On the Stability and Stabilizability of Elastically Suspended Rigid Bodies

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Abstract

This paper deals with the stiffness and stability of an elastically suspended rigid body, pre-loaded by internal and external forces. For this problem the stiffness matrix of the mechanisms is derived, and its symmetry is analyzed. The problem of stable force distribution is stated at the level of force planning. It is shown that an unstable force distribution can be stabilized by a simple control law if the mechanism is not in a singular configuration. Two simple procedures of the design of the feedback gains are sketched. Finally, conditions of the feedback stabilizability in singular configurations are established in the general matrix form and illustrated on simple examples.

1 Introduction

In this paper we analyze the stiffness and the stability of an elastically suspended rigid body, pre-loaded by internal and external forces. A similar subject—stability due to internal forces in mechanisms with closed kinematic chains—was analyzed by Hanafusa and Adli [16] and Yi et al. [32]. In literature on multi-fingered grasps the dependence of the stiffness matrix on the contact forces was studied by Nguyen [25], Cutkosky and Kao [11], Kaneko et al. [20], Choi et al. [6].

In literature on parallel manipulators [24] the stiffness analysis has also received a considerable attention. Gosselin [14] and El-Khasawneh and Ferreira [12] analyzed the Cartesian stiffness mapping under the assumption of no pre-loading (zero driving forces). Whether this is a realistic assumption or not it depends on the specific applications. However, contribution of the pre-loading forces to the total stiffness matrix can be very significant, especially in the case of the force redundant parallel manipulators [4].

Another problem that can be caused by the pre-loading forces is the asymmetry of the resulting stiffness matrix. Pigoski et al. [26] used a planar, three spring, elastic coupling to investigate mapping of a 3×3 asymmetric stiffness matrix. Griffis and Duffy [15] derived an asymmetric stiffness matrix for a Stewart platform-type mechanism with six springs. Ciblak and Lipkin [8] extended the formulation to an arbitrary number of springs. According to Ciblak and Lipkin [8], the stiffness matrix is always asymmetric whenever the body is loaded by the external forces, even only by the gravity force. This does not seem to be correct, since the asymmetry indicates that some forces or moments are of non-conservative nature.

A more correct treatment has been given by Howard et al. [18], who used methods of differential geometry and Lie groups theory to study the Cartesian stiffness matrix of an elastically suspended body in a potential (conservative) force field. This work has been extended by Žefran and Kumar [34], who showed that the stiffness can be asymmetric away from equilibrium when it is defined in terms of screw coordinates (twists and wrenches) and not in terms of generalized coordinates.

In this paper we emply generalized coordinates to derive the stiffness matrix at equilibrium as we are interested mainly in the stability conditions. More specifically, we are interested in stabilization of unstable force distribution by the feedback. This sets the main direction of this paper. To the best of our knowledge, the only paper that treats the stabilizability of in-parallel mechanisms is that by Prattichizo and Bicchi [27]. However, they did it under the assumption of non-singular configurations.

This paper is organized as follows. In Section 2, an analytical expression for the stiffness matrix is derived. The derivation is based solely on the concepts of classical vectorial mechanics, which leads to coordinate-free equations. Also in this section, the conditions for the symmetry of the stiffness matrix are established. Section 3 gives a definition of the stable force distribution and introduces new theorems of stabilization of an unstable force distribution in singular configurations. An illustrative example is given in Section 4. Finally, conclusions are presented in Section 5.

2 Stiffness Matrix

Consider a parallel mechanism shown in Figure 1. Assume that the traveling body is loaded by the constant force $\mathcal{F}$ and the constant moment $\mathcal{M}$. Assume also that the contact forces $\boldsymbol{f}_1, \boldsymbol{f}_2, \dots, \boldsymbol{f}_n$ are applied at the points defined by the radius-vectors $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_n$ drawn from the reference point O.

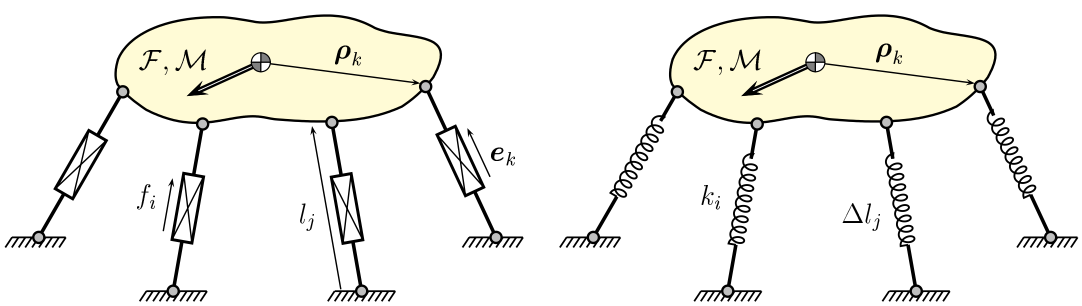

Figure 1: Parallel mechanism and the equivalent elastic system.

The static equations of the traveling body read

$$\mathcal{F} + \sum_{i=1}^{n} \boldsymbol{f}_i = \boldsymbol{0}, \tag{1}$$

$$\mathcal{M} + \sum_{i=1}^{n} \boldsymbol{\rho}_i \times \boldsymbol{f}_i = \mathbf{0}. \tag{2}$$

Let the direction of the i-th leg be given by the unit vector $\boldsymbol{e}_i$, and its length by l_i. The i-th leg is loaded by the contact force $-\boldsymbol{f}_i$, where

$$\boldsymbol{f}_i = f_i\, \boldsymbol{e}_i, \tag{3}$$

and f_i is the driving force.

