Kinematic Analysis of the Hexapod Telescope

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Abstract: Within this paper an analysis of the Hexapod Telescope's kinematic structure is presented. We develop a specially adapted algorithm for the direct kinematics of the device and a new method for the singularity analysis.

1 Introduction

The 1.5-m-HEXAPOD-Telescope (HPT) is a unique construction world-wide. It presents revolutionary new ideas in astronomical telescope design: not only its mechanics but also its optical telescope assembling. It was originally developed by the company VERTEX (former Krupp Industrietechnik) in collaboration with the astronomical institute of the University in Bochum. Instead of the traditional two axes support, a six degree of freedom Stewart-Gough platform (SGP) mechanism is used to permit the telescope the right pointing and tracking of stellar objects. The HPTs primary mirror is realized as a hybrid structure consisting of a light-weight Carbon Fibre Reinforced Plastic (CFRP) structure permanently fixed to a 55 mm thin Zerodur faceplate, produced by Carl Zeiss in Jena, that forms the reflecting surface. Piezo-electrical ceramic positioners, integrated into the CFRP structure, serve as active interface between the CFRP structure and the optical surface. Compared to a classical telescope of the same mirror diameter, the Hexapod-Telescope allows for a weight reduction by a factor of 15! Both the low weight and the extremely good optical quality make the HPT an ideal candidate for larger telescopes in space, the moon and the stratosphere. For the near future the telescope will be placed, for astronomical research, at one of the best astronomical places on the earth; at the Cerro Armazones in Chile (Chini [1]).

From point of view of Kinematics the design of this platform has the classical layout of a 6-6 Stewart-Gough platform manipulator, as it is used for flight simula-

Figure 1: Hexapod Telescope

tors or milling machines. It was built already 10 years ago but because of lack of research money a lot of theoretical issues (like control algorithms, direct kinematics singularity analysis) never were solved and therefore it never went into operation. Recently theoretical research on the kinematics of the HPT was resumed and within this contribution the first results are reported. The paper is organized as follows: The first part deals with an specially for the HPT adapted version of the general solution algorithm of the direct kinematics and the second part reports some results on the singularity theory of this manipulator. We believe that the methods reported herein can be used for all parallel manipulators of a similar design.

2 Direct Kinematics

Within this section we shall adapt the direct kinematics algorithm of a general Stewart-Gough Platform developed in [3] to the HPT. Because of the special geometry of this example we shall have simplifications of the general algorithm and we shall show that some special constraint equations can be produced. All this simplifications will allow us to keep the joint parameters general for a big part of the computation.

The HPT consists of a base and a platform connected by six legs via ball and socket joints at the base and U-joints at the platform. The anchor points of the joints are specified in a geometric special way: They are located on the vertices of a semi-regular hexagon which is generated by the following geometric process: take an equilateral triangle and cut it with a circle centered at the centroid of the triangle. The six points of the intersection of the circle and the triangle are the vertices of the semi-regular hexagon. Coordinate systems are attached to platform and base so that the origin is located at the centroid of the hexagon and the x-axis is aligned with one of the symmetry axes and the y-axis is in the plane of the hexagon. x-,y- and z-axes form a right handed coordinate system (Fig. 2). The left side of Fig. 2 shows the layout of the base anchor points ($\mathbf{B_i}$) and the right side shows the platform anchor points $\mathbf{p_i}$. The dimensions used by the manufacturer (VERTEX) are also shown in this Figure. But for the calculation we need Cartesian coordinates $B_{xi}, B_{yi}, p_{xi}, p_{yi}$ which are listed in the Table 2. Now we employ the kinematic mapping, introduced

