Generation of Architecturally Singular 6-SPS Parallel Manipulators with Linearly Related Planar Platforms

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Abstract: An architecturally singular parallel manipulator is a parallel manipulator with architectural singularities spanning the whole workspace. It should be abandoned in the design process. This paper deals with the generation of architecturally singular 6-SPS parallel manipulators with linearly related planar platforms. The forward displacement analysis of the 6-SPS parallel manipulator with linearly related planar platforms is performed at first. All the architecturally singular 6-SPS parallel manipulators with linearly related planar platforms are then generated. It is found that a 6-SPS parallel manipulator with linearly related planar platforms is architecturally singular if and only if all the spherical joints on the base (or moving platform) are located on a conic section. At last, self-motion analysis of the architecturally singular 6-SPS parallel manipulators with linearly related planar platforms is addressed.

1 Introduction

An architecturally singular parallel manipulator is a parallel manipulator with architectural singularities spanning the whole workspace. Architecturally singular parallel manipulators are useless since they cannot be controlled. Generation of architecturally singular parallel manipulators should be performed in the early stage of design to guarantee that architecturally singular parallel manipulators, if they exist, are abandoned in the design process.

Fruitful results have been obtained on the generation of architecturally singular 6-SPS (Gough-Stewart platform) parallel manipulators [1]–[9]. So far, all the architecturally singular 6-SPS parallel manipulators containing PL, PB and LL components have been revealed [1, 2, 3]. Here, P, L and B denote point, straight line segment and rigid body respectively [2]. A class of architecturally singular 6-SPS parallel manipulators containing the LB component is proposed in [3]. Some architecturally singular 6-SPS parallel manipulators with planar platforms of class BB are revealed in [4, 5, 6, 7]. All the architecturally singular Griffis-Duffy type 6-SPS parallel manipulators are revealed in [8]. In [7], a sufficient condition is also obtained for architecturally singular 6-SPS parallel manipulators with spatial platforms. A general method is given in [9] to construct architecturally singular 6-SPS parallel manipulators.

Different approaches have been proposed to generate architecturally singular 6-SPS parallel manipulators.

a) The first is the method based on the determinant of the Jacobian matrix ([3, 7, 8]. When the parameters of the base platform and moving platform which make the determinant of the Jacobian matrix vanish are found, architecturally singular 6-SPS parallel manipulators are obtained.

b) The second is the geometric method based on linear manifolds of correlations and quadratic transformations [9]. For an architecturally singular 6-SPS parallel manipulator, the spherical joints are conjugate points with respect to 3-dimensional linear manifolds of correlations.

c) The third is the method based on the forward displacement analysis [2, 6]. The generation of architecturally singular 6-SPS parallel manipulators amounts to finding the parameters for the base platform and moving platform which allow continuous solutions to the forward displacement problem for a given set of inputs.

d) The fourth is the component approach [2]. The generation of architecturally singular 6-SPS parallel manipulators is reduced to the generation of over-constrained components or architecturally singular components.

Due to the complexity of the generation of architecturally singular 6-SPS parallel manipulators and the large amount of classes of 6-SPS parallel manipulators [10], it is of practical importance to investigate the 6-SPS parallel manipulators class by class.

This paper deals with the generation of architecturally singular 6-SPS PMslpp (parallel manipulators with linearly related planar platforms) using the method based on the forward displacement analysis. The 6-SPS PMlpp (parallel manipulator with linearly related planar platforms) is proposed in [11, 12]. The characteristic of these manipulators is that the forward displacement analysis can be performed in closed form and the maximum number of isolated real solutions is 8. In this paper, the forward displacement analysis of the 6-SPS PMlpp is performed at first. The architecturally singular 6-SPS PMslpp are then generated. At last, self-motion analysis of the architecturally singular 6-SPS PMslpp is performed.

2 Description of the 6-SPS PMlpp

A general 6-SPS parallel manipulator is shown in Fig. 1. It is constructed by connecting a moving platform ($B_1B_2B_3B_4B_5B_6$) and a base ($A_1A_2A_3A_4A_5A_6$) platform with six SPS legs (A_iB_i). Here S denotes a spherical joint which is passive while P denotes an actuated prismatic joint.