The stiffness matrix of the object relates the infinitesimal changes in applied forces and moments, $\Delta\mathcal{F}$ and $\Delta\mathcal{M}$ with the resulting linear and rotational displacement* of the object, $\Delta\boldsymbol{x}$ and $\boldsymbol{\theta}$. The stiffness matrix can be obtained by linearizing the static equations (1, 2). It gives

$$\begin{aligned} \Delta\mathcal{F} &= -\sum_{i=1}^{n} \Delta\boldsymbol{f}_i, & (4)\\ \Delta\mathcal{M} &= -\sum_{i=1}^{n} \Delta\boldsymbol{\rho}_i \times \boldsymbol{f}_i + \boldsymbol{\rho}_i \times \Delta\boldsymbol{f}_i, & (5) \end{aligned}$$

Taking into account that the differential of a vector $\boldsymbol{\rho}$, constant "in the body," is defined as $\Delta\boldsymbol{\rho} = \boldsymbol{\theta} \times \boldsymbol{\rho}$, one transforms (5) to the following form

$$\Delta\mathcal{M} = \boldsymbol{K}_{\text{con}}\boldsymbol{\theta} - \sum_{i=1}^{n} \boldsymbol{\rho}_i \times \Delta\boldsymbol{f}_i. \tag{6}$$

where the convective term

$$\boldsymbol{K}_{\text{con}} = \sum_{i=1}^{n} \boldsymbol{\Omega}(\boldsymbol{f}_i)\boldsymbol{\Omega}^{\mathrm{T}}(\boldsymbol{\rho}_i), \tag{7}$$

and $\boldsymbol{\Omega}$ is the skew-symmetric operator such that $\boldsymbol{\Omega}(\boldsymbol{a}) \cdot \boldsymbol{b} \equiv \boldsymbol{a} \times \boldsymbol{b}$.

To proceed further, one must take into account the details of the driving system. Taking into account that $\Delta\boldsymbol{e}_i = \boldsymbol{\theta}_i \times \boldsymbol{e}_i$, where $\boldsymbol{\theta}_i$ is the vector of the infinitesimal rotation of the i-th leg, the linearization of (3) gives

$$\Delta\boldsymbol{f}_i = \Delta f_i\, \boldsymbol{e}_i + \boldsymbol{\theta}_i \times f_i \boldsymbol{e}_i. \tag{8}$$

In what follows, the linear spring relationships,

$$\Delta f_i = -k_i\, \Delta l_i, \tag{9}$$

where the active stiffness k_i of the translational motion of the i-th leg is generated by the control system, are employed. Thus, the disturbed system (4, 5) describes equilibrium of an elastically suspended rigid body, pre-loaded by the internal and external forces.

To relate Δl_i and $\boldsymbol{\theta}_i$ with $\Delta\boldsymbol{x}$ and $\boldsymbol{\theta}$, consider the linearized kinematic constraint equations

$$\boldsymbol{\theta}_i \times l_i \boldsymbol{e}_i + \Delta l_i \boldsymbol{e}_i = \Delta\boldsymbol{x} + \boldsymbol{\theta} \times \boldsymbol{\rho}_i, \tag{10}$$

*We do not put Δ in front of $\boldsymbol{\theta}$ as the infinitesimal rotation cannot be represented as a differential of a vector [13] expressing coincidence of the i-th contact point with the tip of the i-th leg under infinitesimal displacements. Next, the legs cannot spin about themselves, and it defines the additional constraints

$$\boldsymbol{\theta}_i \cdot \boldsymbol{e}_i = 0. \tag{11}$$

Solving (10, 11) with respect to Δl_i and $\boldsymbol{\theta}_i$ gives

$$\begin{aligned} \Delta l_i &= \boldsymbol{e}_i \cdot (\Delta \boldsymbol{x} + \boldsymbol{\theta} \times \boldsymbol{\rho}_i), & (12) \\ \boldsymbol{\theta}_i &= \frac{1}{l_i} \boldsymbol{e}_i \times (\Delta \boldsymbol{x} + \boldsymbol{\theta} \times \boldsymbol{\rho}_i). & (13) \end{aligned}$$

Substitution of (12,13) into (8,9), and then into (4,6) yields the resulting stiffness matrix $\boldsymbol{K}$. It has two components,

$$\boldsymbol{K} = \boldsymbol{K}_k + \boldsymbol{K}_f, \tag{14}$$

that are linear functions of, respectively, the active stiffnesses k_i, and the driving forces f_i. The components can be represented in the following form