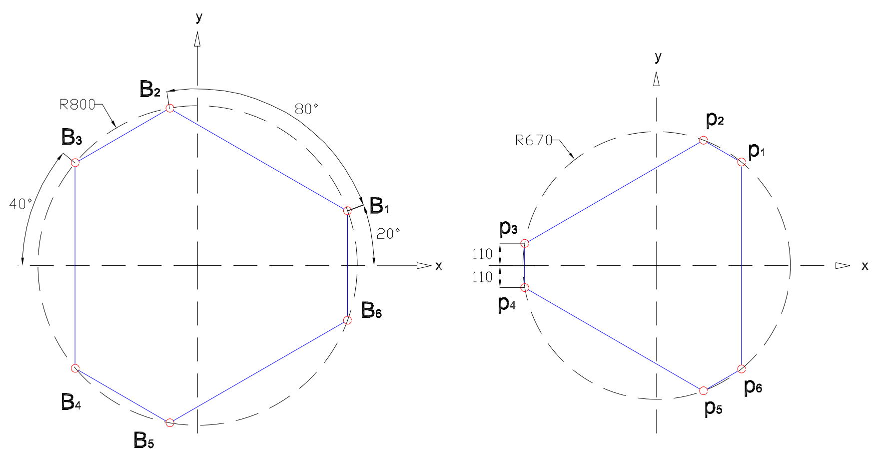

Figure 2: Coordinates of the HPT

by STUDY [8] to map three-dimensional motions using the Study-parameters into a seven-dimensional image space (HUSTY et al. [4]). A general anchor-point in the moving system ${}^{\mathcal{M}}\mathbf{p_i}$ is denoted by:

$$ {}^{\mathcal{M}}\mathbf{p}_i = \begin{bmatrix} 1 & a_i & b_i & 0 \end{bmatrix}^T, \tag{1} $$

joint	base ($\mathbf{B}_i$)			platform ($\mathbf{p}_i$)		
no.	B_{xi}	B_{yi}	B_{zi}	p_{xi}	p_{yi}	p_{zi}
i	[mm]	[mm]	[mm]	[mm]	[mm]	[mm]
1	752	274	0	426	517	0
2	-139	788	0	235	627	0
3	-613	514	0	-661	110	0
4	-613	-514	0	-661	-110	0
5	-139	-788	0	235	-627	0
6	752	274	0	426	-517	0

Table 1: The nominal parameters of the HPT

whereas a general anchor-point in the base system will be denoted by

$$^{\mathcal{B}}\mathbf{B}_i = \begin{bmatrix} 1 & A_i & B_i & 0 \end{bmatrix}^T, \tag{2}$$

or expressed in non-homogeneous coordinates:

$$^{\mathcal{M}}\mathbf{p}_i = \begin{bmatrix} a_i & b_i & 0 \end{bmatrix}^T, \qquad ^{\mathcal{B}}\mathbf{B}_i = \begin{bmatrix} A_i & B_i & 0 \end{bmatrix}^T \tag{3}$$

The transformation of the platform points $^{\mathcal{M}}\mathbf{p}_i$ to the base is given by:

$$^{\mathcal{B}}\mathbf{p}_i = ^{\mathcal{B}}\mathbf{T}_{\mathcal{M}} \cdot ^{\mathcal{M}}\mathbf{p}_i, \tag{4}$$

where Study's parametrization was used for the parametrization of the transformation matrix. Omitting the index i the expanded Eq.4 is:

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_0^2+x_1^2+x_2^2+x_3^2 & 0 & 0 & 0 \\ l & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2+x_0x_3) & 2(x_1x_3-x_0x_2) \\ m & 2(x_1x_2-x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3+x_0x_1) \\ n & 2(x_1x_3+x_0x_2) & 2(x_2x_3-x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \\ c \end{bmatrix} \tag{5}$$

with

$$\begin{aligned} l &= 2(y_1x_0 - y_0x_1 + y_3x_2 - y_2x_3), \\ m &= 2(y_2x_0 - y_0x_2 + y_1x_3 - y_3x_1), \\ n &= 2(y_3x_0 - y_3x_1 + y_2x_1 - y_1x_2). \end{aligned} \tag{6}$$

With this parametrization a point $(x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3)$, $x_i \neq 0$ fulfilling the Study condition $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0$ represents a valid position of the platform coordinate system with respect to the base coordinate system.

