CHAPTER 1

Virtual constraints for the orbital stabilization of the pendubot

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A method for the generation of attractive and neutrally stable limit cycles for nonlinear systems is presented. It consists in designing an output that, when regulated through a suitable feedback, forces the existence of a limit cycle or neutral oscillations in the zero dynamics. Conditions are then given to ensure that those characteristics of the zero-dynamics translate to the whole system. A particular focus is made on the generation of neutrally stable oscillations through that method, because it is not always easy to build an output that results in the existence of a limit-cycle in the zero-dynamics. A special case where such a difficulty arises is given in the analysis of oscillations generation around the upper vertical for the pendubot. The regulation of the output results in neutrally stable oscillations, and we present a method for imposing that those oscillations converge towards the desired ones.
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In many applications, the natural operating mode of a control system is an oscillating one. However, the oscillations are not always present in the open-loop dynamics. Therefore, it is relevant to study new control design methods forcing the internal system dynamics (or a given output) to present a pre-specified limit cycle. Examples of such systems are: walking mechanisms (the full system state should behave periodically), rotating machines (the internal states, i.e. current and flux, are oscillatory if the torque output is kept constant), the synchronization of a vertically landing aircraft with the oscillation of a platform (e.g. an aircraft carrier), etc.

There exist some published works addressing problems in this category. Under the hypothesis of the existence of a limit cycle, Hauser and Chung[8] present a setup for the computation of Lyapunov functions, allowing to determinate if the given limit cycle is exponentially stable. Nevertheless no procedure is presented for the generation of the limit cycle itself. This latter problem has recently been addressed by the works of Marconi *et al*[12], Aracil *et al*[1], Sepulchre and Stan[18], Westervelt *et al*[19] and Canudas de Wit *et al*[4]. In Marconi *et al*[12], the authors deal with the problem of tracking an oscillatory signal. The addressed problem is the motion synchronization of a vertically landing aircraft with the oscillation of a platform. In Aracil *et al*[1], oscillations of the Furuta pendulum are stabilized through an energy-shaping approach, and passivity is used as a mean to generate oscillations in Sepulchre and Stan[18]. Westervelt *et al*[19] and Canudas de Wit *et al*[4] present approaches that have been motivated by the walking mechanism, for which the natural operating mode is a periodic one, with Westervelt studying the zero-dynamics of a controlled biped (which are hybrid due to the impacts) and Canudas de Wit having designed a feedback law that generates globally stable orbits for an underactuated inverted pendulum.

In certain cases, the oscillatory internal behavior is a byproduct of an output regulation problem. An example is the torque and flux norm regulation problem for the induction motor. The linearization of these two outputs (as originally proposed by De Luca and Ulivi[5]), leads to an oscillating behavior in the internal dynamics. Indeed, under this particular frame, the flux and current vector asymptotically converge to a linear stable oscillation[2]. In Canudas de Wit *et al*[4], an output is designed and regulated such that the resulting zero dynamics of the system present a limit cycle. Evenmore, the family of outputs that is proposed allows for enough freedom such that the solutions of the zero dynamics can converge towards any prespecified closed curve. The formalization of this approach, with constructive methods for the design of outputs whose regulation generates oscillating behaviors were then given, along with the corresponding stability analyses[6,16]. The output, once regulated, is called a virtual constraint because it imposes a fixed relation between the states of the model, so that it reduces the number of degrees of freedom of the model.

In this chapter, we will complement the existing results[6,16] and introduce their application on the pendubot, a two-link planar robot whose only actuated joint is the “shoulder”. We wish to generate oscillations for the pendubot around the upward position: the oscillations of the first arm should have a given angular amplitude ($= 2\alpha_s$) and a prespecified period.

This chapter is structured as follows. In Section 1, we present results about the stability of periodic orbits in cascade systems. We then analyze oscillating behaviors in dimension 2 in Section 2. A model of the pendubot is presented in Section 3, followed with the presentation of conditions on the regulated output for the apparition of periodic orbits around the upper vertical. The control problem is then treated, for the pendubot, in Section 4, which is concluded by some simulations. Finally, we give a conclusion.

1. Oscillations in cascade systems

1.1. *Attractive limit sets in cascade systems*

In this section, we will show that a globally attractive limit-cycle in the zero-dynamics of a system could result in a globally attractive limit cycle for the system itself. In order for that result to be valid, some conditions are to be satisfied for the interconnection between the zero dynamics and the rest of the system, and on the stability of the limit cycle in the zero dynamics. The affine system that we consider is analyzed in the normal form

$$\begin{cases} \dot{z} = f(z) + \psi(z,\xi) \\ \dot{\xi} = A\xi + Bu \end{cases} \tag{1}$$

where $\xi \in I\!R^r$, $z \in I\!R^{n-r}$, $u \in I\!R^m$ $(m \leq n)$, and the functions f and ψ are locally Lipschitz continuous on their domains of definition. Function ψ is such that $\psi(z,0) = 0$ for all z (without loss of generality; indeed, if it were not the case, f is redefined as $f(z)+\psi(z,0)$ and ψ as $\psi(z,\xi)-\psi(z,0)$). The zero dynamics are represented by $\dot{z} = f(z)$.

Many stabilization designs are known for this particular normal form (e.g., the global asymptotic stabilization of the origin of the ξ dynamics through a ξ-feedback yields boundedness of the solutions of (1) and global asymptotic stabilization of the origin of (1)[17]). In this section, we impose the same kind of conditions as in Sepulchre *et al.*[17]: the interconnecting term $\psi(z,\xi)$ is linearly bounded in z when $\|z\| > M$ for some $M > 0$, so that finite escape time cannot occur. Also, there exists a positive semidefinite radially unbounded function $W(z)$, that decreases along the solutions of the system when $\|z\|$ is large:

Assumption 1: Suppose that there exists $M \geq 0$, η_1 and η_2 class $\mathcal{K}$ functions such that

$$\|\psi(z,\xi)\| \leq \eta_1(\|\xi\|)\|z\| + \eta_2(\|\xi\|)$$

for $\|z\| \geq M$.

Assumption 2: Suppose that there exists a positive semidefinite radially unbounded function $W(z)$ and positive constants C and M such that, for all $\|z\| > M$, the following holds:

(a) $L_f W(z) \leq 0$;
(b) $\|\frac{\partial W}{\partial z}\| \; \|z\| \leq CW(z)$;

Point (b) of Assumption 2 is classically satisfied by Lyapunov functions $W(z)$ produced by converse theorems for exponentially stable systems[11].

Nonlinear systems can present many different kinds of ω-limit sets other than an equilibrium point[7]. In this section, we are interested in the case where one ω-limit set is a limit cycle. An important property of the limit cycles is that they are compact sets; therefore, we will analyze the situation where one ω-limit set is a compact set γ.

Definition 1: A compact set γ is "almost globally attractive" for the dynamics

$$\dot{x} = f(x) \tag{2}$$

with $x \in I\!R^n$, if it is attractive with basin of attraction containing the whole state space minus a set of Lebesgue measure zero.

In dimension 2, the simple situation where a limit cycle γ attracts every solution except those starting at the mandatory equilibrium inside the area circumscribed by γ fits into Definition 1: γ is almost globally attractive because the equilibrium is of Lebesgue measure zero.

