A class of piecewise linear differential equations arising in biological models

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Introduction

- models of genetic networks
- piecewise linear equations
- step functions
- singular solutions: Filippov solutions
- example
- future works and conclusion

The model

x_i protein concentration

$$x_i' = f_i(x) - \gamma_i x_i \qquad i = 1, \cdots, n \tag{1}$$

$x \in C = \{x : x_i \geq 0, i = 1, \cdots, n\}$ positive

γ_i relative degradation rate

$x = (x_1, \cdots, x_n)$

$\theta_i^1, \cdots, \theta_i^{N_i}$ thresholds for each variable x_i, $i = 1, \cdots, n$

where $\theta_i^0 := 0$ and $\theta_i^{N_i+1} := +\infty$

These thresholds divide the positive cone $C = \{x : x_i \geq 0, i = 1, \cdots, n\}$ into boxes of dimension n

the f_i are constant positive in each box

Piecewise linear system

In the box

box B

value of f is f^B

$x' = f^B - \gamma x$

$$x(t) = \Phi^B + (x(t_0) - \Phi^B)\mathrm{e}^{\gamma(t_0 - t)}, \tag{2}$$

$\Phi^B = \gamma^{-1} f^B$

Theorem 1.1 *If* $\Phi^B \in B$ *then* $x = \Phi^B$ *is an asymptotically stable stationary point*

generic case where the focal point Φ^B does not lie on the threshold hyperplanes

what happens if Φ^B is outside of B?

Filippov solutions

we define the solution on the threshold

Définition 1.1 *A solution in the sense of Filippov is a solution of the differential inclusion*

$$x' \in F(x) - \gamma x$$

that is, an absolutely continuous function $x(t)$ *for which* $x'(t) \in F(x(t)) - \gamma x(t)$ *almost everywhere.*

$F(x) - \gamma x$ is the convex hull of the regular vector fields surrounding x

extension of the vector field on the threshold

Filippov solutions : 2

case where x belongs to a wall W separating two n-boxes B_1 and B_2

transparent, black or white

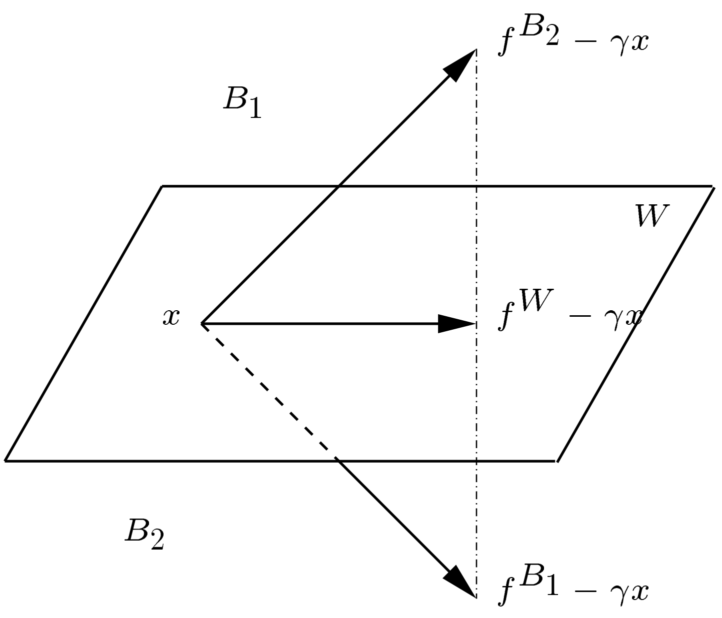


FIG. 1 – The velocity of motion along a threshold hyperplane.

Theorem 1.2 *Assume that* W *is in the hyperplane* $C_i^j = \{x \in C : x_i = \theta_i^j\}$ *and* W *is black or white, that is,* $(f_i^{B_2} - \gamma_i\theta_i^j)(f_i^{B_1} - \gamma_i\theta_i^j) < 0$.

The constant vector f^W *is given by*

$$f^W = \alpha f^{B_1} + (1-\alpha) f^{B_2}, \qquad \alpha = \frac{f_i^{B_2} - \gamma_i\theta_i^j}{f_i^{B_2} - f_i^{B_1}}. \tag{3}$$

Filippov solutions : 3

black wall : stable sliding motion

white wall : unstable sliding motion

a solution which passes through a point of the white wall W at $t = t_0$ may either leave the wall W into the box B_1 or B_2, or remain on W for $t > t_0$...