For purposes of simplification and without loss of generality, two coordinate systems are established. The coordinate system $O - XYZ$ is attached to the base platform ($\mathrm{A_1A_2A_3A_4A_5A_6}$) with O being coincident with A_1 and the X-axis passing through A_2. The coordinate system $O_P - X_PY_PZ_P$ is attached to the moving platform ($\mathrm{B_1B_2B_3B_4B_5B_6}$) with O_P being coincident with B_1 and the X_P-axis passing through B_2. Y (Y_P) is selected so that $z_{A3} = 0$ ($z_{B3}^{(P)} = 0$). The unit vectors along the axes of the coordinate systems are $\mathbf{i} = \{1 \ \ 0 \ \ 0\}^T$, $\mathbf{j} = \{0 \ \ 1 \ \ 0\}^T$, $\mathbf{k}$ and $\mathbf{i}_P = \{i_x \ \ i_y \ \ i_z\}^T$, $\mathbf{j}_P = \{j_x \ \ j_y \ \ j_z\}^T$,$\mathbf{k}_P$ respectively. The position vectors of A_i and B_i in the coordinate system $O-XYZ$ are denoted by $\mathbf{A}_i = \{x_{Ai} \ \ y_{Ai} \ \ z_{Ai}\}^T$ and $\mathbf{B}_i = \{x_{Bi} \ \ y_{Bi} \ \ z_{Bi}\}^T$ (i=1, 2, ..., 6). The position vector of B_i in the coordinate system $O_P - X_PY_PZ_P$ is denoted by $\mathbf{B}_i^P = \{x_{Bi}^{(P)} \ \ y_{Bi}^{(P)} \ \ z_{Bi}^{(P)}\}^T$.

For a 6-SPS PMlpp, there exists

$$\begin{cases} x_{Bi}^{(P)} = t_{11}x_{Ai} + t_{12}y_{Ai} \\ y_{Bi}^{(P)} = t_{22}y_{Ai} \\ z_{Bi}^{(P)} = 0 \\ z_{Ai} = 0 \end{cases} \qquad i = 2, 3, \ldots, 6 \tag{1}$$

where $t_{11}t_{22} \neq 0$.

It should be pointed out that the condition for the 6-SPS PMlpp given here is more accurate than that given in [11, 12]. In the condition given in [11, 12], the deformation of the moving platform is not separated from its overall rotation.

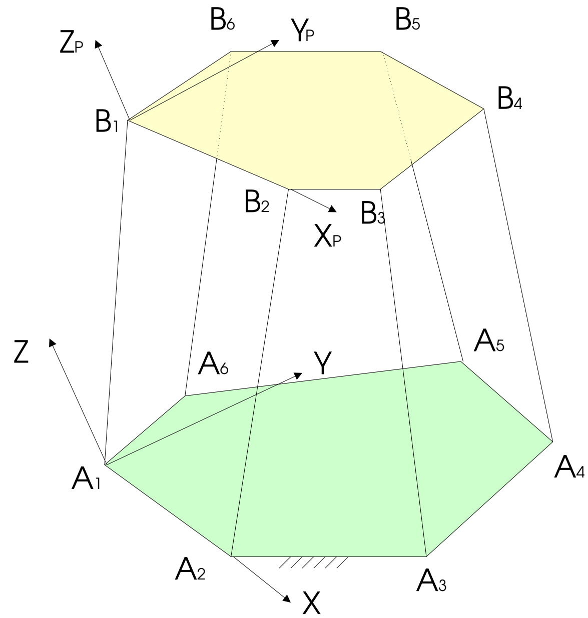

Figure 1. General 6-SPS parallel manipulator.

3 Forward Displacement Analysis of the 6-SPS PMlpp

The forward displacement analysis of a 6-SPS parallel manipulator can be stated as follows: for a given set of inputs $l_i = ||A_i B_i||$, i =1, 2, ..., 6, find the pose (position and orientation) of the moving platform. The position of the moving platform is denoted by $\mathbf{O}_P = \mathbf{B}_1 = \{x \quad y \quad z\}^T$. The orientation of the moving platform is denoted by $\mathbf{i}_P$ and $\mathbf{j}_P$. We have

$$i_x^2 + i_y^2 + i_z^2 = 1 \tag{2}$$

$$j_x^2 + j_y^2 + j_z^2 = 1 \tag{3}$$

$$i_x j_x + i_y j_y + i_z j_z = 0 \tag{4}$$

and

$$\mathbf{B}_1^T \mathbf{B}_1 = l_1^2 \tag{5}$$

In this section, an alternative approach is presented for the forward displacement analysis of the 6-SPS PMlpp. The approach taken here is a modification of that presented in [11] and is more concise and suitable for the generation of architecturally singular 6-SPS parallel manipulators.

