REGULAR POLYHEDRAL LINKAGES

K.Wohlhart
Institute for Mechanics, Graz University of Technology
Kopernikusgasse 24, A-8010 Graz, Austria
T: +43 316 873 7642, Fax: +43 316 873 7641
e-mail: wohlhart@mech.tu-graz.ac.at

Abstract. In this paper new overconstrained linkages are described, which are synthesized by inserting *planar* link groups into the faces of Regular Polyhedra and are interconnected by corresponding multiple gussets. These spatial linkages belong to the category of the so-called paradoxical linkages as they disobey the topological structure formula of Grübler-Kutzbach. In a general position they are mobile with one degree of freedom and shaky with a number of degrees of freedom in special multifurcation positions. As these linkages considerably change in their overall size while deforming, they lend themselves to being used as deployable structures.

1. Introduction

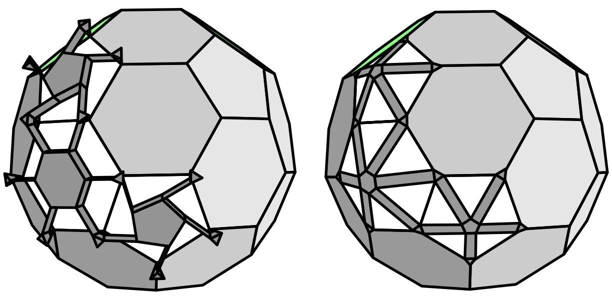

Figure 1. Insertion of planar or spatial link groups into faces of a Regular Polyhedron

In a recent paper [1] it was shown that from any Platonian or Archimedean Polyhedron with regular faces highly overconstrained lin-kages can be derived by implanting *spatial* multilegged spreading link groups into the faces of the polyhedron grid. The linkages synthesized in this way were baptized Polyhedral Star-Transformer as they develop from the polyhedral form to a starlike form. But the implantation of such spatial link groups is not the only way to “mobilize” a Regular Polyhedron. In the following it will be shown that also the insertion of special *planar* link groups into the faces of any Regular Polyhedron leads to overconstrained spatial linkages which are also mobile with one degree of freedom (Fig.1).

2. Regular Polyhedra

Though the term Regular Polyhedra [2] is sometimes used to refer exclusively to the five convex Platonian Solids, we shall subsume under this term all polyhedra which have similar arrangments of non-intersecting regular plane polygonal faces of two or more types about each vertex with all edges of equal length. Therefore we also include under the Regular Polyhedra the four concave (stellated) Kepler-Poinsot Solids, the thirteen Archimedean Solids[3] and finally the Regular Antiprisms and the Regular Prisms constructed with two equal n-sided regular polygons and $2n$ regular triangles or n regular quadrangles. As, however, the range of movability of linkages which can be derived from the concave Kepler-Poinsot Solids is very small, we exclude this type from our considerations. Figure 2 shows all the Regular Poyhedra to which we can apply the same procedure of synthesizing Regular Polyhedra Linkages.

There are several types of planar link groups which can be inserted into the faces of convex Regular Polyhedra in order to "mobilize" them. We shall concentrate on the simplest possible case in which the inserted planar link groups consist of a regular (n-sided) polygonal central body to which (via rotary joints) laminas are articulated at all of its corners. The laminas are of equal size and the central body has a similar polygonal contour as the polygonal face into which the link group is to be inserted. Let us add at each of the vertices of the polyhedron an appropriate multiple gusset-body which consists of as many rotary joints as there are edges on the vertex and which interconnects the planar link groups. This leads to a spatial linkage which, although highly overconstrained, turns out to be mobile with one degree of freedom.