To solve the direct kinematic problem we break up the platform so that the joints and a piece of the platform, together with its mobile-system, hangs at the end of a leg. This mobile-system has five degrees of freedom, because its only constraint is, that the distance between the two anchor-points is constant. So the platform anchor-point is constrained to move on the surface of a sphere, with constant radius. This constraint is reflected by Eq.7, developed in Husty [3].

$$\begin{aligned} sce :& 4y_0x_1a + 2Aax_2^2 + 4y_3^2 + 4y_1^2 + 4y_2^2 - 4Abx_1x_2 - 4y_2bx_0 + 4y_3x_2a - 4y_1x_0a + \\ & 4y_0^2 - 2aAx_1^2 - 2bBx_0^2 - 2bBx_2^2 - 2aAx_0^2 + 4By_2x_0 - 4By_0x_2 + 2Bbx_1^2 - \\ & 4By_3x_1 - 4Bax_1x_2 + 4y_0x_2b - 4y_3x_1b + 4Ay_3x_2 + 4Ay_1x_0 - 4Ay_0x_1 + 2x_3^2bB \\ & - 4x_3ay_2 + 4x_3y_1B + 4x_3by_1 - 4x_3y_2A + 4x_3Abx_0 - 4x_3Bax_0 + \\ & 2x_3^2aAx_1^2R + x_3^2R + x_0^2R + x_2^2R = 0 \end{aligned} \tag{7}$$

In Eq. 7 the coordinates of the base joint center are $(A, B, 0)^T$ and the platform joint center is $(a, b, 0)^T$. Furthermore the abbreviation $R = A^2 + B^2 + a^2 + b^2 - r^2$ was used. Note that this constraint equation is already a serious simplification compared to the general constraint equation in [3], because the z-coordinates of all the anchor points in both base and platform are zero. Now we apply this equation to the HPT. We insert the design parameters of Table 2 into the Eq. 7 and show how the general solution algorithm simplifies because of the symmetries of the design parameters. For each leg we get a constraint equation h_i $(i = 1 \ldots 6)$. One of these equations is displayed, all others are similar.

$$\begin{aligned} h_1 :& 4y_2^2 + 4y_3^2 + 4y_0^2 - 5664x_3y_2 + 5664y_3x_2 + 640y_1x_0 - 640y_0x_1 + 2014272x_3^2 - \\ & 384x_3y_1 - 1496y_0x_2 + 384y_3x_1 + 1496y_2x_0 + 391552x_1x_2 - 1089888x_3x_0 - \\ & x_1^2R_1 - x_2^2R_1 + 4y_1^2 - x_0^2R_1 - x_3^2R_1 + 2144932x_2^2 + 34816x_1^2 + 165476x_0^2 = 0 \end{aligned} \tag{8}$$

We compute differences of the constraint equations h_i and two more helpful equations:

$$\begin{array}{ll} U_1 = h_1 - h_6 & (\text{with } S_1 = (-R_1 + R_6)/64) \\ U_2 = h_2 - h_5 & (\text{with } S_2 = (-R_2 + R_5)/32) \\ U_3 = h_3 - h_4 & (\text{with } S_3 = (-R_3 + R_4)/32) \\ U_4 = h_1 - h_2 & (\text{with } S_4 = (-R_2 + R_1)/4) \\ U_5 = h_5 - h_6 & (\text{with } S_5 = (-R_5 + R_6)/4) \\ U_6 = x_0y_0 + x_1y_1 + y_2x_2 + x_3y_3 & (\text{Study-quadric}) \end{array}$$

Expanded we have the surprisingly simple difference equations:

$$\begin{aligned} U_1 :& -20x_1y_0 - 177x_3y_2 + 177x_2y_3 + 20x_0y_1 + 12236x_2x_1 - 34059x_0x_3 \\ & - S_1(x_0^2 + x_1^2 + x_3^2 + x_2^2) = 0 \end{aligned}$$