The loop closure equation of loop $A_1 B_1 B_i A_i A_1$ is

$$(\mathbf{B}_i - \mathbf{A}_i)^T (\mathbf{B}_i - \mathbf{A}_i) = l_i^2 \quad i = 2, 3, \ldots, 6 \tag{6}$$

where

$$\mathbf{B}_i = \mathbf{B}_1 + x_{Bi}^{(P)} \mathbf{i}_P + y_{Bi}^{(P)} \mathbf{j}_P$$

$$\mathbf{A}_i = x_{Ai}\mathbf{i} + y_{Ai}\mathbf{j}$$

Eq. (6) can be re-written as

$$\begin{aligned} x_{Bi}^{(P)}\mathbf{B}_1^T\mathbf{i}_P + y_{Bi}^{(P)}\mathbf{B}_1^T\mathbf{j}_P - x_{Ai}x_{B1} - y_{Ai}y_{B1} \\ -x_{Bi}^{(P)}x_{Ai}i_x - x_{Bi}^{(P)}y_{Ai}i_y - y_{Bi}^{(P)}x_{Ai}j_x - y_{Bi}^{(P)}y_{Ai}j_y = L_i \quad i = 2, 3, \ldots, 6 \end{aligned} \tag{7}$$

where

$$L_i = \frac{1}{2}(l_i^2 - x_{Bi}^{(P)2} - y_{Bi}^{(P)2} - x_{Ai}^2 - y_{Ai}^2 - l_1^2)$$

Substitution of Eq. (1) into Eq. (7) yields

$$x_{Ai}u_1 + y_{Ai}u_2 + x_{Ai}^2u_3 + y_{Ai}^2u_4 + x_{Ai}y_{Ai}u_5 = L_i \quad i = 2, 3, \ldots, 6 \tag{8}$$

where

$$u_1 = t_{11}\mathbf{B}_1^T\mathbf{i}_P - x_{B1} \tag{9}$$

$$u_2 = t_{12}\mathbf{B}_1^T\mathbf{i}_P + t_{22}\mathbf{B}_1^T\mathbf{j}_P - y_{B1} \tag{10}$$

$$u_3 = -t_{11}i_x \tag{11}$$

$$u_4 = -t_{12}i_y - t_{22}j_y \tag{12}$$

$$u_5 = -t_{12}i_x - t_{11}i_y - t_{22}j_x \tag{13}$$

Eq. (8) can be regarded as a set of linear equations in u_j, j=1,2, ..., 5. For a non-architecturally-singular 6-SPS parallel manipulator, there should be unique set of solutions to u_i when the inputs are specified, i. e., the determinant of the coefficient matrix should not be zero.

After the u_i's have been obtained, the pose of the moving platform can be found easily following the procedure given in [11]. The remaining steps are omitted here as they have no relation to the generation of architecturally singular 6-SPS parallel manipulators.

4 Generation of Architecturally Singular 6-SPS PMslpp

For an architecturally singular 6-SPS PMlpp, there should be continuous solutions to the u_i's when the inputs are specified, i. e., the five linear equations in Eq. (8) should be linearly dependent. The linear dependence of the five linear equations in Eq. (8) means that the determinant of the coefficient matrix in Eq. (8) should be zero. i. e.,

$$\begin{vmatrix} x_{A2} & y_{A2} & x_{A2}^2 & y_{A2}^2 & x_{A2}y_{A2} \\ x_{A3} & y_{A3} & x_{A3}^2 & y_{A3}^2 & x_{A3}y_{A3} \\ x_{A4} & y_{A4} & x_{A4}^2 & y_{A4}^2 & x_{A4}y_{A4} \\ x_{A5} & y_{A5} & x_{A5}^2 & y_{A5}^2 & x_{A5}y_{A5} \\ x_{A6} & y_{A6} & x_{A6}^2 & y_{A6}^2 & x_{A6}y_{A6} \end{vmatrix} = 0 \tag{14}$$

Eq. (14) is met if and only if

$$k_1 x_{Ai} + k_2 y_{Ai} + k_3 x_{Ai}^2 + k_4 y_{Ai}^2 + k_5 x_{Ai} y_{Ai} = 0 \quad i = 2, 3, \ldots, 6 \tag{15}$$

where the k_i's are arbitrary constants which cannot all vanish at the same time.