We now consider the case where the union of the ω-limit sets of (2) is made of an almost globally attractive compact set γ and an equilibrium $\bar{x}$ such that γ and $\bar{x}$ are disjoint.

Remark 2: Note that $\bar{x}$ is unstable. Indeed, if $\bar{x}$ is stable without being asymptotically stable, there are solution whose ω-limit set is neither $\bar{x}$ nor γ. On the other hand if $\bar{x}$ is asymptotically stable, the regions of attraction of $\bar{x}$ and γ are open, non empty, connected sets[11]. Therefore, $I\!R^n$ needs to be covered by two disjoint, non empty, open sets, which is a contradiction. Therefore $\bar{x}$ is unstable.

If we consider the existence of an almost globally attractive compact set γ in the z-dynamics, the next theorem gives conditions for the almost globally attractivity of γ to translate into almost global attractivity of $\Gamma \equiv \{(z, \xi) \in \mathbb{R}^n | z \in \gamma \text{ and } \xi = 0\}$ in the interconnected system.

Theorem 3: Suppose that Assumptions 1 and 2 are satisfied for system (1) for which $f : \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$ and $\psi : \mathbb{R}^{n-r} \times \mathbb{R}^r \to \mathbb{R}^{n-r}$ are locally Lipschitz continuous functions. Let the only invariant sets of $\dot{z} = f(z)$ be $z = 0$ and γ, respectively an equilibrium and an almost globally attractive compact set ($0 \notin \gamma$). If (A, B) is controllable, then any feedback in the form $u = k(\xi)$ guaranteeing that the origin of $\dot{\xi} = A\xi + Bk(\xi)$ is Globally Asymptotically Stable and Locally Exponentially Stable (GAS-LES) yields

(i) convergence of the solutions of

$$\begin{cases} \dot{z} = f(z) + \psi(z, \xi) \\ \dot{\xi} = A\xi + Bk(\xi) \end{cases} \tag{3}$$

to the unstable equilibrium $(z, \xi) = (0, 0)$ or to the compact set Γ.

(ii) If the set of initial conditions of solutions converging to $(0, 0)$ is of Lebesgue measure zero, the compact set Γ is almost globally attractive.

(iii) If the equilibrium of $\dot{z} = f(z)$ is hyperbolic, and if the global stable manifold of the origin $(0, 0)$ is a manifold whose dimension is globally defined and constant, the compact set Γ is almost globally attractive.

(iv) If γ is an exponentially stable periodic orbit for $\dot{z} = f(z)$, then Γ is an exponentially stable periodic orbit for (3).

Proof: Boundedness of the state along the solutions is a direct consequence of the boundedness of W along the solutions and the radial-unboundedness of $W(z)$[17].

Every solution of (3) converges to the set $E = \{(z, \xi) \in \mathbb{R}^n | \xi = 0\}$. LaSalle's invariance principle then asserts that every bounded solution converges to the largest invariant set of E. This set is defined by the largest invariant set of $\dot{z} = f(z)$: the origin $z = 0$ and the compact set γ. Because every solution of (3) is bounded, every solution either converges towards the origin $(0, 0)$ or towards the compact set Γ. Moreover, Remark 2 implies that $z = 0$ is an unstable equilibrium in the z-dynamics. The fact that $(z, \xi) = (0, 0)$ is unstable directly follows, which shows (i).

The basin of attraction of the origin $(0, 0)$ is the set of initial conditions of solutions not converging to Γ. Because it is of Lebesgue measure zero, the compact set Γ is almost globally attractive, which shows (ii).

If the origin $z = 0$ is a hyperbolic fixed point for $\dot{z} = f(z)$, the set of eigenvalues of the Jacobian linearization $\frac{\partial f}{\partial z}(0)$ contains $n_u \geq 1$ (resp. $n_s \leq n-r-1$) eigenvalues with positive (resp. negative) real parts ($n_s+n_u = n-r$).

Since the origin $\xi = 0$ is locally exponentially stable for $\dot{\xi} = A\xi + Bk(\xi)$, the Jacobian linearization $A + B\frac{\partial k}{\partial \xi}(0)$ has r eigenvalues with negative real parts. The Jacobian linearization of the complete system (3) is then

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial z}(0) + \frac{\partial \psi}{\partial z}(0,0) & \frac{\partial \psi}{\partial \xi}(0,0) \\ 0 & A + B\frac{\partial k}{\partial \xi}(0) \end{pmatrix}$$

Because $\psi(z,0) = 0$ for all z, $\frac{\partial \psi}{\partial z}(0,0) = 0$. Therefore, the eigenvalues of $\mathcal{J}$ are those of $\frac{\partial f}{\partial z}(0)$ and of $A + B\frac{\partial k}{\partial \xi}(0)$. $\mathcal{J}$ has then $n_s + r \leq n - 1$ eigenvalues with negative real parts and $n_u \geq 1$ eigenvalues with positive real parts. The stable manifold theorem for a fixed point[7] then states that there exists a local stable manifold $\mathcal{M}_s$ of dimension $n_s + r$ and a local unstable manifold $\mathcal{M}_u$ of dimension n_u at the origin. If the dimension of the global stable manifold is globally defined and constant, this global stable manifold also has a dimension $n_s + r$. Therefore the set of initial conditions of solutions that converge to the origin $(0,0)$ (and not to Γ) lies in a manifold of dimension smaller or equal to $n-1$. Because a manifold of dimension smaller or equal to $n-1$ is of Lebesgue measure zero in $\mathbb{R}^n$, the compact set Γ is almost globally attractive, which proves (iii).

Exponential stability of γ implies the existence of a Lyapunov function $V_1(z)$ that is decreasing along the solutions of $\dot{z} = f(z)$. This function satisfies

$$L_f V_1 \leq -k_1 \|z\|_\gamma^2$$

where $\|z\|_\gamma = \inf_{y \in \gamma} \|z - y\|$[8]. On the other hand, a similar Lyapunov function $V_2(\xi)$ can be found for the ξ subsystem ($m_1\|\xi\|^2 \leq V_2(\xi) \leq m_2\|\xi\|^2$, $L_{A\xi+Bk(\xi)}V_2 \leq -m_3\|\xi\|^2$). As in Khalil[11], we then define the Lyapunov function:

$$V(z,\xi) = V_1(z) + 2k\sqrt{V_2(\xi)}$$

with $k > 0$, whose derivative is

$$\begin{aligned} \dot{V} &= L_f V_1(z) + L_\psi V_1(z) + k\frac{L_{A\xi+Bk(\xi)}V_2}{\sqrt{V_2}} \\ &\leq -k_1\|z\|_\gamma^2 + \frac{dV_1}{dz}\psi(z,\xi) - km_3\frac{\|\xi\|^2}{\sqrt{V_2}} \\ &\leq -k_1\|z\|_\gamma^2 - \frac{km_3}{m_1}\|\xi\| + \frac{dV_1}{dz}\psi(z,\xi) \\ &\leq -k_1\|z\|_\gamma^2 - \frac{km_3}{m_1}\|\xi\| + M\|\xi\| \end{aligned}$$

where the last inequality is valid in a small neighborhood of Γ because $\frac{dV_1}{dz}\psi(z,\xi)$ is continuously differentiable. Taking $k > \frac{m_1 M}{m_3}$ ensures negative definiteness of the derivative of V. The closed orbit Γ is therefore exponentially stable. This proves (iv). □

The origin of the complete closed-loop system behaves like a saddle point: it is attractive in some directions and repulsive in others. This is illustrated in Figure 1, where γ is a limit cycle, $n = 3$, and $r = 1$. In this figure, the z system is of dimension 2 with an almost globally attractive limit cycle γ. In order to clarify the figure, we suppose that the stable manifold $\mathcal{M}_s$ of the origin is the ξ axis. Therefore, every solution starting on that axis converge to the origin $(0,0,0)^T$. All the other solutions converge to the limit cycle because the origin is only attractive in the ξ direction.