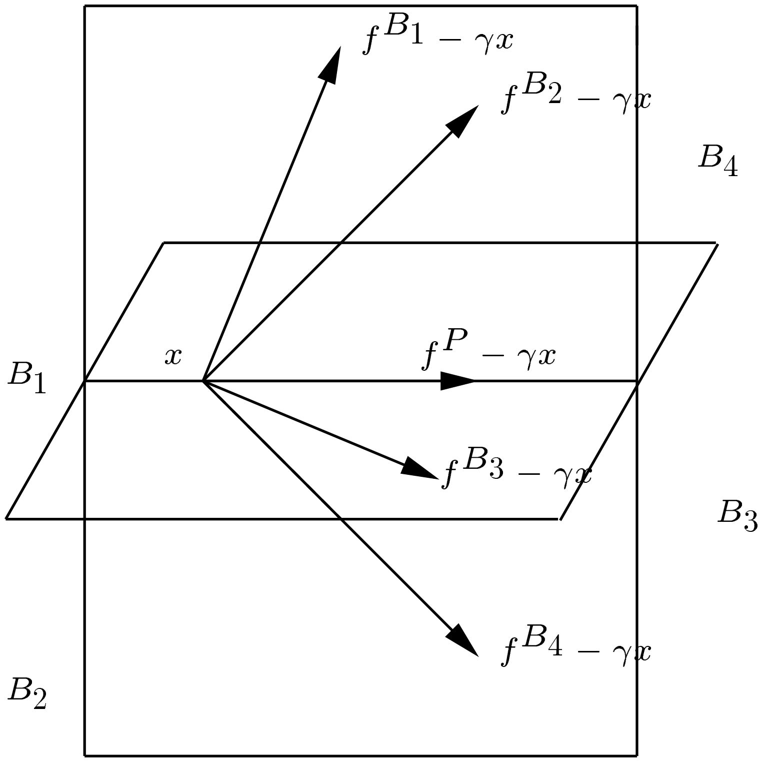
...and may leave it at **any** moment

in some sense, the solution is not wholly determined

but the event "leave the wall" is unique

Filippov solutions : 4

$(n-2)$-box P intersection of two threshold hyperplanes



$$F(x) = \mathrm{conv}(f^{B_1}, f^{B_2}, f^{B_3}, f^{B_4})$$

not unique : a **set**

Filippov solutions : SSP

Singular Steady State

on the threshold

$\Phi^W = \gamma^{-1} f^W$

Theorem 1.3 *Let $\Phi^W \in W$. If W is a black wall, then $x = \Phi^W$ is an asymptotically stable stationary point (in the sense of Filippov).*

If W is a white wall, then $x = \Phi^W$ is an unstable stationary point (in the sense of Filippov).

Transition graph

1. If there is a solution lying in B and terminating in B', then the state transition graph contains the directed edge $B \to B'$.
2. If there is a solution lying in B and beginning in B', then the state transition graph contains the directed edge $B' \to B$.

– if W is a transparent wall separating two n-boxes B_1 and B_2, then the state transition graph contains the edges

$$B_1 \to W \to B_2 \quad \text{or} \quad B_1 \leftarrow W \leftarrow B_2$$

– if W is a black wall separating two n-boxes B_1 and B_2, then the state transition graph contains the edges

$$B_1 \to W \leftarrow B_2$$

– if W is a white wall separating two n-boxes B_1 and B_2, then the state transition graph contains the edges