For a (simply connected) polyhedron with V vertices, E edges and F faces, the Euler formula [4] states: $V - E + F = 2$. The linkage synthesized in the decribed way then consists of $B = V + F + nF$ bodies interconnected by $J = 2\sum n_\alpha F_\alpha$ rotary joints. In the latter formula F_α denotes the number of the n_α-sided regular polygonal faces in the Regular Polyhedron. The internal degree of freedom dof of the linkage will then be with the number of fundamental loops $L = J - B + 1$ and the Euler formula $V - E + F = 2$:

$$dof = \sum f_\alpha - 6L = J - 6(J - B + 1) = -4\sum n_\alpha F_\alpha + 6(E+1). \tag{1}$$

3. Overconstrainedness of Polyhedral Linkages

For the five Platonian Polyhedra the relation $\sum n_\alpha F_\alpha = nF = 2E$ holds and therewith formula (1) reduces to: $dof = 6 - 2E$. The degree of overconstrainedness is defined as $c = 1 - dof$. The linkages which can be derived from the five Platonian Solids are then overconstrained to the following degrees: the Tetrahedral Linkage: $c = 7$, the Hexahedral- and the Octahedral Linkage: $c = 19$, and finally the Dodecahedral Linkage and the Icosahedral Linkage: $c = 55$. Also for each linkage derived in the described way from onof the thirteen Archimedean Solids we could find the degree of coverconstrainedness by simply counting the edges and the number of equal faces on it. We shall do this only for one representative case: the Football Polyhedron. This polyhedron can be obtained by cutting off the vertices of a dodekahedron so that in the final polyhedron all edges are again of equal length. This polyhedron is therefore also called the Truncated Dodecahedron.

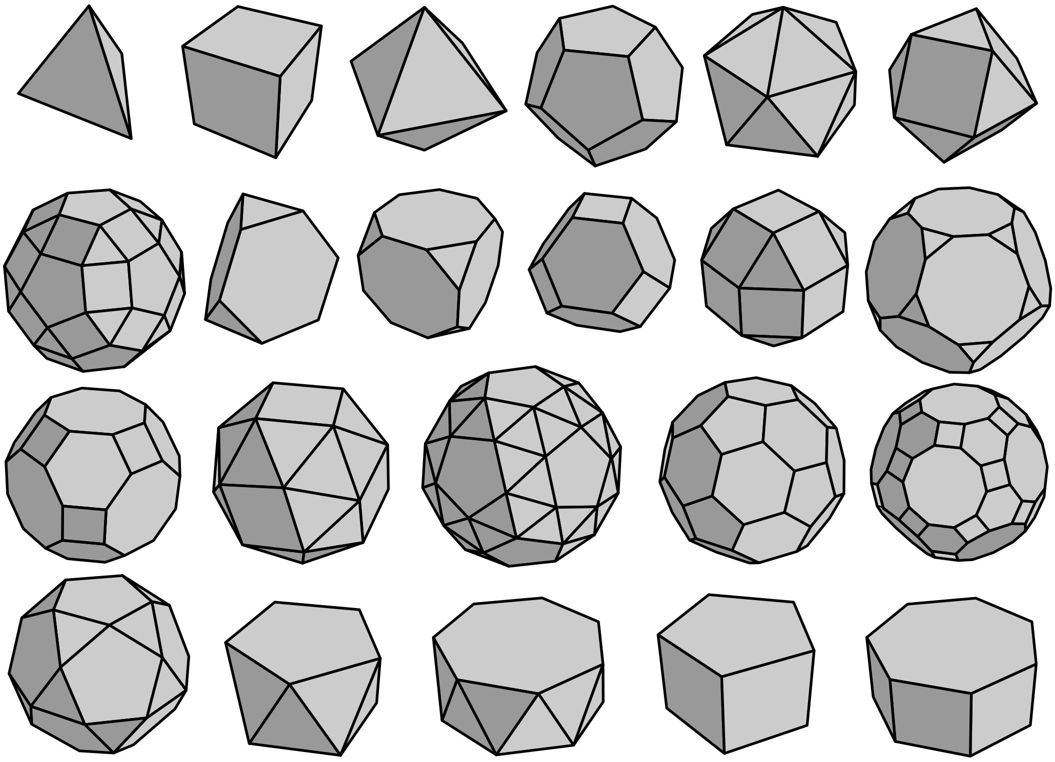

Figure 2. Regular Polyhedra: The five Platonian Solids, the thirteen Archimedean Polyhedra and two examples of Antiprisms and Prisms