$$
\begin{aligned}
U_2 :& -101x_1y_0 - 156x_3y_2 + 156x_2y_3 + 101x_0y_1 - 101796x_2x_1 + 68081x_0x_3 \\
& - S_2(x_0^2 + x_1^2 + x_3^2 + x_2^2) = 0 \\
U_3 :& 198x_3y_2 - 198y_3x_2 + 61y_1x_0 - 61y_0x_1 - 126565x_1x_2 - \\
& 68203x_3x_0 - S_3(x_0^2 + x_3^2 + x_1^2 + x_2^2) = 0 \qquad (9) \\
U_4 :& -792x_3y_2 + 792y_3x_2 - 244y_1x_0 + 244y_0x_1 + 455x_3^2 - 1370x_3y_1 - \\
& 422y_0x_2 + 1370y_3x_1 + 422y_2x_0 + 505072x_1x_2 - 544796x_3x_0 + 438313x_2^2 - \\
& 437869x_1^2 - 11x_0^2 + S_4(x_3^2 + x_0^2 + x_2^2 + x_1^2) = 0 \\
U_5 :& -792x_3y_2 + 792y_3x_2 - 244y_1x_0 + 244y_0x_1 - 455x_3^2 + 1370x_3y_1 + \\
& 422y_0x_2 - 1370y_3x_1 - 422y_2x_0 + 505072x_1x_2 - 544796x_3x_0 - 438313x_2^2 + \\
& 437869x_1^2 + 11x_0^2 - S_5(x_2^2 + x_1^2 + x_3^2 + x_0^2) = 0 \\
U_6 :& x_0y_0 + x_1y_1 + y_2x_2 + x_3y_3 = 0
\end{aligned}
$$

We add the following helpful equations:

$U_7 : h_1 - h_2 + h_3 - h_4 + h_5 - h_6 = 0 \quad (\text{with } \mathrm{W}_1 = (\mathrm{R}_2 - \mathrm{R}_1 - \mathrm{R}_3 + \mathrm{R}_4 - \mathrm{R}_5 + \mathrm{R}_6)/32)$
$U_8 : h_4 - h_5 = 0 \quad (\text{with } \mathrm{S}_8 = \mathrm{R}_5 - \mathrm{R}_4)$

In our example:

$$U_7 : 297x_1x_2 + 204402x_3x_0 + W_1(x_0^2 + x_2^2 + x_3^2 + x_1^2) = 0 \qquad (10)$$

Note that y_0, y_1, y_2, y_3 are linear in the difference equations and so we take U_2, U_3, U_4 and U_6 and solve for y_0, y_1, y_2, y_3. After that only the Euler parameters remain to be solved for. At this point we still have the option to normalize the Euler parameters and this case it is reasonable to set $x_0 = 1$. With this assignment we have eliminated a theoretical possible, but practical useless set of solutions where the telescope platform would be rotated by 180°. Now the solutions for y_i are substituted into the remaining equations. Essentially only three different equations remain: U_7, U_8, h_1. U_7 is still of degree 2 and has not changed because it did not contain y_i. U_8 is a polynomial of degree 4 and h_1 is of degree 8. Using resultant method we eliminate x_1 from these three polynomials and create two new polynomials $T_1(x_2, x_3, R_i)$ and $T_2(x_2, x_3, R_i)$. T_1 is of degree 6 in the variable x_2 and T_2 is of degree 16 in x_2. But in both polynomials x_2 appears only in even powers:

$$
\begin{aligned}
T_1 :&\ k_0 + k_1x_2^2 + k_2x_2^4 + k_3x_2^6 = 0 \\
T_2 :&\ p_0 + p_1x_2^2 + p_2x_2^4 + p_3x_2^6 + p_4x_2^8 + p_5x_2^{10} + p_6x_2^{12} + p_7x_2^{14} + p_8x_2^{16} = 0,
\end{aligned}
$$

In T_1 and T_2 $k_i, i = 0, \ldots, 3$ and $p_i, i = 0 \ldots 8$ are functions of the variable x_3 and the leg parameters R_i. Note that the leg parameters are still general! It is not difficult to calculate the resultant of T_1 and T_2 to eliminate x_2. We get a polynomial in k_i and p_i which is of degree 11 squared. But unfortunately in back substituting for k_i and p_i we get to big expressions. On the other hand there is no problem to do this back substitution for a given set of leg parameters R_i. This results in the final univariate polynomial which is of degree 18 (squared). It should be noted that the final polynomial also could be computed by substituting the leg parameters into T_1 and T_2 and then performing the elimination of the variable x_2. The remarkable fact is that T_1 and T_2 can be stored and the computation of the univariate polynomial for the direct kinematics of the HPT consists of substituting the leg parameters into T_1 and T_2, computing one resultant and a factoring of the resultant![1]