$$\begin{aligned} \boldsymbol{K}_k &= \sum_{i=1}^{n} k_i \begin{bmatrix} \boldsymbol{e}_i \boldsymbol{e}_i^{\mathrm{T}} & \boldsymbol{e}_i \boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{\Omega}^{\mathrm{T}}(\boldsymbol{\rho}_i) \\ \boldsymbol{\Omega}(\boldsymbol{\rho}_i) \boldsymbol{e}_i \boldsymbol{e}_i^{\mathrm{T}} & \boldsymbol{\Omega}(\boldsymbol{\rho}_i) \boldsymbol{e}_i \boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{\Omega}^{\mathrm{T}}(\boldsymbol{\rho}_i) \end{bmatrix}, & (15) \\ \boldsymbol{K}_f &= \sum_{i=1}^{n} \frac{f_i}{l_i} \begin{bmatrix} \Omega^2(\boldsymbol{e}_i) & \Omega^2(\boldsymbol{e}_i) \Omega^{\mathrm{T}}(\boldsymbol{\rho}_i) \\ \Omega(\boldsymbol{\rho}_i) \Omega^2(\boldsymbol{e}_i) & \Omega(\boldsymbol{\rho}_i) \Omega^2(\boldsymbol{e}_i) \Omega^{\mathrm{T}}(\boldsymbol{\rho}_i) \end{bmatrix} + f_i \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \Omega(\boldsymbol{e}_i) \Omega^{\mathrm{T}}(\boldsymbol{\rho}_i) \end{bmatrix}, & (16) \end{aligned}$$

where $\boldsymbol{O}$ stands for the 3×3 zero matrix. Note that the expressions for the stiffness matrix are obtained in the coordinate-free form, leaving us with the freedom to choose the most convenient frames of reference for the computations.

As can be seen from (15-16) the tensor $\boldsymbol{K}_k$ is symmetric while the tensor $\boldsymbol{K}_f$ is generally not. The asymmetry of the total stiffness tensor comes from the convective term (7). One can prove that the skew-symmetric part of $\boldsymbol{K}_{\text{con}}$, $\boldsymbol{K}_{\text{con}} - \boldsymbol{K}_{\text{con}}^{\mathrm{T}} = \boldsymbol{\Omega}(\sum_{i=1}^{n} \boldsymbol{\rho}_i \times \boldsymbol{f}_i)$, and it is $-\boldsymbol{\Omega}(\mathcal{M})$ by the static equations (2). Therefore, $\boldsymbol{K}$ is symmetric at the static equilibrium as long as $\mathcal{M} = \mathbf{0}$. It is known [1, 7] that moments defined in either body-centered or space-centered frames are not conserving. The asymmetry of the stiffness matrix $\boldsymbol{K}$ under non-zero moment loading is the direct consequence of this fact. In the rest of this paper, unless otherwise is specified, we will be dealing with the symmetric stiffness matrix.

3 On the Stability of the Static Equilibrium

If the system is not conservative its stability is influenced by the mass matrix of the system (Bolotin [2]). For the conservative systems the stability of the equilibrium (1,2) depends on whether the stiffness matrix $\boldsymbol{K}$ is positive definite or not.

Note that it is meaningful to discuss the stability property even if $k_i = 0$ (no feedback). In this case, the mechanism is considered on the position/force planning level. Indeed, $\boldsymbol{K}_f$ explicitly depends on the driving forces f_i, and it is reasonable define the stability of the force distribution. A particular force distribution scheme, satisfying the static equations, can be called stable if $\boldsymbol{K}_f$ is positive definite. If such a force distribution is found it will always guarantee the total stability of the system because the feedback contribution, the matrix $\boldsymbol{K}_k$ is always positive semi-definite. However, in some situations, especially in case of non-redundant parallel manipulators, it is not always possible to produce stable $\boldsymbol{K}_f$. Some necessary and sufficient conditions for the stability of force distributions are given in our accompanying paper [29].

If the stability conditions for the matrix $\boldsymbol{K}_f$ are not achievable, the feedback stabilization by the "control springs" k_i is necessary. Note that if $\boldsymbol{K}_k$ is non-singular one can always make the total stiffness matrix $\boldsymbol{K}$ positive definite by simple increasing the gain coefficients $k_i > 0$ and ensuring that

$$\lambda_{\min}(\boldsymbol{K}_k) + \lambda_{\min}(\boldsymbol{K}_f) > 0. \tag{17}$$

More specific assignment of the feedback coefficients is sketched in the next subsection.

3.1 Synthesis of the "control springs"

1°. Assume that the force distribution $\boldsymbol{f}$ is fixed and the desired operational stiffness is given by the symmetric matrix $\boldsymbol{K}_d$. The trace of the off-diagonal block of $\boldsymbol{K}$ in (14) is zero, and if the joints are decoupled (the joint stiffness matrix is diagonal) it takes 20 springs to realize $\boldsymbol{K}_d$ (Lončarić [23]). To reduce the number of springs, some procedures of synthesis include $\boldsymbol{e}_i$ into the design variables [9, 19, 28]. This is applicable to the design of mechanical devices. For the same purpose, instead of varying the mechanism configuration, we will assume that the joints can be coupled by the feedback signals.

The matrix $\boldsymbol{K}_k$ can be represented as $\boldsymbol{K}_k = \boldsymbol{J}\mathcal{K}\boldsymbol{J}^{\mathrm{T}}$, where $\mathcal{K} \in \mathbb{R}^{n\times n}$ is the joint stiffness matrix that is not supposed to be diagonal, and

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \dots & \boldsymbol{e}_n \\ \boldsymbol{\rho}_1 \times \boldsymbol{e}_1 & \boldsymbol{\rho}_2 \times \boldsymbol{e}_2 & \dots & \boldsymbol{\rho}_n \times \boldsymbol{e}_n \end{bmatrix}, \tag{18}$$

is the Jacobian matrix, mapping the driving forces $\boldsymbol{f}$ to the external forces and moments (by duality $\boldsymbol{J}^{\mathrm{T}}$ maps the object displacements to the joint displacements Δl_i).