The Football Polyhedron consists of $F_5 = 12$ regular pentagons and of $F_6 = 20$ regular hexagons and its number of edges is $E = 90$. From the formula (1) we then find for its degree of freedom: $dof = -4\sum n_\alpha F_\alpha + 6(E+1) = -4(5F_5 + 6F_6) + 6(E+1) = -174$, and for its degree of overconstrainedness: $c = 1 - dof = 175$.
A Regular Antiprism, constructed with two n- sided polygons and $2n$ triangles. has $E = 4n$ edges and the degree of freeedom of the linkage derived from it will therefore be $dof = 6(4n+1) - 4(2n + 2n \times 3) = -8n + 6$, and its degree of overconstrainedness consequently: $c = 1 - dof = 8n - 5$. For $n = 3$ the Antiprism is simply the Octahedron and we get for it asabove: $c = 19$.
Finally, a Regular Prism, constructed with two n- sided polygons and n square faces, has $E = 3n$ edges and therewith we get for the degree of freedom of the linkage derived: $dof = 6(3n+1) - 4(n2 + 4n) = -6n + 6$and for $c = 6n - 5$. For $n = 4$ the Regular Prism is identical with the Hexahedron and, as above, we obtain for this linkage: $c = 19$.
As a rule it turns out that the more complex the Regular Polyhedron from which the linkage is derived, the higher is their degree of overconstrainedness.

4.The Planar Link Groups

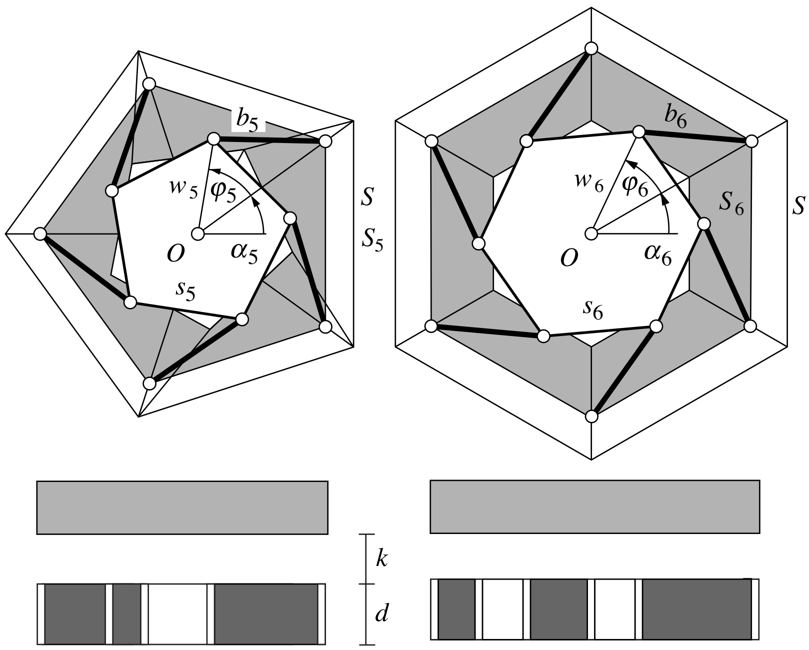

Figure 3. Two faces of a Regular Polyhedron together with the inserted planar link groups in front- and top view. (k measures the elevation of the link groups over the faces)

In a Regular Polyhedron with different faces different planar link groups are to be inserted. But also into the faces of the Platonian Solids with their equal faces varied planar link groups might be implanted. A link group loosely inserted into an n-sided polygonal face of a Regular Polyhedron has $n+2$ degrees of freedom within the face plane. However, if its center body can only rotate about the center of the polygonal face and the ends of the laminas attached to it can only slide along radial lines, its degree of freedom will be reduced to one. The position of the link group is then determined e.g., by the rotation angle φ of the center body. Figure 3 shows two link group $(n_k=5, n_l=6)$ in two positions. The relation between the position angle φ_k of the central body and the side length S_k of the face polygon is given by:

