \notin \mathcal{W}$ [22], [23]. In general, this amounts to solving 7 DGPs, namely, 1 DGP with 3 cables in tension, 3 DGPs with 2 cables in tension and 3 DGPs with a 1 cable in tension. The solution of the problem with a single active cable is trivial, whereas the DGP of a CDPR suspended by 2 cables is solved in [23].

VII. Conclusions

This paper studied the kinematics and statics of under-constrained cable-driven parallel robots with three cables, in crane configuration. For such robots, kinematics and statics are intrinsically coupled and they must be dealt with simultaneously. This poses major challenges, since position problems gain remarkable complexity with respect to those of analogous rigid-link robots, such as the Gough-Stewart manipulator. This paper presented an original geometrico-static model that allowed the direct position analysis to be effectively performed in analytical form. The task consists in determining the platform posture and the cable tensions once the cable lengths are assigned. By a hybrid procedure relying on Groebner bases and Sylvester's dialytic method, it was shown that the problem admits, in general, 156 complex solutions, with results being confirmed by homotopy continuation.

The mentioned hybrid procedure appears to be innovative, in order to obtain a least-degree univariate polynomial from a given polynomial ideal. Indeed, finding a Groebner basis suitable for elimination purposes may be a highly demanding task. Even by using computationally efficient monomial orders (such as $\mathrm{grevlex}$) for initial computations and suitable algorithms (such as the FGLM one) to convert bases from the initial orders to the desired ones, memory usage and calculation times may be so large that performing a full elimination may easily prove unfeasible, even for problems of moderate complexity. The technique presented in this paper, encompassing three steps, considerably reduced computation requirements, in terms of both memory and time. First, a Grobner basis G was calculated with respect to an efficient monomial order (such as $\mathrm{grevlex}$). Then, a subset of the original unknowns was eliminated by computing, by way of the FGLM algorithm, a Groebner basis G_l of a suitable elimination ideal. Finally, a least-degree univariate polynomial in one of the remaining unknowns was computed by applying Sylvester's dialytic method to the polynomials of G_l. The method is tailored to the particular structure of the ideal emerging from the DGP of the 33-CDPR, but there are chances to generalize it to fit more general cases.

It must be observed that the reported number of solutions does not take into account constraints imposed by the stability of equilibrium and the sign of cable tensions. Once such constraints are imposed and the solutions are sifted, the number of *feasible* configurations drastically reduces.

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TABLE II

REAL SOLUTIONS OF THE DGP OF A 33-CDPR

Geometric dimensions and load: $\mathbf{a}_2 = [10;0;0]$, $\mathbf{a}_3 = [0;12;0]$, $\mathbf{b}_1 = [1;0;0]$, $\mathbf{b}_2 = [0;1;0]$, $\mathbf{b}_3 = [0;0;1]$, $(\rho_1, \rho_2, \rho_3) = (7.5, 10, 9.5)$, $Q = 10$.

Conf.	$(e_1, e_2, e_3;\ x, y, z)$	(τ_1, τ_2, τ_3)	$\mathbf{H}_r$
1	$(-4.2220216376218525374, -5.9041632869515210360, -0.4719284164260346102;$ $+1.6804603696020390943, +3.5743047536049493407, +5.5605475750988856764)$	$(+6.84, +3.05, +6.14)$	$<>$
2	$(-3.3553981637732204646, +0.5425359168641715099, +1.7110227662077546889;$ $+2.9313331749199504570, +4.0768903590846968732, +6.0451905744644536057)$	$(+5.26, +5.11, +5.81)$	$>$
3	$(-2.6616890629909497781, +0.4160373487571940226, +0.9655548628886102991;$ $+2.5977352480361477511, +3.8457865212868645040, -4.8661048045758031135)$	$(-5.71, -4.85, -5.59)$	$<>$
4	$(-2.5291311336353393166, +7.3670838551717188775, -3.0436947470784328872;$ $+4.3757198849572551337, +5.8522722689950264632, -4.0010370837572794347)$	$(-1.40, -9.30, -9.83)$	$>$
5	$(-1.1658499286472699650, -1.2731250301592223731, -1.0066002786209496830;$ $+1.3992607683511133116, +3.2794852510182088478, +5.5312834538826469464)$	$(+6.76, +2.51, +4.86)$	$<>$
6	$(-0.5483498696623835987, -0.4877188327940637588, -1.2105960172659885404;$ $(+1.8159313811036966479, +4.3022189513770458215, +5.5516371755216273886)$	$(+5.46, +3.25, +5.50)$	$<>$
7	$(-0.5044737581189470443, +2.5903097146888037712, -1.2479550929596409397;$ $+3.5231344366003843222, +5.5320236367500482920, +5.2626779413057278297)$	$(+2.89, +7.87, +9.12)$	$<>$
8	$(-0.3252555841337169146, -0.8891989606461705137, -1.6130562813850683595;$ $+2.5760653793782856615, +4.3924466541541392403, -6.7527508537785241857)$	$(-4.61, -4.12, -5.65)$	$<$
9	$(+0.5434332197723969320, -0.1455056574282349313, +0.5696219999523911064;$ $+3.0240954483208687602, +4.7309738515237873056, +3.3019215367593690362)$	$(+5.90, +7.83, +9.56)$	$<$
10	$(+0.6844447542486557310, -0.0996112288836193264, +0.5262976928395059876;$ $+2.8401864910572365897, +4.8133317875987522652, -4.4534720523569757781)$	$(-6.01, -7.61, -9.53)$	$<>$

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