Given the desired operational stiffness $\boldsymbol{K}_d$, the joint stiffness matrix $\mathcal{K}$ (the "control" stiffness) can be found from the following linear matrix equation

$$\boldsymbol{J}\mathcal{K}\boldsymbol{J}^{\mathrm{T}} + \boldsymbol{K}_f = \boldsymbol{K}_d \tag{19}$$

Rearranging the elements of $\mathcal{K}$ into the vector form, $\mathrm{vec}(\mathcal{K}) = \{k_{11}, \dots, k_{n1}, \dots, k_{1n}, \dots, k_{nn}\}^{\mathrm{T}}$, one can represent (19) in the following form

$$(\boldsymbol{J} \otimes \boldsymbol{J})\ \mathrm{vec}(\mathcal{K}) = \mathrm{vec}(\boldsymbol{K}_d - \boldsymbol{K}_f) \tag{20}$$

where $\otimes$ denotes the tensor (Kroneker) product. Assume that $\boldsymbol{J}$ has full rank. Taking the general solution for (20) and translating it into the matrix form gives

$$\mathcal{K} = \boldsymbol{J}^{+}(\boldsymbol{K}_d - \boldsymbol{K}_f)(\boldsymbol{J}^{+})^{\mathrm{T}} + \{\boldsymbol{Z} - (\boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{Z}(\boldsymbol{J}^{+}\boldsymbol{J})^{\mathrm{T}}\}, \tag{21}$$

where $\boldsymbol{J}^{+} = \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})^{-1}$ is the Moore-Penrose pseudoinverse of $\boldsymbol{J}$ and the matrix $\boldsymbol{Z} \in \mathbb{R}^{n\times n}$ is arbitrary. The first component of (21) minimizes $\mathrm{tr}\mathcal{K}\mathcal{K}^{\mathrm{T}} = \mathrm{vec}^{\mathrm{T}}(\mathcal{K})\mathrm{vec}(\mathcal{K})$ under the constraint (19), while the second component of (21) belongs to the matrix null space of $\boldsymbol{J}$ defined as $\{\mathcal{K} : \boldsymbol{J}\mathcal{K}\boldsymbol{J}^{\mathrm{T}} = \boldsymbol{O}\}$. Note that the first component of (21) is always singular for $n > 6$.

In practical applications it might be desirable to have $\mathcal{K}$ symmetric and positive-definite. However, it is not clear how to choose $\boldsymbol{Z}$ in order to guarantee the positive-definiteness of $\mathcal{K}$. For this purpose, it is more convenient to use non-redundant parameterization of the matrix null space of $\boldsymbol{J}$. In such a parameterization, the general solution of (19) can be represented as

$$\mathcal{K} = \begin{bmatrix} \boldsymbol{J}^{+} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_d - \boldsymbol{K}_f & \boldsymbol{V} \\ \boldsymbol{U} & \boldsymbol{W} \end{bmatrix} \begin{bmatrix} (\boldsymbol{J}^{+})^{\mathrm{T}} \\ \boldsymbol{N}^{\mathrm{T}} \end{bmatrix}, \tag{22}$$

where $\boldsymbol{N} \in \mathbb{R}^{n\times r}$ is the base matrix of the vector null space of $\boldsymbol{J}$, $r = n - 6$ is the degree of redundancy, and $\boldsymbol{U} \in \mathbb{R}^{r\times 6}, \boldsymbol{V} \in \mathbb{R}^{6\times r}, \boldsymbol{W} \in \mathbb{R}^{r\times r}$ are arbitrarily specified matrices. For the matrix $\boldsymbol{\mathcal{K}}$ partitioned as (22) the stability conditions can be defined in more details (see, for example, Horn and Johnson [17]). Assuming $\boldsymbol{K}_f = \boldsymbol{K}_f^{\mathrm{T}}$ and choosing $\boldsymbol{K}_d = \boldsymbol{K}_d^{\mathrm{T}}$, $\boldsymbol{U} = \boldsymbol{V}^{\mathrm{T}}$ and $\boldsymbol{W}$ in such a way that $\boldsymbol{K}_d - \boldsymbol{K}_f$ and $\boldsymbol{W} - \boldsymbol{V}^{\mathrm{T}}(\boldsymbol{K}_d - \boldsymbol{K}_f)^{-1}\boldsymbol{V}$ are both positive definite, one can guarantee the positive definiteness of the joint stiffness matrix $\boldsymbol{\mathcal{K}}$.

It should be noted that a somewhat similar problem—synthesis of the joint stiffness matrix for the redundant serial manipulators—was addressed in [5, 21, 33]. In particular, Choi et al. [5] used the representation similar to (22) with $\boldsymbol{U} = \boldsymbol{V}^{\mathrm{T}} = 0$ and diagonal $\boldsymbol{W}$, and called it the orthogonal stiffness decomposition.