$$\cos\varphi_k = \frac{S_k^2 + s_k^2 - (2\,b_k \sin\alpha_k)^2}{2\,S_k\,s_k}, \tag{2}$$

where s_k denotes the side length of the (n_k-sided) polygonal center body, b_k the length of the laminas and α_k stands for π / n_k. The position angles φ_k and φ_l in two neighbouring faces of the polyhedron are related by the condition:

$$S_k(\varphi_k) = S_l(\varphi_l). \tag{3}$$

The minimum overall size of the link group can be obtained by equalizing the length of the laminas to the polygon side length of the center body:

$$b_k = s_k . \tag{4}$$

The minimum size will then be reached at the angle $\varphi_k = 2\alpha_k$. If we want the link groups to reach their minimum size in all faces simultaneously, we have to give all the polygon sides of the center bodies the same length. This follows from $S_k(2\alpha_k) = s_k$ together with condition (3).
The maximum size of the link group is reached at the angle $\varphi_k = 0$. The maximum size of all link groups in the different faces can only be reached simulteneously, if all faces of the Regular Polyhedron are equal, i.e., if it is a Platonic Solid.

5. The Multiple Rotary Joints (Gussets)

Fig.4 shows how the gussets are to be constructed. Geometrically speaking they are truncated pyramids with as many side faces as faces meet at the vertex of the polyhedron on which the gusset is built. On each of the intersecting edges of the gusset rotary joints (pairing elements) are fixed into which the laminas of the link groups can be articulated. In Fig.4 four polyhedron faces $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ meet at one polyhedron vertex. Of each face the outer normal unit vector n_1, n_2, n_3, n_4 can be determined from the geometry of the polyhedron. With the position vector x_V of the polyhedron vertex V, the link group shift k and the breadth d of the attached laminas in the link group, the geometry of the gusset body will be given by its corner points : $x_\alpha = x_V + k\, n_\alpha , y_\alpha = x_V + (k+d)\, n_\alpha , \cdots \alpha = 1 \div 4$. Shifting of the link group is necessary ($k \neq 0$) to avoid link interference while the linkage is deforming. As in a Regular Polyhedron the configuration of the faces about each vertice is the same, all the gussets necessary to combine the link groups are identical.

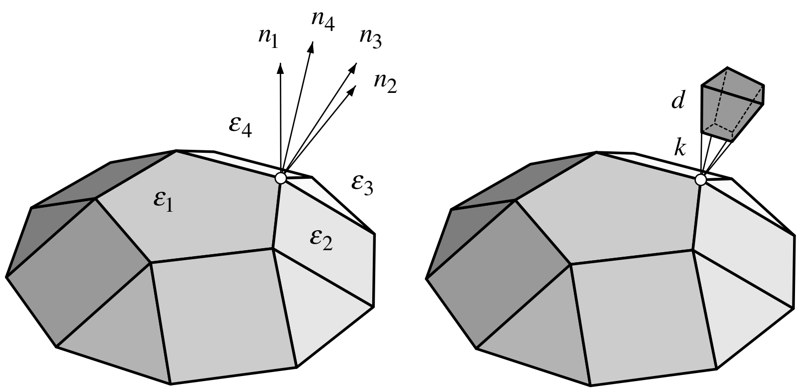

Figure 4. Construction of a multiple rotary joint (gusset)

6. The Platonian Linkages

Fig. 5 and Fig. 6 represent the spatial linkages which have been derived from the five the Platonian Solids. Each of the five linkages is shown in three positions: at its maximum extension $(\varphi = 0)$, in a medium position and in the closed position. The shift of the link group is measured by k (see Fig.3 and Fig.4). With $k = 0$ all the Platonian Linkages would close up completely at the position angle $\varphi = 2\alpha$. But in order to avoid link interference it is necessary to make $k \geq k_{\min}$ if the center body in the inserted link group is triangular, i.e.. in the case of the Tetrahedral Linkage, the Octahedral Linkage and the Icosahedral Linkage. This is due to the fact that for the region of the position angle: $\pi/3 \leq \varphi \leq 2\pi/3$one corner of the triangular center body goes beyond the contour of the triangular face. The distance of this corner from the contour line is given by:

$$p(\varphi) = \frac{s}{2}(\sin\varphi - \cos\alpha\sqrt{(2\sin\alpha)^2 - (\sin\varphi)^2}). \tag{5}$$