Eq. (15) is the necessary and sufficient condition which an architecturally singular 6-SPS PMlpp should meet. Geometrically speaking, a 6-SPS PMlpp is an architecturally singular 6-SPS parallel manipulator if and only if all the spherical joints on the base (or moving platform) are located on a conic section.

It is clear that Eq. (6) is a specific case of the sufficient condition, given in [7], for architecturally singular 6-SPS parallel manipulators with planar platforms.

5 Self-Motion Analysis of Architecturally Singular 6-SPS PMslpp

For an architecturally singular 6-SPS PMlpp, Eq. (15) is met. The constraint equations in Eq. (8) are linearly dependent. It is usually the case that one equation in Eq. (8) is linearly dependent on the other four. Thus, five inputs are independent and the remaining one is determined by the five specified inputs. The self-motion of the architecturally singular 6-SPS PMslpp can thus be performed by solving four equations in Eq. (8) in conjunction with Eqs. (2)–(5). To perform the self-motion analysis, one unknown, e.g. i_x, should be chosen as the independent coordinate. The leg with the dependent input can be removed virtually from the architecturally singular 6-SPS parallel manipulator.

5.1 Solution of i_y and i_z

Taking the first four equations in Eq. (8) as linear equations in u_1, u_2, u_4 and u_5, we have

$$J \begin{Bmatrix} u_1 \\ u_2 \\ u_4 \\ u_5 \end{Bmatrix} = t_{11} i_x \begin{Bmatrix} x_{A2}^2 \\ x_{A3}^2 \\ x_{A4}^2 \\ x_{A5}^2 \end{Bmatrix} + \begin{Bmatrix} L_2 \\ L_3 \\ L_4 \\ L_5 \end{Bmatrix} \tag{16}$$

where

$$J = \begin{bmatrix} x_{A2} & y_{A2} & y_{A2}^2 & x_{A2} y_{A2} \\ x_{A3} & y_{A3} & y_{A3}^2 & x_{A3} y_{A3} \\ x_{A4} & y_{A4} & y_{A4}^2 & x_{A4} y_{A4} \\ x_{A5} & y_{A5} & y_{A5}^2 & x_{A5} y_{A5} \end{bmatrix}$$

Solving Eq. (16), we obtain

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_4 \\ u_5 \end{Bmatrix} = i_x \mathbf{C} + \mathbf{D} \tag{17}$$

where

$$\mathbf{C} = t_{11} J^{-1} \begin{Bmatrix} x_{A2}^2 \\ x_{A3}^2 \\ x_{A4}^2 \\ x_{A5}^2 \end{Bmatrix}$$

$$\mathbf{D} = J^{-1}\begin{Bmatrix} L_2 \\ L_3 \\ L_4 \\ L_5 \end{Bmatrix} \tag{18}$$

From Eqs. (12) and (13), j_x and j_y can be expressed using i_x and i_y.

$$j_y = -(u_4 + t_{12} i_y)/t_{22} \tag{19}$$

$$j_x = -(u_5 + t_{12} i_x + t_{11} i_y)/t_{22} \tag{20}$$

Substitution of Eq. (17) into Eqs. (19) and (20) gives

$$j_x = e_1 i_y + f_1 i_x + g_1 \tag{21}$$

where

$$\begin{aligned} e_1 &= -t_{11}/t_{22} \\ f_1 &= -(c_4 + t_{12})/t_{22} \\ g_1 &= -d_4/t_{22} \end{aligned}$$

and

$$j_y = e_2 i_y + f_2 i_x + g_2 \tag{22}$$

where

$$\begin{aligned} e_2 &= -t_{12}/t_{22} \\ f_2 &= -c_3/t_{22} \\ g_2 &= -d_3/t_{22} \end{aligned}$$