2°. It is of interest to modify the synthesis problem in such a way that would allow to define the driving forces $\boldsymbol{f}$ and the joint stiffness $\boldsymbol{\mathcal{K}}$ at the same time. One possible approach is to minimize the weighted sum of the Euclidian norms of $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{f}$ under the constraints (19) and those given by the static equations

$$\boldsymbol{J}\boldsymbol{f} = \Phi, \tag{23}$$

where $\Phi = \{\mathcal{F}^{\mathrm{T}}, \mathcal{M}^{\mathrm{T}}\}^{\mathrm{T}}$. The objective function of the corresponding unconditional minimization problem can be written down as

$$L = \alpha\, \mathrm{tr}\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^{\mathrm{T}} + \beta \boldsymbol{f}^{\mathrm{T}}\boldsymbol{f} + \mathrm{tr}(\boldsymbol{J}\boldsymbol{\mathcal{K}}\boldsymbol{J}^{\mathrm{T}} + \boldsymbol{K}_f - \boldsymbol{K}_d)\boldsymbol{\Lambda}^{\mathrm{T}} + (\boldsymbol{J}\boldsymbol{f} - \Phi)^{\mathrm{T}}\boldsymbol{\lambda}, \tag{24}$$

where α and β are given scalars, and $\boldsymbol{\Lambda} \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{\lambda} \in \mathbb{R}^n$ are the Lagrange multipliers. One can show that the minimum value of L is attained under

$$\boldsymbol{\mathcal{K}} = -\frac{1}{\alpha}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{J}, \tag{25}$$

$$\boldsymbol{f} = \boldsymbol{J}^{+}\Phi - \frac{1}{\beta}(\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{\mathcal{K}}_f^{\mathrm{T}}\mathrm{vec}(\boldsymbol{\Lambda}), \tag{26}$$

where

$$\mathrm{vec}(\boldsymbol{\Lambda}) = \left\{\frac{1}{\alpha}(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}) \otimes (\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}) + \frac{1}{\beta}\boldsymbol{\mathcal{K}}_f(\boldsymbol{I} - \boldsymbol{J}^{+}\boldsymbol{J})\boldsymbol{\mathcal{K}}_f^{\mathrm{T}}\right\}^{-1}\boldsymbol{\mathcal{K}}_f\boldsymbol{J}^{+}\Phi - \mathrm{vec}(\boldsymbol{K}_d), \tag{27}$$

and the matrix $\boldsymbol{\mathcal{K}}_f \in \mathbb{R}^{36\times n}$ is defined as $\boldsymbol{\mathcal{K}}_f = \begin{bmatrix} \mathrm{vec}(\boldsymbol{K}_{f_1}) & \mathrm{vec}(\boldsymbol{K}_{f_2}) & \dots & \mathrm{vec}(\boldsymbol{K}_{f_n}) \end{bmatrix}$, so that $\mathrm{vec}(\boldsymbol{K}_f) = \mathrm{vec}(\sum_{i=1}^{n}\boldsymbol{K}_{f_i}f_i) = \boldsymbol{\mathcal{K}}_f\boldsymbol{f}$. In this solution $\boldsymbol{f}$ has the internal force component even if $\Phi = 0$. Also note that the resulting $\boldsymbol{\mathcal{K}}$ is singular when $n > 6$. To establish non-singular joint stiffness, more physically sound rather than simply mathematically tractable criteria of optimality should be defined. In this connection the ideas of Tsuji et al. [30] might give additional insight.

3.2 On the Feedback Stabilization in Singular Configurations

If the matrix $\boldsymbol{K}_k$ is singular then $\lambda_{\min}(\boldsymbol{K}_k) = 0$ and inequality (17) does not depend on $\boldsymbol{K}_k$. In this case a a special study on the structure of $\boldsymbol{K}_k$ and $\boldsymbol{K}_f$ is required. In literature (see, for example, Wen and Wilfinger [31]), singular configurations of the parallel mechanisms are often associated with instability. However, the singular configurations are not necessarily unstable in the Lyapunov sense. In fact, the singular configurations can be stable or unstable depending on the type of singularity and on the force distribution.

In general, the judgement on the stability of a singular configuration is established as follows. Let $\mathcal{N}(\boldsymbol{K}_k) \neq \boldsymbol{0}$ be the null space of the matrix $\boldsymbol{K}_k$. If the disturbance $\boldsymbol{y} \notin \mathcal{N}(\boldsymbol{K}_k)$ the stiffness matrix $\boldsymbol{K}$ can be made stable by increasing the gain coefficients k_i. However, it is not the case when $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{K}_k)$. Denote by $\boldsymbol{N}_k$ the base matrix of $\mathcal{N}(\boldsymbol{K}_k)$ so that $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{K}_k)$ can be represented as $\boldsymbol{y} = \boldsymbol{N}_k \boldsymbol{z}$. The change of the potential energy is $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{y} = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{N}_k^{\mathrm{T}} \boldsymbol{K}_f \boldsymbol{N}_k \boldsymbol{z}$, and there holds the following

Theorem 1 *A force distribution in a singular configuration is stabilizable by the "control" springs if the matrix $\boldsymbol{N}_k^T \boldsymbol{K}_f \boldsymbol{N}_k$ is positive definite, and it is not stabilizable if at least one of the eigen-values of $\boldsymbol{N}_k^T \boldsymbol{K}_f \boldsymbol{N}_k$ is negative.*

In some cases the stabilizability judgement can be done without exploring the eigen-values of $\boldsymbol{N}_k^{\mathrm{T}} \boldsymbol{K}_f \boldsymbol{N}_k$. Let $\mathcal{R}^+(\boldsymbol{K}_f)$ and $\mathcal{R}^-(\boldsymbol{K}_f)$ be the ranges spanned by the eigen-vectors corresponding to, respectively, the positive and the negative eigen-values of $\boldsymbol{K}_f$.