This formula shows that for polygons with more than three sides in the whole region of the position angle $(0 \leq \varphi \leq 2\alpha)$ we get: $p \leq 0$, and in the case of the triangle p reaches a positive maximum for the position angle $\varphi = \pi/2 : p_{\max} = s(1/2 - 1/\sqrt{6})$. The necessary minimum shift $k_{\min}$ depending on the angle between neighbouring faces, i.e. the angle between the normal vectors of neighbouring faces, can then be determined by:

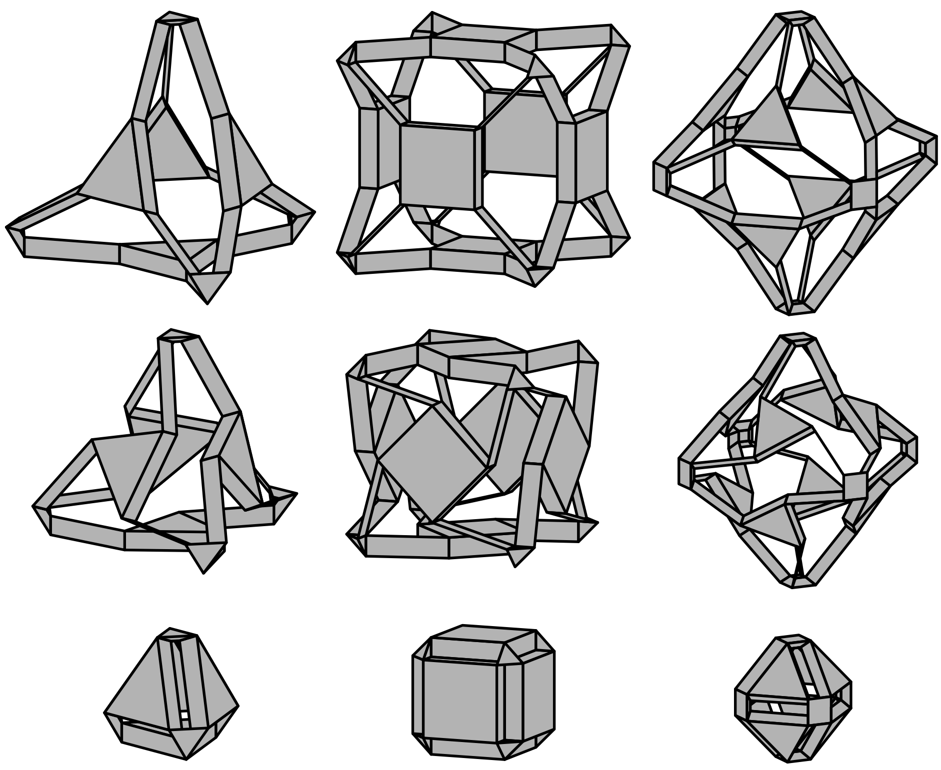

Figure 5. Different phases of the Tetrahedral Linkage, the Hexahedral Linkage and the Octahedral Linkage

$$k_{\min} = p_{\max}\cot(\frac{\gamma}{2}) = s\,(\frac{1}{2} - \frac{1}{\sqrt{6}})\cot(\frac{\gamma}{2}). \tag{6}$$

For the Thetrahedral Linkage we get: $k_{\min} = s(1/2-1/\sqrt{6})/\sqrt{2} = 0.06488\ s$, for the Octahedral Linkage: $k_{\min} = s(1/2-1/\sqrt{6})\sqrt{2} = 0.1298\ s$ and finally for the Icosahedral Linkage: $k_{\min} = s(1/2-1/\sqrt{6})(3+\sqrt{5})/2 = 0.2402\ s$.
Though the main reason for a shift is avoidance of link interference within the range of mobility of the linkage, there is yet another reason to make the shift k greater than $k_{\min}$ as the dimensions of the gussets depend on the shift: the gussets are easier to manufacture if they have a reasonable size.