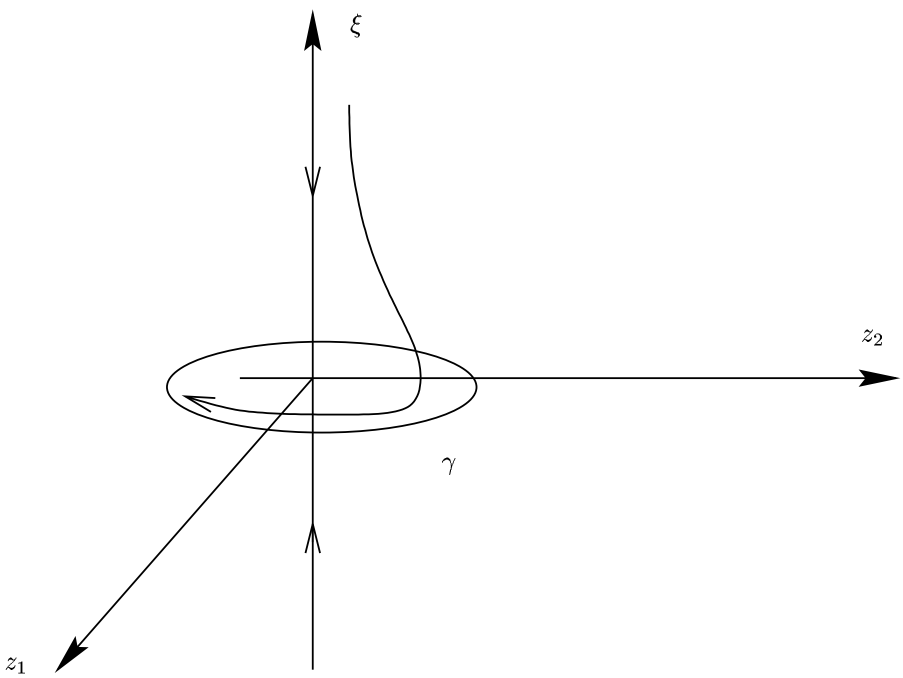

Fig. 1. Almost global attractivity of a limit cycle. Solutions starting on the ξ axis converge to the origin. All others converge to γ.

The method that is then used for the generation of oscillations in an affine system in the form

$$\dot{x} = F(x) + G(x)u$$

then consists in finding an output y that will be such that this system put into the normal form with $\xi_1 = y$ has the form (1) with an almost globally attractive limit-cycle in its zero-dynamics. Such a method has been presented in Grognard and Canudas de Wit[6], but its application is difficult. Also, it is sometimes easier to design an output such that the oscillations in the zero-dynamics are neutrally stable; in other words, the equilibrium is surrounded by a continuum of closed, periodic, orbits (in dimension 2).

1.2. *Neutrally stable oscillations in cascade systems*

As we will see in the case of the pendubot, the design of an output whose regulation generates an attractive limit cycle (as in the previous section) is not always an easy task. It is sometimes easier to find an output whose zero-dynamics are neutrally stable, with an equilibrium surrounded by a continuum of periodic orbits. We will generalize this behavior to invariance of the level sets of a radially unbounded positive semidefinite Lyapunov function and to the following cascade system:

$$\begin{cases} \dot{z} = f(z) + \psi_z(z, \xi, u) \\ \dot{\xi} = \psi_\xi(\xi, u) \end{cases} \tag{4}$$

We therefore need two new assumptions that are closely related to the previous ones:

Assumption 3: Suppose that there exists $M \geq 0$, η_1 and η_2 class $\mathcal{K}$ functions such that

$$\|\psi_z(z, \xi, u)\| \leq \eta_1(\|(\xi, u)\|)\|z\| + \eta_2(\|(\xi, u)\|)$$

for $\|z\| \geq M$.

Assumption 4: Suppose that there exists a positive semidefinite radially unbounded function $W(z)$ such that, for all $\|z\|$:

$$L_f W(z) = 0$$

and we obtain a weaker result than Theorem 3:

Theorem 4: Suppose that Assumptions 3 and 4 are satisfied for system (4) for which $f : \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$, $\psi_z : \mathbb{R}^{n-r} \times \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^{n-r}$ and

$\psi_\xi : I\!R^r \times I\!R^m \to I\!R^r$ are locally Lipschitz continuous functions. If the origin of the ξ subsystem is finite-time stabilizable, then any feedback in the form $u = k(\xi)$ guaranteeing that the origin of $\dot{\xi} = \psi_\xi(\xi, k(\xi))$ is finite-time stabilized ensures that each solution of

$$\begin{cases} \dot{z} = f(z) + \psi_z(z, \xi, k(\xi)) \\ \dot{\xi} = \psi_\xi(\xi, k(\xi)) \end{cases}$$

reaches, in finite time, a region where $W(z)$ is constant and $\xi = 0$

Proof: For any initial condition $(z(0), \xi(0))$, the controller $k(\xi)$ ensures that there exists a time $T(z(0), \xi(0))$ and some $Z(z(0), \xi(0)) \in I\!R^{n-r}$ such that $(z(T(z(0), \xi(0))), \xi(T(z(0), \xi(0)))) = (Z, 0)$ (Assumption 3 ensures that no escape of z in finite time of z can take place during that time-span so that Z is finite). For all $t \geq T$, we then have $\xi(t) = 0$, so that the the remaining dynamics are

$$\dot{z} = f(z)$$

and the solution is then such that $W(z)$ stays constant (at the value $W(Z)$). □

It is interesting to see that, if (4) can be rewritten in the following form

$$\begin{cases} \dot{z} = f(z) + \psi_z(z, \xi)\xi + \psi_{uz}(z, \xi, u)u \\ \dot{\xi} = \psi_\xi(\xi) + \psi_{u\xi}(z, \xi, u)u \end{cases}$$

with, among other conditions, $\dot{z} = f(z)$ neutrally stable, $\dot{\xi} = \psi_\xi(\xi)$ globally exponentially stable and $\psi_z(z, 0) = 0$, the classical forwarding technique[13,10] can be applied to achieve stabilization of the solutions of the closed-loop system to a prespecified level-set. We do not develop this method because, in the pendubot case, we do not have $\psi_z(z, 0) = 0$.

Also note that, if (4) is a normal form[9], with u scalar and the ξ subsystem a chain of integrators, it is easy to build a finite-time controller (the time-optimal controller for the ξ subsystem, for example). However, the resulting controller is not very satisfying, because the level of $W(z)$ that is reached cannot be tuned. For the pendubot, we will present heuristics that ensure convergence of the solutions to the desired level set (and desired oscillations) after having designed an output that ensures neutral stability of the oscillations in dimension 2. In the following section, we will be interested in analyzing those neutrally stable oscillations in 2D-systems.