Theorem 2 *A force distribution in a singular configuration is stabilizable by the "control" springs if $\mathcal{N}(\boldsymbol{K}_k) \perp \mathcal{R}^-(\boldsymbol{K}_f)$ and $\mathcal{N}(\boldsymbol{K}_k) \cap \mathcal{N}(\boldsymbol{K}_f) = \boldsymbol{0}$.*

Proof of the theorem is straightforward. Represent $\boldsymbol{K}_f = \sum \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^{\mathrm{T}} + \sum \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^{\mathrm{T}}$, where the eigen-vectors $\boldsymbol{u}_i$ correspond to $\lambda_i > 0$ and $\boldsymbol{v}_i$ do to $\lambda_i < 0$. Next, $\forall \boldsymbol{y} \in \mathcal{N}(\boldsymbol{K}_k)$ one has $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{y} = 0$. If $\boldsymbol{y} \perp \boldsymbol{v}_i$ and $\boldsymbol{y} \notin \mathcal{N}(\boldsymbol{K}_f)$ then $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{y} = \sum \lambda_i (\boldsymbol{y}^{\mathrm{T}} \boldsymbol{u}_i)^2 > 0$ and the equilibrium is stable.

Similarly, we can state the following

Theorem 3 *A force distribution in a singular configuration is not stabilizable by the "control" springs if $\mathcal{N}(\boldsymbol{K}_k) \perp \mathcal{R}^+(\boldsymbol{K}_f)$ and $\mathcal{N}(\boldsymbol{K}_k) \cap \mathcal{R}^-(\boldsymbol{K}_f) \neq \boldsymbol{0}$.*

The importance of the theorems established is in that they allow to estimate the stabilizability of a force distribution without direct calculation of the stability conditions. Indeed, the matrix $\boldsymbol{K}_k = \boldsymbol{J} \boldsymbol{\mathcal{K}} \boldsymbol{J}^{\mathrm{T}}$, and there holds the following

Theorem 4 *$\mathcal{N}(\boldsymbol{K}_k) = \mathcal{N}(\boldsymbol{J}^T)$ if $\mathcal{R}(\boldsymbol{J}^T) \cap \mathcal{N}(\boldsymbol{\mathcal{K}}) = \boldsymbol{0}$.*

Proof of this statement is straightforward. If $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{J}^{\mathrm{T}}) \Longrightarrow \boldsymbol{y} \in \mathcal{N}(\boldsymbol{K}_k)$. Assume now that $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{K}_k)$ and put $\boldsymbol{z} = \boldsymbol{J}^{\mathrm{T}} \boldsymbol{y}$, i.e. $\boldsymbol{z} \in \mathcal{R}(\boldsymbol{J}^{\mathrm{T}})$. Then $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{y} = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{\mathcal{K}} \boldsymbol{z} = \boldsymbol{0}$. Therefore if $\mathcal{R}(\boldsymbol{J}^{\mathrm{T}}) \cap \mathcal{N}(\boldsymbol{\mathcal{K}}) = \boldsymbol{0} \Longrightarrow \boldsymbol{z} = \boldsymbol{0}$ and $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{J}^{\mathrm{T}})$.

Therefore, if only $\boldsymbol{\mathcal{K}}$ is non-singular, no information about the actual stiffness of the "control springs" is necessary to judge the stabilizability. In other words, the judgement on the the stabilizability can be made at the force planning level.

To complete the analysis, one should mention the critical cases when the stabilizability judgement cannot be made based on the linearized model. That is the case when $\mathcal{N}(\boldsymbol{K}_f) \neq \boldsymbol{0}$ and the matrix $\boldsymbol{N}_k^{\mathrm{T}} \boldsymbol{K}_f \boldsymbol{N}_k$ is positive semi-definite. The critical cases can be recognized by the condition $\mathcal{N}(\boldsymbol{K}_k) \perp \mathcal{R}(\boldsymbol{K}_f)$ where $\mathcal{R}(\boldsymbol{K}_f) = \mathcal{R}^+(\boldsymbol{K}_f) \cup \mathcal{R}^-(\boldsymbol{K}_f)$. The most simple example when this condition is satisfied is the case of the unloaded system ($f_i = 0$). In this case the instability is associated with "large" motions not changing the potential energy[†], and the non-linear term analysis is required.

4 Illustrative Example

1°. Consider a Gough-Stewart platform with n legs and assume that the base points of the legs and the connection points on the platform form regular polyhedrons with n vertices. Assume also that the mechanism is in the configuration shown in Figure 2, where the base is parallel to the platform and the lengths of all the legs are same, $l_i = l$, $i = 1, \ldots, n$.

[†]A trivial example is when the bases of all the legs are centered at one point. The rotation of the platform is uncontrollable and therefore unstable.

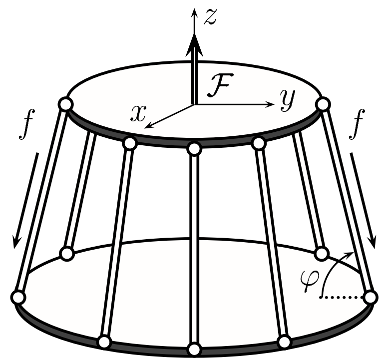

Figure 2: Illustrative example.