From Eqs. (2), (3) and (4), we get

$$(1 - i_x^2 - i_y^2)(1 - j_x^2 - j_y^2) - (i_x j_x + i_y j_y)^2 = 0$$

Substitution of Eqs. (21) and (22) into the above equation, we get a polynomial in i_y of degree 4.

$$h_4 i_y^4 + 2h_3 i_y^3 + h_2 i_y^2 + 2h_1 i_y + h_0 = 0 \tag{23}$$

where

$$\begin{aligned} h_4 &= e_1^2 \\ h_3 &= (e_1 f_1 - e_1 e_2) i_x + e_1 g_1 \\ h_2 &= (-2 f_1 e_2 - 2 e_1 f_2 + e_2^2 + f_1^2) i_x^2 + 2(f_1 g_1 - e_1 g_2 - g_1 e_2) i_x + g_1^2 - 1 - e_1^2 - e_2^2 \\ h_1 &= (-f_1 f_2 + e_2 f_2) i_x^3 + (-g_1 f_2 + e_2 g_2 - f_1 g_2) i_x^2 - (e_1 f_1 + g_1 g_2 + e_2 f_2) i_x - e_1 g_1 - e_2 g_2 \\ h_0 &= i_x^4 f_2^2 + 2 i_x^3 f_2 g_2 + (-f_2^2 - f_1^2 + g_2^2 - 1) i_x^2 - 2(f_1 g_1 + f_2 g_2) i_x + 1 - g_1^2 - g_2^2 \end{aligned}$$

For a given value of i_x, i_y can be solved by solving the above equation. i_z can be obtained by solving Eq. (2) as

$$i_z = \pm(1 - i_x^2 - i_y^2)^{1/2} \tag{24}$$

5.2 Solution of j_x, j_y and j_z

j_x and j_y can be calculated using Eqs. (21) and (22). j_z can be got by solving Eq. (4) as

$$j_z = -(i_x j_x + i_y j_y)/i_z \tag{25}$$

5.3 Solution of x, y, and z

Eqs. (9) and (10) can be regarded as linear equations in x, y, and z. Solving Eqs. (9) and (10) for x and y, we get

$$\begin{cases} x = p_1 z + q_1 \\ y = p_2 z + q_2 \end{cases} \tag{26}$$

where

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = Q^{-1} \begin{Bmatrix} -t_{11} i_z \\ -(t_{12} i_z + t_{22} j_z) \end{Bmatrix}$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = Q^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$Q = \begin{bmatrix} t_{11} i_x - 1 & t_{11} i_y \\ t_{12} i_x + t_{22} j_x & t_{12} i_y + t_{22} j_y - 1 \end{bmatrix}$$

Substitution of Eq. (26) into Eq. (5) gives a quadratic equation in z

$$r_2 z^2 + r_1 z + r_0 = 0 \tag{27}$$

where

$$\begin{aligned} r_2 &= p_1^2 + p_2^2 + 1 \\ r_1 &= 2q_1 p_1 + 2q_2 p_2 \\ r_0 &= q_1^2 + q_2^2 - l_1^2 \end{aligned}$$

z can be obtained by solving Eq. (27). Then, x and y can be got using Eq. (26).

5.4 Numerical Example

Consider a 6-SPS PMlpp. The parameters of the base are $x_{A2} = 5$, $y_{A2} = -5$, $x_{A3} = 9$, $y_{A3} = -3$, $x_{A4} = 10$, $y_{A4} = 0$, $x_{A5} = 9$, $y_{A5} = 3$, $x_{A6} = 5$ and $y_{A6} = 5$. The other parameters are $t_{11} = 1$, $t_{12} = 1$ and $t_{22} = 1$.

The parallel manipulator considered is architecturally singular as $\mathbf{A}_i$'s are all located on the conic section $(x-5)^2 + y^2 - 25 = 0$.

The inputs of the manipulators are l_1=33.17, l_2=32.02, l_3=35.12, l_4=37.57, l_5=39.01 and l_6=38.32.

It should be noted that only five of the inputs are independent. The remaining one is dependent on the five specified inputs and can be determined in performing the self-motion analysis of the architecturally singular parallel manipulator.