2. Oscillations in the plane

As stated earlier, the oscillations that we will generate will come from the zero-dynamics, which is very interesting because the analysis of the cycles can then be made in the dimension of the zero-dynamics (which is smaller than the dimension of the original system). By construction, this dimension will often be two. Therefore, we give two results for the analysis of cycles in two dimensional systems.

Lemma 5: *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that for all $(z_1, z_2) \in \mathbb{R}^2$ the function f is such that $f(z_1, z_2) = f(z_1, -z_2)$. Let the system*

$$\ddot{z} + f(z, \dot{z}) = 0 \tag{5}$$

If there exists $\bar{z} \in \mathbb{R}$ such that $f(\bar{z}, 0) = 0$ (($\bar{z}, 0$) is an equilibrium of (5)) and there exist $z_{min} < \bar{z} < z_{max}$ such that $f(z, 0) < 0$ in $[z_{min}, \bar{z})$ and $f(z, 0) > 0$ in $(\bar{z}, z_{max}]$, then there exists a neighborhood of $(z, \dot{z}) = (\bar{z}, 0)$ such that all solutions in that neighborhood are periodic orbits (the hypotheses are illustrated on Figure 2).

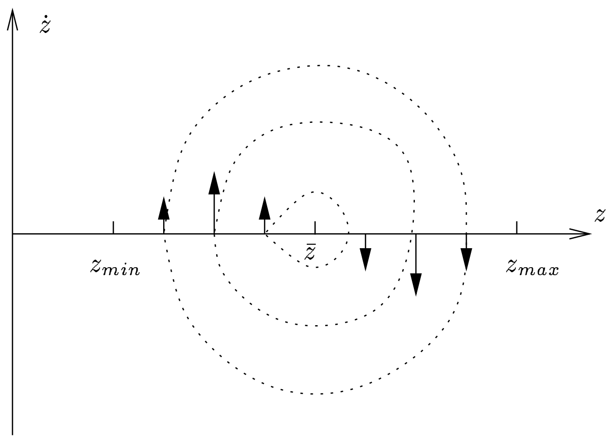

Fig. 2. Generic form of behavior for systems satisfying the hypotheses of Lemma 2. The arrows indicate the direction of the field when $\dot{z} = 0$.

Proof: Let us first show that the phase plane is symmetric with respect to the z axis. We first define $(z_1, z_2) = (z, \dot{z})$. System (5) can then be rewritten as

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -f(z_1, z_2) \end{cases} \tag{6}$$

The symmetry has to be shown with respect to the z_1 axis. Let us now reverse the time (replace t by $\tau = -t$) and replace z_2 by $-z_2$. The dynamics (6) then becomes

$$\begin{cases} \frac{dz_1}{d\tau} = z_2 \\ \frac{dz_2}{d\tau} = -f(z_1, -z_2) = -f(z_1, z_2) \end{cases}$$

The dynamics are unchanged. This means that, if $(z_1(t), z_2(t))$ is solution of (6), then $(z_1(t), -z_2(t))$ also is. The main difference is that, if $(z_1(t), z_2(t))$ is solution with t increasing, then $(z_1(t), -z_2(t))$ is solution with t decreasing: they run in opposite directions.

We will now constructively show the existence of a cycle for (6) around the equilibrium $(z_1, z_2) = (\bar{z}, 0)$.

Let us consider $(z_1(0), z_2(0)) = (z_{max}, 0)$. From (6), we see that $\dot{z}_2(0) < 0$; this means that z_2 starts decreasing as well as z_1. We will now show that $(z_1(t), z_2(t))$ reaches the axis $z_1 = \bar{z}$ in finite time.

First, we see that, as long as $z_1(t)$ is inside the interval $[\bar{z}, z_{max}]$, z_1 decreases. Indeed, if it were to increase at some time, this would mean that z_2 has become positive, so that it is gone through 0 with $z_1(t)$ inside the interval. However, $\dot{z}_2 < 0$ for $z_2 = 0$ and z_1 in the interval, which prevents z_2 from becoming positive and thus z_1 from increasing.

We now show that $(z_1(t), z_2(t))$ has to reach the axis $z_1 = \bar{z}$ at time T with $z_2(T) < 0$. We will show this by contradiction:

- Let us suppose that $z_1(t)$ converges to $z_1^* > \bar{z}$ and never converges to $\bar{z}$. To make sure that z_1 does not keep decreasing beyond z_1^* (so that z_1^* is the limit in finite or infinite time of $z_1(t)$), there must exist $z_2^* \leq 0$ such that (z_1^*, z_2^*) is an equilibrium, which is not the case. Therefore, $z_1(t)$ has to converge to $\bar{z}$ in finite or infinite time.
- Let us suppose that $(z_1(t), z_2(t))$ converges to $(\bar{z}, 0)$ in finite or infinite time. This would create a solution with initial condition at $(z_{max}, 0)$ and going to $(\bar{z}, 0)$ through the region where z_2 is negative. By symmetry, there would exists a reversed-time solution with initial condition in $(z_{max}, 0)$ and going to $(\bar{z}, 0)$ through the region where z_2 is positive. The concatenation of both solutions in positive time creates a homoclinic curve. For the existence of a homoclinic curve, an equilibrium is required in the interior of

the region defined by the curve, which is not the case. Therefore a solution starting at $(z_{max}, 0)$ cannot converge to $(\bar{z}, 0)$.

Therefore, there exists $z_2^+ < 0$ such that $(z_1(t), z_2(t))$ converges to $(\bar{z}, z_2^+)$ (See Figure 3). This takes place in finite time. Indeed, for a convergence to take place in infinite time, $(\bar{z}, z_2^+)$ must be an equilibrium, which is not the case.

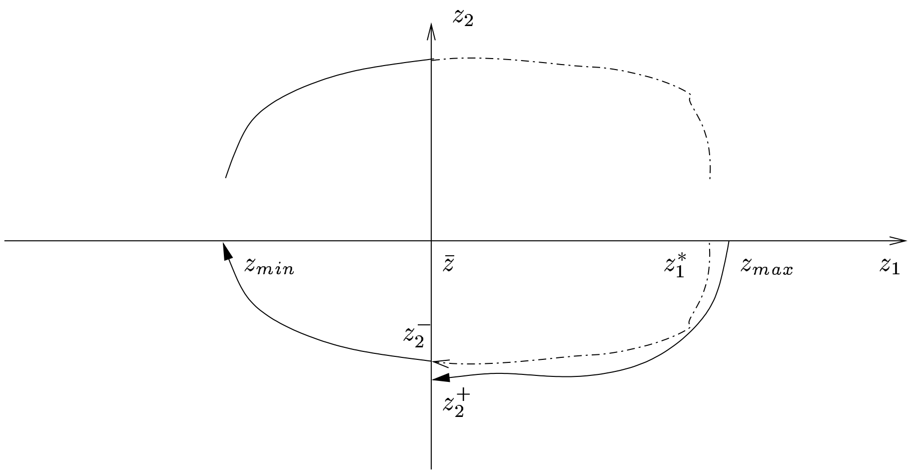

Fig. 3. Construction of a cycle for equation (5).