Let us calculate the stiffness matrix at the center of the reference frame of the platform. The vectors $\boldsymbol{e}_i$ and $\boldsymbol{\rho}_i$, expressed in the projections onto the axes of the reference frame, can be represented as

$$\boldsymbol{\rho}_i = \begin{bmatrix} r\cos\psi_i \\ r\sin\psi_i \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_i = \begin{bmatrix} -\cos\psi_i\cos\varphi \\ -\sin\psi_i\cos\varphi \\ \sin\varphi \end{bmatrix}, \tag{28}$$

where $\psi_i = \psi_0 + 2\pi i/n$, and $\psi_0 = \text{const}$. The angle of inclination φ is defined so that $h = l\sin\varphi$, $R - r = l\cos\varphi$, where h is the height of the platform, R and r are the radii of, respectively, the base and the platform.

The configuration under consideration is singular as the mechanism cannot resist the moment about the axis z. Assume that the mechanism is loaded by the force $\mathcal{F} = \{0, 0, F\}^{\mathrm{T}}$ and the force distribution is uniform: $f_i = f$, $i = 1, \ldots, n$. Neglecting the mass of the legs, solving the static equations gives $f = -F/n\sin\varphi$.

It is known (see, for example, Coxeter [10]) that the following identities hold for a regular polyhedron with n vertices: $\sum_{i=1}^{n}\sin\psi_i = \sum_{i=1}^{n}\cos\psi_i = \sum_{i=1}^{n}\sin\psi_i\cos\psi_i = 0$, $\sum_{i=1}^{n}\sin^2\psi_i = \sum_{i=1}^{n}\cos^2\psi_i = n/2$. Taking them into account, one obtains the following expression for the stiffness matrix:

$$\boldsymbol{K} = \begin{bmatrix} a_x & 0 & 0 & 0 & c_z & 0 \\ 0 & a_y & 0 & -c_z & 0 & 0 \\ 0 & 0 & a_z & 0 & 0 & 0 \\ 0 & -c_z & 0 & b_x & 0 & 0 \\ c_z & 0 & 0 & 0 & b_y & 0 \\ 0 & 0 & 0 & 0 & 0 & b_z \end{bmatrix}, \tag{29}$$

where

$$a_x = a_y = \frac{n}{2}\left\{(k + \frac{f}{l})\cos^2\varphi - 2\frac{f}{l}\right\}, \tag{30}$$

$$b_x = b_y = \frac{nr}{2}\left\{(k + \frac{f}{l})\, r\sin^2\varphi - R\frac{f}{l}\right\}, \tag{31}$$

$$a_z = n\left\{(k + \frac{f}{l})\sin^2\varphi - \frac{f}{l}\right\}, \tag{32}$$

$$b_z = -nrR\frac{f}{l}, \tag{33}$$

$$c_z = \frac{nr}{2}(k + \frac{f}{l}) \sin\varphi \cos\varphi. \tag{34}$$

2°. *Stability of the mechanism.* The stability conditions for the matrix $\boldsymbol{K}$ of the form (29) are defined as follows:

$$a_x > 0, \quad a_z > 0, \quad b_z > 0, \quad b_x > c_z^2/a_x. \tag{35}$$

The coefficient b_z is not influenced by the active control stiffness k, and it gives

$$f < 0. \tag{36}$$

The translational stability ($a_x > 0$, $a_z > 0$) is guaranteed for any $k > 0$ providing that $f < 0$. The exact estimate for k is found from the last condition of the set (35), which, after some transformations, can be represented as

$$\frac{R}{r} > \frac{(k + \frac{f}{l}) \sin^2\varphi}{\frac{f}{l} - \frac{1}{2}(k + \frac{f}{l}) \cos^2\varphi}. \tag{37}$$

The graphical illustration of the stability area in the plane $R/r, k$ is shown in Figure 3. Note that the condition (37) is always satisfied if $k > -f/l$ and $f < 0$ (the sufficient conditions). Resolving (37) with respect to k gives

$$k > -\frac{f}{l}\left(1 - \frac{1}{\frac{1}{2}\cos^2\varphi + \frac{r}{R}\sin^2\varphi}\right). \tag{38}$$

The conditions (36,38) are necessary and sufficient for the stability of the platform under consideration.

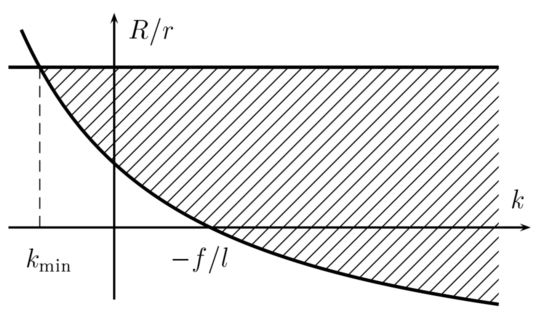

Figure 3: Stability area.

3°. *Stability and stabilizability of the force distribution.* Put $k = 0$ and consider the stability of the uniform force distribution. The first condition is still given by (36) while the second condition (37) can be transformed now to the following form:

$$\cot^2\varphi > 2(r - R)/R. \tag{39}$$

This condition is always satisfied if the radius of the base, R, is larger than that of the platform, r. If, however, $R < r$ then there exists a critical angle φ at which the force distribution looses

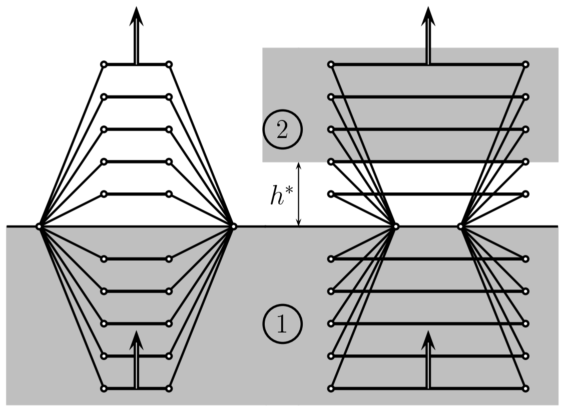

Figure 4: Stability of the force distribution.

its stability. This behavior is illustrated graphically in Figure 4, where the areas of instability are shown in gray color. One can see that if $R < r$ and the platform exceeds the critical height $h^* = \sqrt{R(r-R)/2}$, the force distribution becomes unstable.