The self-motion of the architecturally singular 6-SPS PMlpp is obtained using the above procedure and shown in Fig. 2. The results are verified by checking all the six leg lengths. In performing the numerical analysis here, link interference is neglected. It is shown that for the architecturally singular 6-SPS PMlpp, there are eight assembly modes for a specified value of i_x when the inputs are locked. These assembly modes fall on two circuits. A circuit here refers to a continuous locus of the location of the moving platform. In Fig. 2, each assembly mode is represented with the location of the moving platform which is denoted by a triangle composed of O_P and $\mathbf{i}_P$ and $\mathbf{j}_P$. Each circuit is represented by a continuous locus of the position of the moving platform for clarity reason.

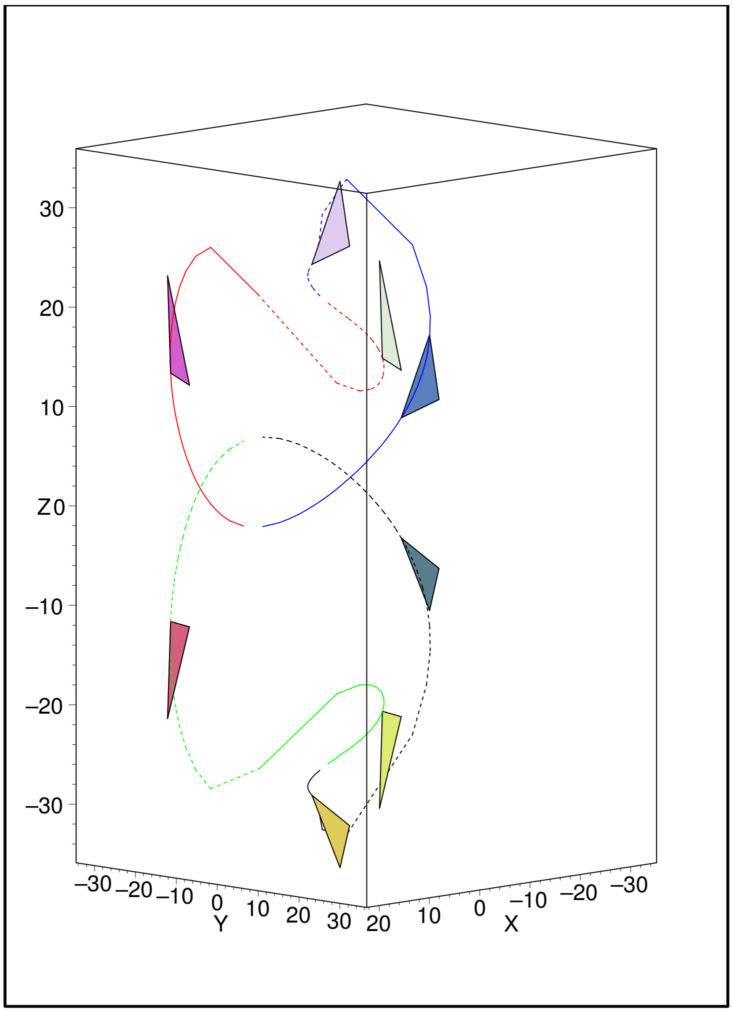

Figure 2. Self-motion of the architecturally singular 6-SPS parallel manipulator.

6 Conclusions

All the architecturally singular 6-SPS PMslpp have been generated. It has been shown that a 6-SPS PMlpp is an architecturally singular 6-SPS parallel manipulator if and only if all the spherical joints on the base (or moving platform) are located on a conic section.

The above result can also be obtained using either the method based on the determinant of the Jacobian matrix or the geometric method based on linear manifolds of correlations and quadratic transformations. For the specific 6-SPS parallel manipulator considered here, the approach presented in this paper is more concise.

Self-motion analysis of the architecturally singular 6-SPS PMslpp has also been addressed. The analysis shows that the problem can be solved analytically and that two circuits exist for the self-motion of the architecturally singular 6-SPS PMlpp when the inputs are locked.

Moreover, the mathematical condition for the 6-SPS PMslpp has been clarified. The forward displacement analysis of the 6-SPS PMslpp has been also simplified.

The results in this paper should be useful in the design of 6-SPS parallel manipulators. It is still open to generate other new classes of architecturally singular 6-SPS parallel manipulators.

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