Let us now consider an initial condition $(z_1(t), z_2(t)) = (z_{min}, 0)$ for system (6) in reverse time:

$$\begin{cases} \frac{dz_1}{d\tau} = -z_2 \\ \frac{dz_2}{d\tau} = f(z_1, z_2) \end{cases}$$

The same reasoning can be held to show that there exists $z_2^- < 0$ such that $(z_1(\tau), z_2(\tau))$ reaches $(\bar{z}, z_2^-)$ in finite (reversed) time.

Let us now suppose, without loss of generality, that $z_2^+ < z_2^- < 0$. We will now consider all solutions starting at $(z_1(0), 0)$ with $z_1(0) \in [\bar{z}, z_{max}]$. All those solutions cross the axis $z_1 = \bar{z}$ in finite time. If $z_1(0) = \bar{z}$, this crossing takes place at $z_2 = 0$; if $z_1(0) = z_{max}$, this crossing takes place at $z_2 = z_2^+$. By continuity, for all $z_2^* \in [z_2^+, 0]$, there exists $z_1(0)$ such that the solution cross the axis $z_1 = \bar{z}$ with $z_2 = z_2^*$. Let us now pick $z_2^* = z_2^-$, and rename the corresponding $z_1(0)$ "z_1^*" (see the dash-dotted line on Figure 3).

We now have a solution going from $(z_1^*, 0)$ to $(\bar{z}, z_2^-)$ in finite time with $z_2 < 0$. We can then concatenate this solution with the solution going from $(\bar{z}, z_2^-)$ to $(z_{min}, 0)$ in finite time (which we had discovered in reversed time). We now have a solution going from $(z_1^*, 0)$ to $(z_{min}, 0)$ in finite time with $z_2 < 0$. By symmetry of the phase plane, we have a solution linking $(z_{min}, 0)$ to $(z_1^*, 0)$ in finite time with $z_2 > 0$. This creates a cycle Γ.

A similar reasoning can be held for any initial condition inside the region circumscribed by Γ (except the equilibrium). Γ defines the border of a neighborhood of the equilibrium inside which all solutions are cycles. □

Remark 6: The condition of Lemma 5 concerning the sign of $f(z, 0)$ for z belonging to an interval $[z_{min}, z_{max}]$ is satisfied if f is differentiable at $(\bar{z}, 0)$ and $\frac{\partial f}{\partial z}(\bar{z}, 0) > 0$.

The simplest example of this kind of system is the harmonic oscillator

$$\ddot{z} + \omega^2 z = 0 \tag{7}$$

The evenness of the function f with respect of $\dot{z}$ is obvious. The *lone* equilibrium is $z = 0$, and $\frac{df}{dz} = \omega^2 > 0$.

A stronger result can be given when f has a particular structure, which results in the following form for (5):

$$\alpha(z)\ddot{z} + \beta(z)\dot{z}^2 + \gamma(z) = 0 \tag{8}$$

where we suppose that $\frac{\beta}{\alpha}$ and $\frac{\gamma}{\alpha}$ are Lipschitz continuous. This form is central in this chapter as it will be the one that the zero-dynamics will take when generating oscillations for the pendubot through the regulation of a linear output.

For (8), we first have a direct consequence of Lemma 5

Corollary 7: *If $\frac{\beta}{\alpha}$ and $\frac{\gamma}{\alpha}$ are Lipschitz continuous functions defined on $I\!R$ then, for any root $\bar{z}$ of $\gamma(z)$, there exists a neighborhood of $(z, \dot{z}) = (\bar{z}, 0)$ such that all solutions in that neighborhood are cycles if γ is differentiable at $\bar{z}$ and*

$$\frac{d\gamma}{dz}(\bar{z})\alpha(\bar{z}) > 0$$

Note that, if the opposite condition ($\frac{d\gamma}{dz}(\bar{z})\alpha(\bar{z}) < 0$) is satisfied at an equilibrium, this equilibrium is unstable (it suffices to analyze the linearization of the system around this equilibrium).

We have also shown in Perram *et al.*[14] that a general integral of system (8) could be built. It is described in the following result:

Theorem 8: Let $[z(t),\ \dot{z}(t)]$be the solution of system (8) with given initial conditions $[z_0,\ \dot{z}_0]$. If the function

$$\begin{aligned} I\left(z,\dot{z},z_0,\dot{z}_0\right) &= \dot{z}^2 - \exp\left\{-2\int_{z_0}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}_0^2 \\ &+ \exp\left\{-2\int_{z_0}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\int_{z_0}^{z}\exp\left\{2\int_{z_0}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}\,ds \end{aligned} \tag{9}$$

exists, then it is finite and preserves its value (= 0) along the solution $[z(t),\ \dot{z}(t)]$.

Coming from this full integral, and as hinted in Shiriaev and Canudas de Wit[16], we will almost always be able to build a first integral of system (8) (independent of the initial condition) as shown in the following result:

Lemma 9: *Given any solution $(z(t),\dot{z}(t))$ of (8) (with initial condition $(z_0,\dot{z}_0)$) and any $z_1^*, z_2^* \in I\!R$ such that $\alpha(s)\neq 0$ for s belonging to the intervals $[z_0, z_1^*]$ (or $[z_1^*, z_0]$) and $[z_0, z_2^*]$ (or $[z_2^*, z_0]$), the function*

$$\begin{aligned} V_{(z_1^*,z_2^*)}(z,\dot{z}) &= exp\left\{2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}^2 \\ &+ \int_{z_2^*}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds \end{aligned} \tag{10}$$

is constant along the solution $(z(t),\dot{z}(t))$ that are such that $\alpha(z(t))\neq 0$ for all $t\geq 0$.

Proof: This result is a direct consequence of (9). Using the equality

$$exp\left\{-2\int_{z_0}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\} = exp\left\{-2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}exp\left\{2\int_{z_1^*}^{z_0}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}$$

which is valid for any $z_1^* \in I\!R$ satisfying the condition given in the Lemma, we can multiply (9) by $exp\left\{2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}$ and obtain:

$$\begin{aligned} &\dot{z}^2 exp\left\{2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\} - exp\left\{2\int_{z_1^*}^{z_0}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}_0^2 \\ &+exp\left\{2\int_{z_1^*}^{z_0}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\int_{z_0}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}exp\left\{-2\int_{z_1^*}^{z_0}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds = 0 \end{aligned}$$

along the solution starting at $(z_0,\dot{z}_0)$.