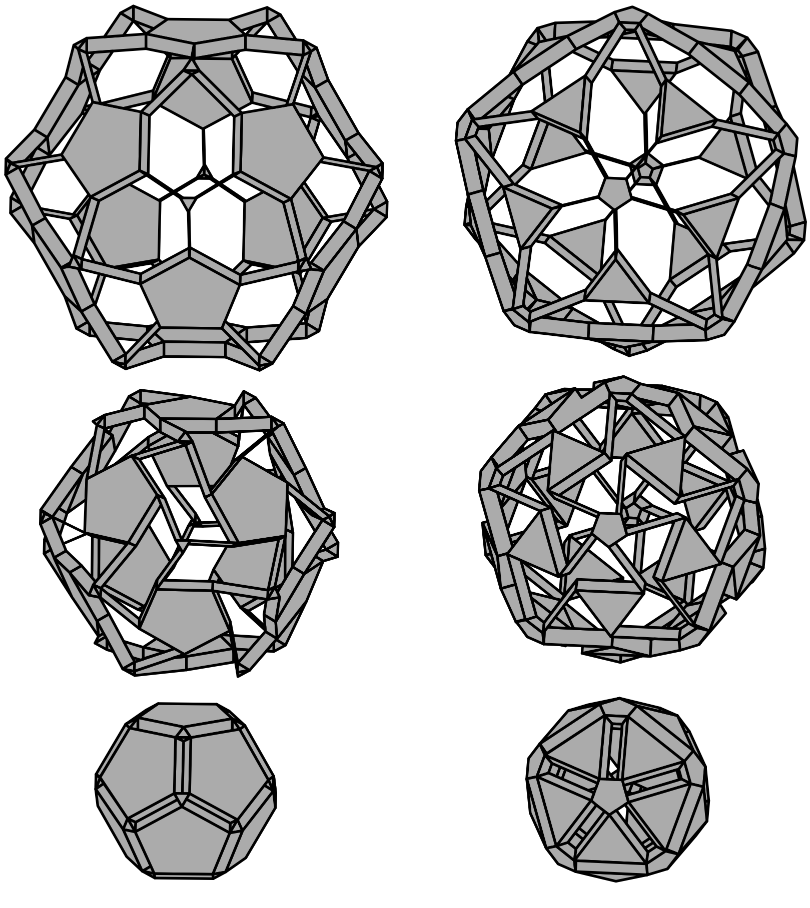

Figure 6. Different phases of the Dodecahedral Linkage
and the Icosahedral Linkage

7. An Archimedean Linkage

Beyond the Platonian Polyhedra any other polyhedron shown in Fig.2 can serve as a basis for the synthesis of a Polyhedral Linkage. From the Archimedean Polyhedra we shall choose as an example the "Football Polyhedron". As from this polyhedron in [1] a linkage has been derived by inserting of spatial spreading link groups into the polyhedral faces, we shall get the opportunity to compare the two different kinds of linkage construction derived from the same polyhedron. With $s_5 = s_6 = s$ and $b_5 = b_6 = b$ we obtain for the polygonal side lengths of the polyhedral face :

$$S_5 = s\,\{\cos\varphi_5 + \sqrt{[2\sin(\pi/5)]^2 - [\sin\varphi_5]^2}\}$$

$$S_6 = s\{\cos\varphi_6 + \sqrt{[2\sin(\pi/6)]^2 - [\sin\varphi_6]^2}\} = 2\,s\,\cos\varphi_6\,,$$