$$B_1 \leftarrow W \to B_2$$

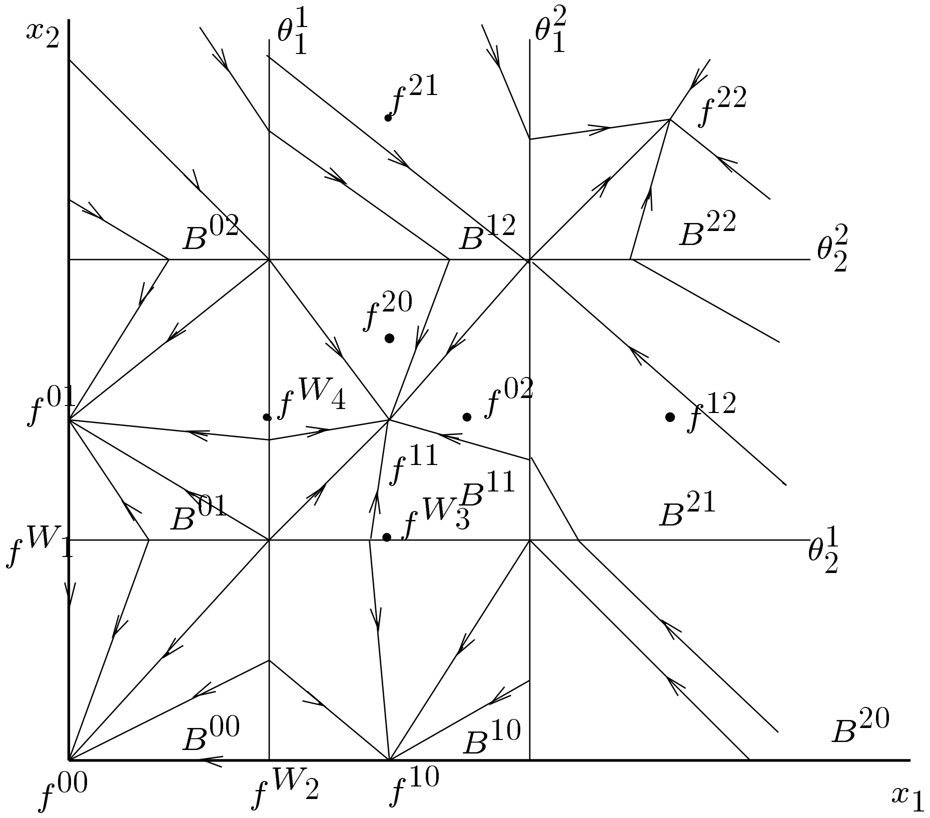
Example

E. H. Snoussi, R. Thomas,

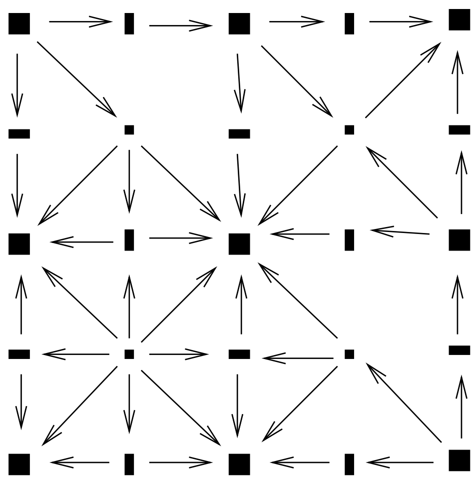
Logical identification of all steady states: the concept of feedback loop characteristic states,

Bulletin of Mathematical Biology, Vol. 55 (1993), 973-991.

$$\begin{aligned} x_1' &= k_1 z_1^1 + k_3 z_2^2 - x_1, \\ x_2' &= k_2 z_1^2 + k_4 z_2^1 - x_2. \end{aligned} \tag{4}$$



x_2
θ_1^1
θ_1^2
f^{21}
f^{22}
B^{02}
B^{12}
B^{22}
θ_2^2
f^{20}
f^{01}
f^{W_4}
f^{02}
f^{12}
f^{11}
B^{11}
B^{01}
f^{W_3}
B^{21}
f^{W_1}
θ_2^1
B^{00}
B^{10}
B^{20}
f^{00}
f^{W_2}
f^{10}
x_1



Remarks

analyze a nonlinear system in a qualitative way

many other approaches...

differential inclusion : we could choose between the behaviours...

....logoids... stochastic....

...but do we want to choose ?

Future works

work with H. de Jong (et al...) : qualitative model

threshold inequalities

equilibrium inequalities

behaviour of a family of models

toward algorithmic methods : steady states, stability

more interested by the events (not the time of the event...)

not classical dynamical systems

no classical answers...