The second term is constant, so that it can be put on the right hand side of the equality. The third term can be simplified (the first and third exponentials in that term are inverse of each other) so that we now have:

$$\dot{z}^2 exp\left\{2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}+\int_{z_0}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds = exp\left\{2\int_{z_1^*}^{z_0}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}_0^2 \tag{11}$$

The second term of the left hand side can now be split into the sum of two integrals:

$$\begin{aligned}\int_{z_0}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds &= \int_{z_0}^{z_2^*} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds \\ &+ \int_{z_2^*}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\end{aligned}$$

with $\int_{z_0}^{z_2^*} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds$ going to the right hand side of (11), so that (11) becomes:

$$\begin{aligned}&exp\left\{2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}^2 + \int_{z_2^*}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds = \\ &exp\left\{2\int_{z_1^*}^{z_0}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}_0^2 + \int_{z_2^*}^{z_0} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\end{aligned}$$

which shows the proposition: the function

$$V_{(z_1^*,z_2^*)}(z,\dot{z}) = exp\left\{2\int_{z_1^*}^{z}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\dot{z}^2 + \int_{z_2^*}^{z} exp\left\{2\int_{z_1^*}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds$$

stays constant along the solution of (8) with initial condition $(z_0, \dot{z}_0)$. □

In this proposition, we present a family of functions which stay constant along any solution of the system: the parameter z_1^* and z_2^* can be chosen almost freely independently of the initial conditions (with the restriction that the condition given in Lemma 9 is satisfied). The main difference with the function I is that the function V is independent of the initial condition, but that the constant value at which the function stays is not zero: it depends on the initial condition. In order to confirm this result, it can easily be computed that $\dot{V} = 0$ along the solutions.

Finally, it is interesting to notice that, behind the infinite number of functions that are integral functions for system (8) (defined by the different values that z_1^* and z_2^* can take), lies a single function. In fact, for any two pairs of parameters (z_1^*, z_2^*) and (z_a^*, z_b^*), there exists real constants A and B such that $V_{(z_1^*,z_2^*)}(z,\dot{z}) = AV_{(z_a^*,z_b^*)}(z,\dot{z}) + B$ for all $(z,\dot{z})$.

Considering the harmonic oscillator (7), we see that $\alpha(z) = 1$, $\beta(z) = 0$ and $\gamma(z) = \omega^2 z$. The construction (10) results in the function

$$V(z,\dot{z}) = \dot{z}^2 + \omega^2 z^2 + \omega^2 z_2^{*2}$$

As z_2^* can be chosen freely, we take $z_2^* = 0$, so that V is the classical Lyapunov function for the harmonic oscillator.

The existence of function V along which the solutions are constant is not sufficient to ensure the presence of cycles. Only if the constant levels of this function represent closed curve does it result into cycles. This function will later be useful to create attractive limit cycles for the pendubot.

In Section 1, we have shown that an efficient method to generate an oscillating behavior in a nonlinear control system was the construction and regulation of an output that ensures the presence of oscillations in the zero-dynamics of the system. After regulation of this output, the remaining dynamics are oscillating, so that the whole system presents oscillations. We have first exposed a case where the zero-dynamics result in an attractive limit cycle, and have then shown that we could obtain a (weaker) result when those zero-dynamics are simply neutrally stable: the resulting closed-loop dynamics present oscillations of unknown amplitude (depending on the initial condition). In Section 2, we have then presented tools for the analysis of those neutral oscillations in a special case of zero-dynamics: planar oscillations in the form (5) or (8); sufficient conditions for the existence of neutrally stable cycles and a first integral were given. We will now use these tools for the control and analysis of a particular mechanical system, the pendubot, for which we will show how to go from neutrally stable to exponentially stable oscillations.

3. Control of the pendubot

The following section will be devoted to the description and desired behavior of the pendubot: a two-links planar robot with a motor at the shoulder and no motor at the elbow (see Figure 4).

A classical mechanical model for this robot is:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = B\tau \tag{12}$$

where $q = [q_1\ q_2]^T \in I\!R^2$, and q_1 represents the angle of the first link with the lower vertical axis and q_2 the angle of the second link with the first link. If a motor is available at both joints, the robot is fully actuated and the solution of the problem is trivial. For the pendubot, a torque can only be applied at the first joint: therefore, $\tau \in I\!R$, which means that the pendubot is underactuated. The matrices that define the model are the following :

$$M(q) = \begin{pmatrix} m_1 l_1^2 + m_2(l_2^2 + L_1^2 + 2L_1 l_2 cos(q_2)) & m_2(l_2^2 + L_1 l_2 \cos(q_2)) \\ m_2(l_2^2 + L_1 l_2 \cos(q_2)) & m_2 l_2^2 \end{pmatrix}$$

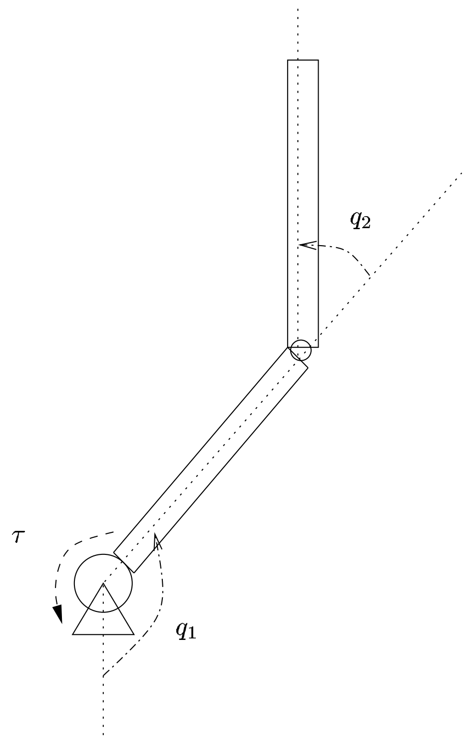

Fig. 4. Coordinates position on the pendubot.

$$C(q,\dot{q}) = \begin{pmatrix} -m_2L_1l_2\sin(q_2)\dot{q}_2 & -m_2L_1l_2\sin(q_2)(\dot{q}_1+\dot{q}_2) \\ m_2L_1l_2\sin(q_2)\dot{q}_1 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} g\left((m_1l_1+m_2l_2)\sin(q_1)+m_2l_2\sin(q_1+q_2)\right) \\ gm_2l_2\sin(q_1+q_2) \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We wish to generate oscillations for the pendubot around the upward position. It is desired for the oscillations of the first arm (linked to the shoulder) to have a given angular amplitude ($= 2\alpha_s$) and a prespecified period (T_s). Also, we wish that the oscillations take place with the second arm close to the vertical. A simple description of the oscillatory behavior would be to say that the robot goes back and forth between a rest position at the left of the vertical and a rest position at the right of the vertical axis. In term of coordinates, this means that q_1 oscillates around π and q_2 oscillates around 0.

If we take the desired behavior to the limit, we would like to have the first link oscillate around the upper vertical while the second one stays vertical. For the model, this translates into oscillations of q_1 around π, while $q_1 + q_2$ stays constant at the value π (this also results in $\dot{q}_1 = -\dot{q}_2$). If a control law is built that achieves this objective, the behavior of the complete system is given by the evolution of q_1 or q_2 once $q_1 + q_2 = \pi$: it represents the zero-dynamics. For the pendubot, we will concentrate on the evolution of q_2 based on the second equation of the model

$$\begin{aligned} 0 &= -m_2(l_2^2 + L_1 l_2 \cos(q_2))\ddot{q}_2 + m_2 l_2^2 \ddot{q}_2 + m_2 L_1 l_2 \sin(q_2)\dot{q}_2^2 \\ &= -\cos(q_2)\ddot{q}_2 + \sin(q_2)\dot{q}_2^2 = 0 \end{aligned}$$

where we have introduced the constraint $q_1 + q_2 = \pi$. This system is not in the appropriate form to apply Lemma 5. If oscillations are present, there is a rest position on the right of the vertical axis ($q_2 < 0$ and $\dot{q}_2 = 0$). At that point, the derivative of the angle satisfies $\dot{q}_2 = 0$, so that $\ddot{q}_2$ is forced to be zero by the zero-dynamics: the whole system is at rest and cannot move afterwards. This is in contradiction with the fact that we were considering an oscillating solution. Therefore, no oscillation can be generated with the second link staying vertical.