3 Singularity Analysis

Within this section a singularity analysis of the HPT is presented. Singular configurations are defined as configurations where the screws of the lines $\mathbf{p}_i\mathbf{B}_i$ are linearly dependent. This means that the determinant of the of the matrix $\mathbf{J}$ consisting of the six Plücker vectors of the lines is equal to zero. We refer to the general explanation of this matrix in Karger [5], where an detailed explanation of the structure of matrix $\mathbf{J}$ and its determinant was given. Generally this determinant represents a five dimensional surface K on the six dimensional hyper surface given by the Study condition. From [5] it is known that K is of degree eight and that this surface for a fixed position of the platform is of degree three (see [6]). For the HPT it is no problem to compute these surfaces for any position or orientation. Both surfaces can be visualized because they live either in the three dimensional space $(x_0 : x_1 : x_2 : x_3)$ when a position is given or in the three dimensional space (l, m, n) when an orientation is given. Translation parameters l, m, n can be computed by solving the linear system Eq. 6.

Fig. 4 shows a degree three singularity surface belonging to a fixed orientation of the platform. The chosen orientation is shown by the tripod. Fig. 3 shows a degree eight singularity surface for a given orientation of the HPT. This surface is always inside the unit sphere because the Euler parameters $(x_0 : x_1 : x_2 : x_3)$ have been normalized $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. This singularity surface shows an obvious void around the origin of the coordinate system. This void means that the platform has a certain

[1]A complete maple note book with this algorithm can be found on the first author's web page http://techmat.uibk.ac.at/geometrie/husty/husty.html. In this note book the computation of all polynomials and one complete solution for a set of leg parameters with back substitution and verification is shown.

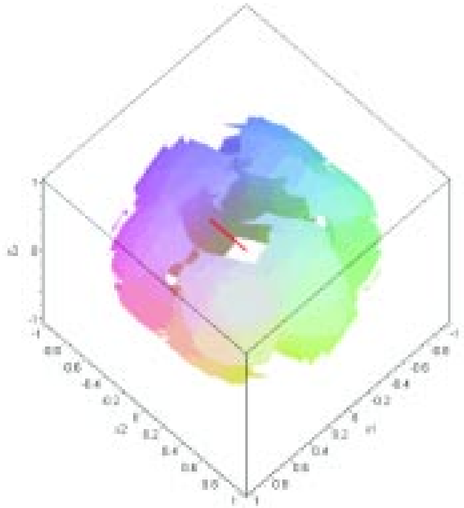

Figure 3: Singularity surface for given position

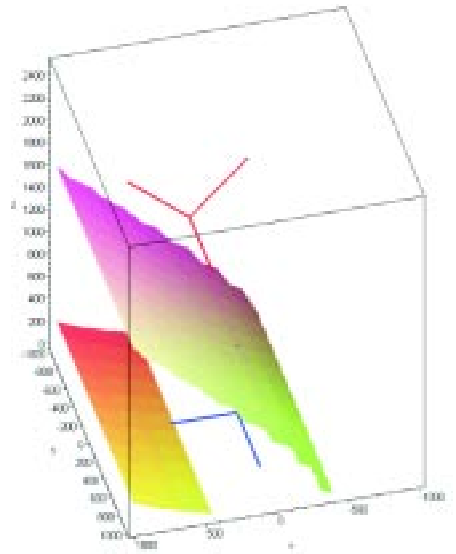

Figure 4: Singularity surface for given orientation

orientability about the origin without hitting a singularity. For a closer inspection of this fact we prove the following:

Lemma: *Whenever the platform and the base do not coincide, then for every position there exists a ball in the Euler parameter space with center* $(1:0:0:0)$ *and radius* $r>0$ *which does not intersect the orientation singularity surface in real points.*