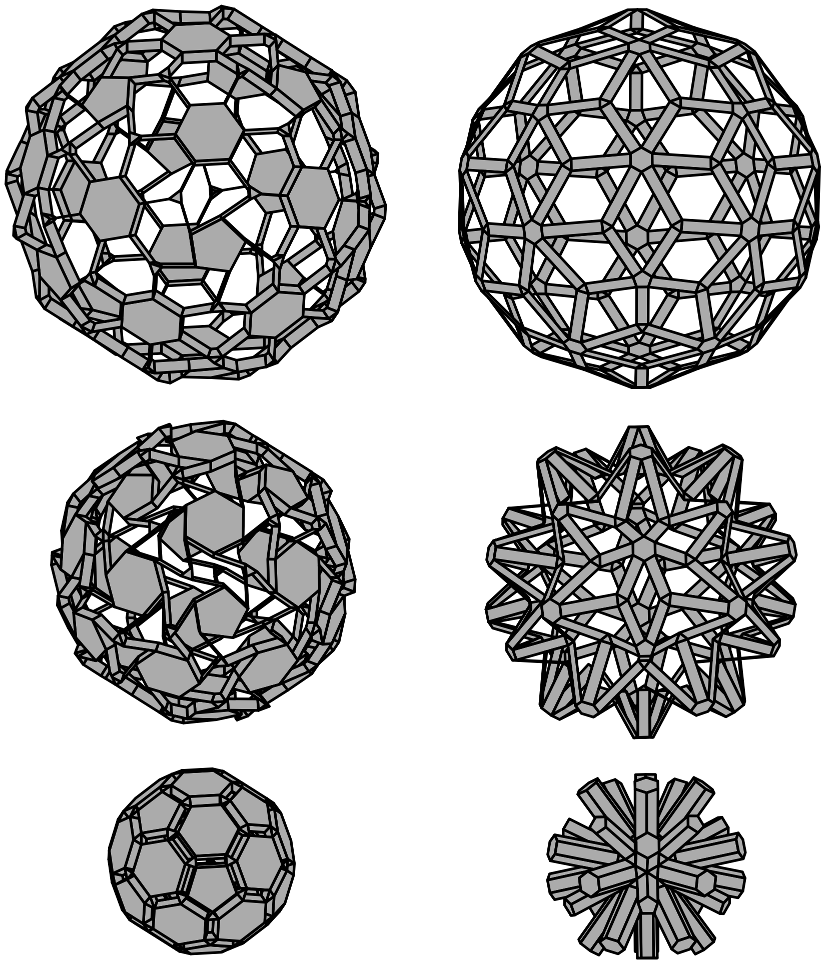

Figure 7. The Football Linkage constructed by inserting planar or spatial link groups

and obtain for the relation between the two position angles φ_5 and φ_6 from the condition informula (3) $S_5(\varphi_5) = S_6(\varphi_6)$:

$$\varphi_5 = \arccos[\cos\varphi_6 - \frac{3-\sqrt{5}}{8\cos\varphi_6}]. \qquad (7)$$

For the angle $\varphi_6 = 2\pi/3$ this formula yields: $\varphi_5 = 2\pi/5$. The linkage then closes up totally. However, while the planar link group in the six-sides faces is at its greatest extension ($\varphi_6 = 0$), the position angle in the five-sided faces is $\varphi_5(\varphi_5 = 0) = 25.2428^\circ$.
Fig.7 allows comparing the two ways of synthesizing a mobile linkage from the same polyhedron. Evidently, the size reduction from full extension to minimum extension is the same in both lingages while the enclosed volume goes back to one eighth. From the Fig.7 it is furtheron clear that the insertion of planar link groups leads to a more compact linkage. This is quite important from the technological viewpoint and is especilly obvious if one compares the two linkages in their closed position.

8. Polyhedral Linkage Complexes

Bycombining Polyhedral Linkages we obtain linkage complexes of which we shall give two examples. By piling up a number of Regular Prisms we get a concave polyhedron which can again serve as a basis for the synthesis of a Polyhedral Linkage. Piling up Regular Antiprisms would give a concave polyhedron, badly suited for the synthesis of a linkage. Figure 8 shows a Polyhedral Linkage Complex derived from a polyhedron which consists of five Regular Prisms put on top of each other.Condition $S_4(\varphi_4) = S_6(\varphi_6)$ gives for the relation between φ_4 and φ_6:

$$\varphi_4 = \arccos[\cos\varphi_6 + \frac{1}{4\cos\varphi_6}]. \qquad (8)$$

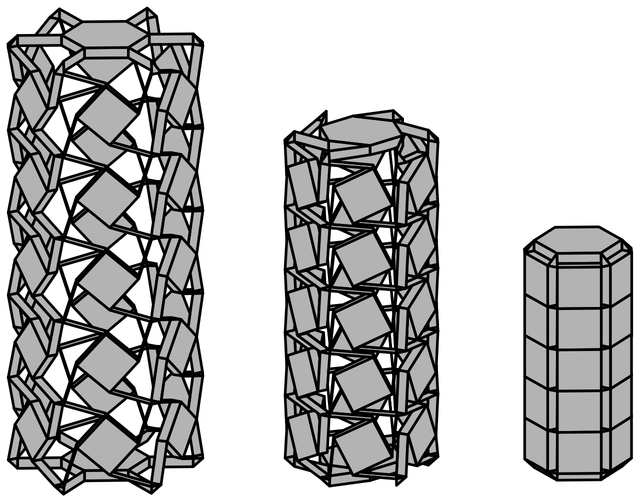

Figure 8. Polyhedral Linkage Complex derived from polyhedron consisting of piled up Regular Prisms

In our last example a Hexahedral Linkage (second linkage in Figure 5) serves as a module to generate a Complex Linkage. Two neighbouring Hexahedral Linkages have one link group in common. Clearly one can merge Hexahedral Linkages even in three different directions in this way. In Figure 9 the addition of fourteen module linkages has been made in two directions so that a closed structure is obtained.

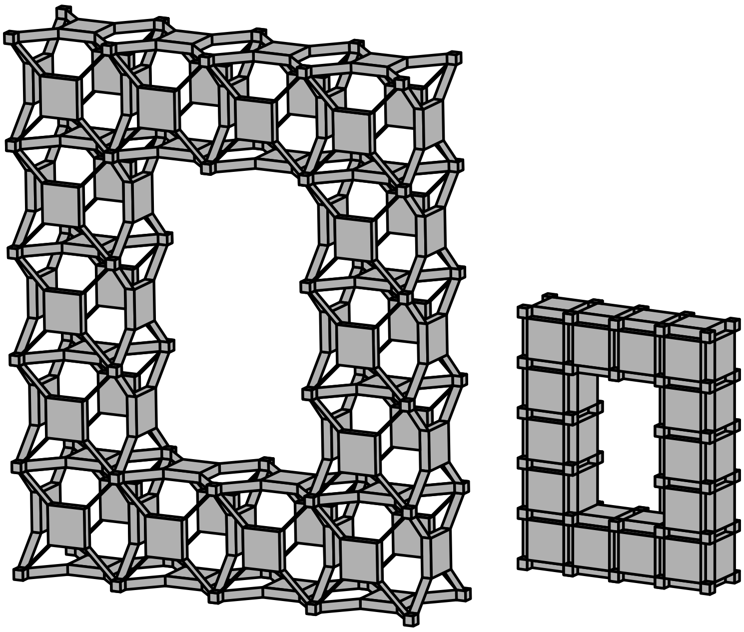

Figure 9. Polyhedral Linkage Complex generated by merging link groups of Hexahedral Linkages

In synthesizing new coverconstrained linkages in this paper we have used the simplest kind of planar link groups which can be inserted into the faces of a Regular Polyhedron. There are other more complex planar link groups which could be used for the same purpose. In the faces of a Rhombododecahedron a planar link group consisting of a pair of triangles has been inserted able to “open” the Rhombododecahedron {4}. In a similar way one could proceed to mobilize any one of the convex Regular Polyhedra. For example into the triangular faces of a Tetrahedron, an Octahedron or an Icosahedron one could insert a link group consisting of two orthogonal triangles or four equilateral sub-triangles.

A first IUTAM-IASS symposium devoted exclusively to “Depoyable Structures: Theory and Applications” recently took place in Cambridge [6]. The proceedings of this symposium are currently the best source of information about foldable structures which have increasingly been attracting the attention of space researchers, kinematicians and architects in the last few decades.

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