3.1. *Sufficient conditions for oscillations*

Because an upward oscillatory behavior is not achievable when keeping the second link vertical, we will not impose such a strong constraint and the second link will also have to oscillate. Instead of having $q_1 + q_2 = \pi$, we will impose a more general constraint

$$q_1 + \varphi(q_2) = 0$$

by building an output $y = q_1 + \varphi(q_2)$ that we will regulate so that the system presents the desired oscillations. In other words, the zero-dynamics must present oscillations.

The zero-dynamics then take the following form (we study the second equation of the model when $y = 0$):

$$\begin{aligned}
0 &= -m_2(l_2^2 + L_1 l_2 \cos(q_2))(\frac{d^2\varphi}{dq_2^2}\dot{q}_2^2 + \frac{d\varphi}{dq_2}\ddot{q}_2) + m_2 l_2^2 \ddot{q}_2 \\
&\quad + m_2 L_1 l_2 \sin(q_2)(\frac{d\varphi}{dq_2})^2 \dot{q}_2^2 + g m_2 l_2 \sin(q_2 - \varphi(q_2)) \\
&= \left[(1 - \frac{d\varphi}{dq_2}) m_2 l_2^2 - \frac{d\varphi}{dq_2} m_2 L_1 l_2 \cos(q_2)\right] \ddot{q}_2 \\
&\quad + \left[m_2 L_1 l_2 \sin(q_2)(\frac{d\varphi}{dq_2})^2 - m_2(l_2^2 + L_1 l_2 \cos(q_2))\frac{\partial^2\varphi}{dq_2^2}\right] \dot{q}_2^2 + g m_2 l_2 \sin(q_2 - \varphi(q_2)) \\
&= \left[(1 - \frac{d\varphi}{dq_2}) l_2 - \frac{d\varphi}{dq_2} L_1 \cos(q_2)\right] \ddot{q}_2 \\
&\quad + \left[L_1 \sin(q_2)(\frac{d\varphi}{dq_2})^2 - (l_2 + L_1 \cos(q_2))\frac{\partial^2\varphi}{dq_2^2}\right] \dot{q}_2^2 + g \sin(q_2 - \varphi(q_2)) \qquad (13)
\end{aligned}$$

These zero-dynamics are exactly in the form (8)

$$\alpha(q_2)\ddot{q}_2 + \beta(q_2)\dot{q}_2^2 + \gamma(q_2) = 0$$

that was considered in Corollary 7. For any system in the form (13), we can then perform the analysis based on this result, and an integral function can be built based on Lemma 9. If this integral function is radially unbounded and there is a single equilibrium, then every solution is a cycle.

Analyzing (13) in the light of Corollary 7 will impose conditions on the function $\varphi(q_2)$ that will make sure that there are oscillating solutions around the upper vertical where $q_1 = \pi$ and $q_2 = 0$. In order to have an equilibrium in $q_2 = 0$, we must have $\gamma(0) = 0$, which translates into

$$\varphi(0) = k\pi$$

for some integer k. This equilibrium must correspond to $q_1 = \pi$, so that we must pick $k = -1$. The second condition that should be satisfied for the application of Corollary 7 is

$$\frac{d\gamma(0)}{dq_2}\alpha(0) > 0$$

which here becomes:

$$g(1 - \frac{d\varphi}{dq_2}(0))\left[(1 - \frac{d\varphi}{dq_2}(0))l_2 - \frac{d\varphi}{dq_2}(0)L_1\right] > 0$$

that is

$$(l_2 + L_1)\frac{d\varphi}{dq_2}(0)^2 - (2l_2 + L_1)\frac{d\varphi}{dq_2}(0) + l_2 > 0$$

so that, cycles occur around the upper vertical if

$$\varphi(0) = -\pi \ \text{ and } \ \frac{l_2}{L_1 + l_2} < \frac{d\varphi}{dq_2}(0) < 1 \tag{14}$$

3.2. *Oscillations shaping*

Instead of looking at the problem of analyzing equation (13) for a given function φ, we can shape φ so that (13) has a desired form:

$$\alpha_d(q_2)\ddot{q}_2 + \beta_d(q_2)\dot{q}_2^2 + \gamma_d(q_2) = 0 \tag{15}$$

so that the oscillations of the zero-dynamics follow a prespecified behavior. In order to have equivalence of the behaviors, we should first notice that we can multiply (15) by an arbitrary function $\phi(q_2) \neq 0$ without changing the behavior of the desired system. We can then match (13) and (15):

$$\begin{cases} (1 - \frac{d\varphi}{dq_2})l_2 - \frac{d\varphi}{dq_2}L_1\cos(q_2) & = \alpha_d(q_2)\phi(q_2) \\ L_1\sin(q_2)(\frac{d\varphi}{dq_2})^2 - (l_2 + L_1\cos(q_2))\frac{\partial^2\varphi}{dq_2^2} & = \beta_d(q_2)\phi(q_2) \\ g\sin(q_2 - \varphi(q_2)) & = \gamma_d(q_2)\phi(q_2) \end{cases}$$

and, with $\phi(q_2)$ fixed, we obtain a set of ordinary differential equations with q_2 as independent variable and φ the unknown solution. If this set of equations is solvable, it is very difficult to find an analytic expression for these solutions for given $\alpha_d, \beta_d, \gamma_d$. In order to avoid the use of a numerical solution of this equation, we simply consider the case where the output is linear.

3.3. *Linear output*

A particular case of the previous constraint is the case where φ is affine: $\varphi(q_2) = aq_2 - b$, so that the regulation of the output y yields $q_1 + aq_2 - b = 0$. It directly comes from the condition for oscillations (14) that we must have

$$b = \pi \ \text{ and } \ \frac{l_2}{l_2 + L_1} < a < 1$$

In order to stay close to the case where $q_1 + q_2 = \pi$, we rewrite a as $a = 1 - \epsilon$ (with ϵ small), so that the constraint is rewritten as $q_1 + q_2 = \epsilon q_2 + \pi$ and the condition for oscillation becomes

$$0 < \epsilon < \frac{L_1}{L_1 + l_2} \tag{16}$$

From the constraint $q_1 + q_2 = \epsilon q_2 + \pi$, it results that, when $q_1 < \pi$, we have $q_2 > 0$ and $q_1 + q_2 > \pi$, so that, when the robot is on the right of the vertical axis, the second link leans slightly to the left (and conversely, when $q_1 > \pi$). The evolution of the pendubot during a half-cycle follows the illustration of Figure 5.