Proof: To proof the lemma we simply substitute $x_0=1, x_1=0, x_2=0, x_3=0$ into the equation of the singularity surface $\det \mathbf{J}=0$ and this yields the result:

$$y_3^3=0 \tag{11}$$

The interpretation of this equation is as follows: By substituting $x_0=1, x_1=0, x_2=0, x_3=0$ into the singularity equation we look for all singular positions of the platform when platform and base have the same orientation. After substitution of $x_0=1, x_1=0, x_2=0, x_3=0$ and $y_3=0$ into Eq. 6 it turns out that the third order singularity surface for this case is degenerated into the plane $n=0$. So whenever the platform and base coordinate system have the same orientation ("the home orientation") and the origin of the platform coordinate system is not in the plane of the base, then there exists a ball with center $(1:0:0:0)$ and radius $r>0$ which does not intersect the singularity surface in real points.[2] $\square$

[2]This lemma can be proven even more generally: it is valid for any SGP whose anchor points are in planes [7].

Practical considerations show that the above mentioned case is impossible for the real HPT, because platform and base can never coincide. For any position of the origin of the platform coordinate system outside of the plane $z = 0$ we have the fact that the singularity surface has a void around the origin of the x_1, x_2, x_3-coordinate system. The size of this void gives an important design information. We can try to blow up the ball until it touches the singularity surface. We shall denote the touching ball with B_t. All points inside B_t represent singularity free orientations of the platform for a given position. The radius $r = \sqrt{x_1^2 + x_2^2 + x_2^3}$ of B_t gives the maximum angle of rotation $\varphi = 2 \arccos \sqrt{1 - r^2}$ about which the platform can rotate singularity free for a given position.

With this method we investigated the following box of the positional workspace of the HPT, measured in [mm]:

$$x = -500 \dots 500, \quad y = -500 \dots 500 \quad z = 1500 \dots 2200,$$

and at every node for a grid of 10 cm we tested for the orientation capability of the manipulator. To do this we blew up a ball until it touched the singularity surface. As result we can report, that the absolute of the radius of B_t was between 0.4023 and 0.5884. Therefore we can conclude that in the given positional workspace box rotations (starting from the home orientation) without singularities are possible with an angle of at least ±47 degree about any axis and in the best case up to ±72 degree about any axis. In Table 2 the results of orientation capacity at positions on different z-constant levels of the workspace are listed. Additionally we observed that the larger

level: z [mm]	r_{min}	$angle_{min}$	r_{max}	$angle_{max}$
1500	0.4023581068	$\pm 47.45^o$	0.5396207416	$\pm 65.31^o$
1600	0.4166886075	$\pm 49.25^o$	0.5488342830	$\pm 66.57^o$
1700	0.4299324505	$\pm 50.92^o$	0.5571138682	$\pm 67.71^o$
1800	0.4421872357	$\pm 52.48^o$	0.5645915371	$\pm 68.74^o$
1900	0.4535425184	$\pm 53.94^o$	0.5713760913	$\pm 69.69^o$
2000	0.4640799095	$\pm 55.30^o$	0.5775578769	$\pm 70.55^o$
2100	0.4738734229	$\pm 56.57^o$	0.5832124657	$\pm 71.35^o$
2200	0.4829899558	$\pm 57.76^o$	0.5884034885	$\pm 72.08^o$

Table 2: Possible rotation angles about arbitrary axes in the given workspace box

rotation angle values were at positions close to the z-axis, whereas the smaller rotation capabilities occur in the positions at the lateral boundaries of the given workspace box.

4 Conclusions

In this paper we have presented a specially adapted algorithm to solve the direct kinematics of the Hexapod Telescope. This algorithm enables keeping the leg parameters general almost to the end of the computation and allows a very fast computation of the univariate polynomial which governs the direct kinematics. Furthermore a singularity analysis of the platform was performed. This analysis yielded the remarkable result that within a certain positional workspace box the platform can be rotated singularity free from the home orientation about any axis with at least $\pm 47°$.

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