Note that if the condition $f < 0$ is violated (area 1 in Figure 4) the system cannot be stabilized as it is impossible to change b_z by the active control stiffness k. If, however, only the condition (39) is violated (area 2 in Figure 4) the system can be stabilized by choosing k in accordance with (38).

Let us now show that the judgement on the stabilizability can be made without establishing the exact estimate (38). The null space of the matrix $\boldsymbol{J}^{\mathrm{T}}$ (that is the null space of $\boldsymbol{K}$ when $f = 0$ and $k = 1$) is parameterized by the vector

$$\boldsymbol{n} = \{0,\, 0,\, 0,\, 0,\, 0,\, 1\}^{\mathrm{T}}. \tag{40}$$

Next, the eigen-values of the matrix $\boldsymbol{K}_f$ are

$$\lambda_1 = \lambda_2 = (a_x + b_x - d)/2, \quad \lambda_3 = a_z, \tag{41}$$
$$\lambda_4 = \lambda_5 = (a_x + b_x + d)/2, \quad \lambda_6 = b_z, \tag{42}$$

where $d = \sqrt{(a_x - b_x)^2 + 4c_z^2}$. The corresponding eigen-vectors (not normalized) are defined as

$$\boldsymbol{w}_1 = \{(a_x - b_x - d)/2c_z,\, 0,\, 0,\, 0,\, 1,\, 0\}^{\mathrm{T}}, \tag{43}$$
$$\boldsymbol{w}_2 = \{0,\, (b_x - a_x + d)/2c_z,\, 0,\, 1,\, 0,\, 0\}^{\mathrm{T}}, \tag{44}$$
$$\boldsymbol{w}_3 = \{0,\, 0,\, 1,\, 0,\, 0,\, 0\}^{\mathrm{T}}, \tag{45}$$
$$\boldsymbol{w}_4 = \{(a_x - b_x + d)/2c_z,\, 0,\, 0,\, 0,\, 1,\, 0\}^{\mathrm{T}}, \tag{46}$$
$$\boldsymbol{w}_5 = \{0,\, (b_x - a_x - d)/2c_z,\, 0,\, 1,\, 0,\, 0\}^{\mathrm{T}}, \tag{47}$$
$$\boldsymbol{w}_6 = \{0,\, 0,\, 0,\, 0,\, 0,\, 1\}^{\mathrm{T}}. \tag{48}$$

Note that $\boldsymbol{n} \perp \boldsymbol{w}_i$, $i = 1, \ldots, 5$, and $\boldsymbol{n} = \boldsymbol{w}_6$. If $f > 0$ then $\lambda_6 < 0$ and $\mathcal{N}(\boldsymbol{J}^{\mathrm{T}}) \cap \mathcal{R}^-(\boldsymbol{K}_f) \neq \boldsymbol{0}$. Therefore, such a force distribution is not stabilizable by Theorem 3. If, on the other hand, $f < 0$ then $\lambda_6 > 0$ and $\mathcal{N}(\boldsymbol{J}^{\mathrm{T}}) \perp \mathcal{R}^-(\boldsymbol{K}_f)$ regardless of what specific eigen-values become negative. In this case $\mathcal{N}(\boldsymbol{J}^{\mathrm{T}}) \cap \mathcal{N}(\boldsymbol{K}_f) = \boldsymbol{0}$ and the force distribution is stabilizable by Theorem 2.

The stabilizability theorems established in Section 3 are based on the use of the matrix $\boldsymbol{K}$ taken in the non-partitioned form. As a result, they are not convenient for an analytical treatment (design situations) as the eigen-values of $\boldsymbol{K}$ mix different dimensions and are not invariant under the change of the reference frame. However, the signs of the eigen-values are preserved under the rigid body transformations [3] and the theorems can be useful for numerical checking stabilazability based on the geometric structure of the $\boldsymbol{J}^{\mathrm{T}}$ and $\boldsymbol{K}_f$.

More convenient forms of the stabilazability conditions, suitable for the analytical treatment, should be established from the partitioned (into the translational, rotational, and cross-coupled parts) form of the matrix $\boldsymbol{K}$. The partitioning can also be helpful in estimating the degree of stabilazability. The results of Lin et al. [22] may be of interest for the research in this direction.

5 Conclusions

The stability problem for an elastically suspended rigid body, pre-loaded by internal and external forces, has been considered in this paper. For this problem the stiffness matrix has been derived, and its symmetry has been analyzed. The problem of stable force distribution has been stated at the level of force planning. It has been shown that an unstable force distribution can be stabilized by a simple control law if the mechanism is not in a singular configuration, and two simple procedures for the design of the feedback gains have been sketched out. As to the singular configurations, the stabilizability depends on the type of singularity and on the force distribution. The conditions of the feedback stabilizability has been established in the general matrix form.

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