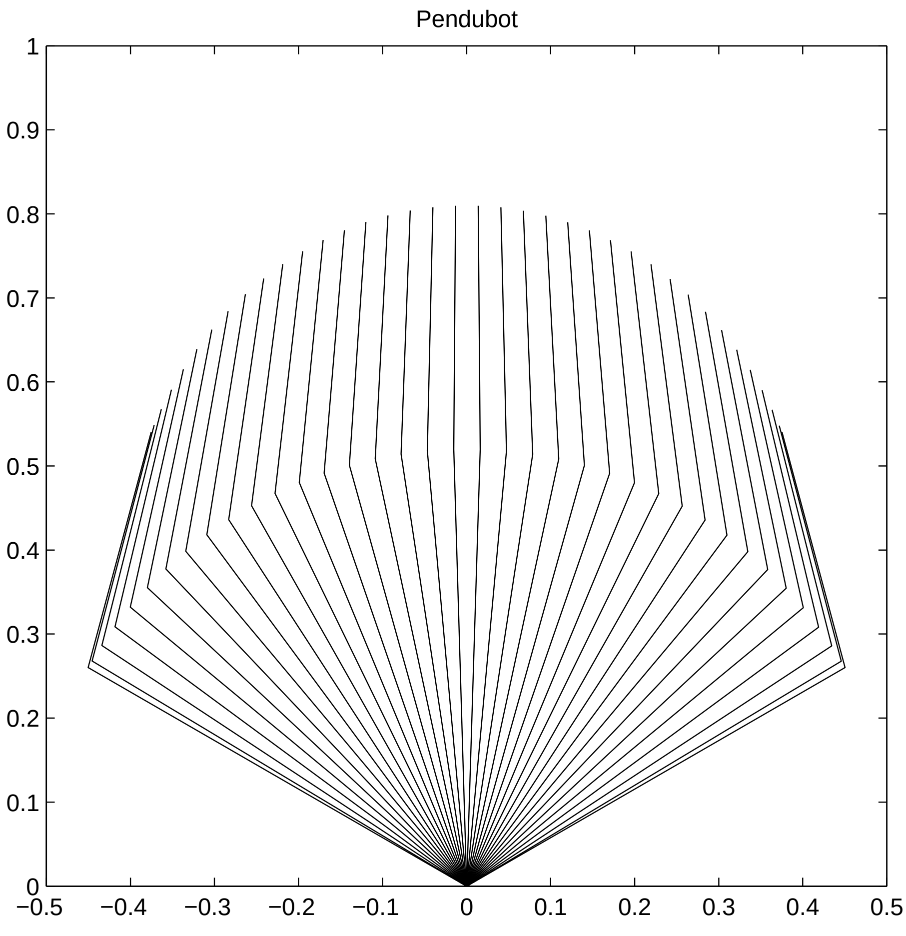

Fig. 5. Evolution of the pendubot during a half-cycle for $\epsilon = 0.2$ and $\alpha_s = \frac{\pi}{3}$.

The notations of (13) then simplify and the zero dynamics become:

$$(\epsilon l_2 - (1-\epsilon)L_1\cos(q_2))\,\ddot{q}_2 + L_1\sin(q_2)(1-\epsilon)^2\dot{q}_2^2 + g\sin(\epsilon q_2 + \pi) = 0$$

so that the dynamics are rewritten as:

$$((1-\epsilon)L_1\cos(q_2) - \epsilon l_2)\,\ddot{q}_2 - L_1\sin(q_2)(1-\epsilon)^2\dot{q}_2^2 + g\sin(\epsilon q_2) = 0 \quad (17)$$

A first consequence of this constraint (16) on ϵ, is that $\alpha(q_2) = (1-\epsilon)L_1 cos(q_2) - \epsilon l_2$ is not sign definite, so that the phase plane of (17) will be made of vertical strips separated by vertical lines corresponding to the roots of $\alpha(q_2)$. In the case where ϵ does not satisfy (16), we have the behavior of the zero-dynamics that is illustrated on Figure 6. Numerous cycles are observed around different equilibria. However, no cycle is observed around the equilibrium $q_2 = 0$ (which corresponds to the equilibrium $q_1 = \pi$), so that the desired behavior is not achieved.

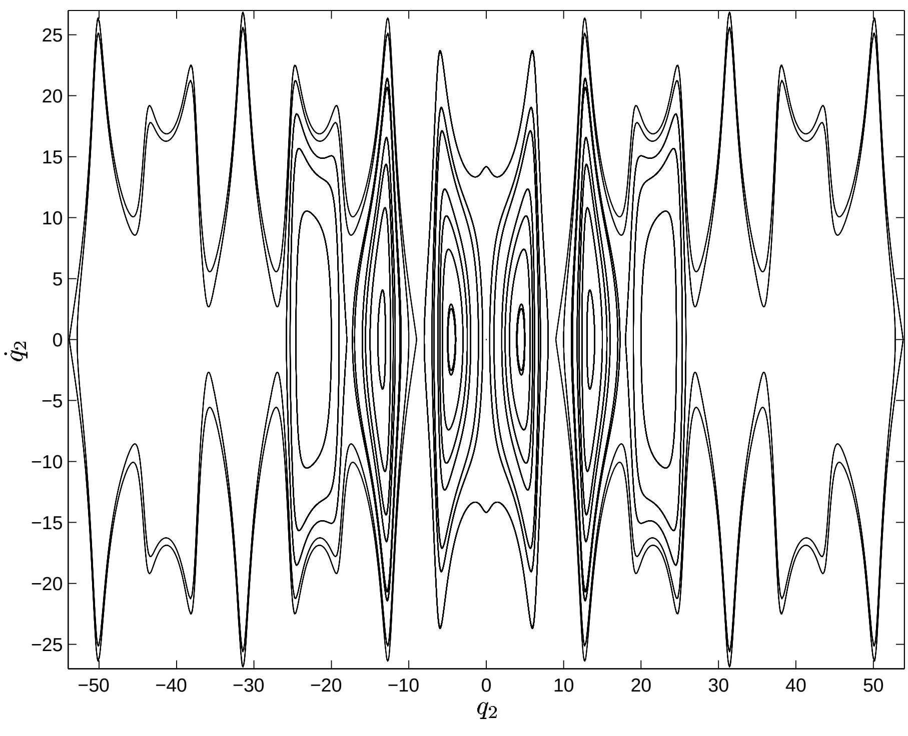

Fig. 6. $(q_2, \dot{q}_2)$ phase plane when constraint (16) is not satisfied by ϵ

We will now concentrate on the oscillations of (17) in the region surrounding $(q_2, \dot{q}_2) = (0, 0)$ where $\alpha(q_2) \neq 0$. This constraint on α imposes $\epsilon l_2 < (1-\epsilon)L_1\cos(q_2)$ for all q_2 in the region. For any ϵ, this constraint can only be satisfied if q_2 does not reach $\frac{\pi}{2}$: for a given ϵ, q_2 must belong to an interval:

$$-arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right) < q_2 < arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right) \quad (18)$$

From this, we see that, the smaller ϵ is, the larger the angle α can be: as ϵ tends to zero, the interval tends to $]-\frac{\pi}{2},\frac{\pi}{2}[$.

The conditions of Lemma 5 and 9 are then well satisfied for the equilibrium $\bar{q}_2 = 0$, so that oscillations take place around the upper vertical axis.

On Figure 7, the symmetries of the phase planes are illustrated through the picture of the cycles. The vertical dotted lines represent the limit induced by (18).

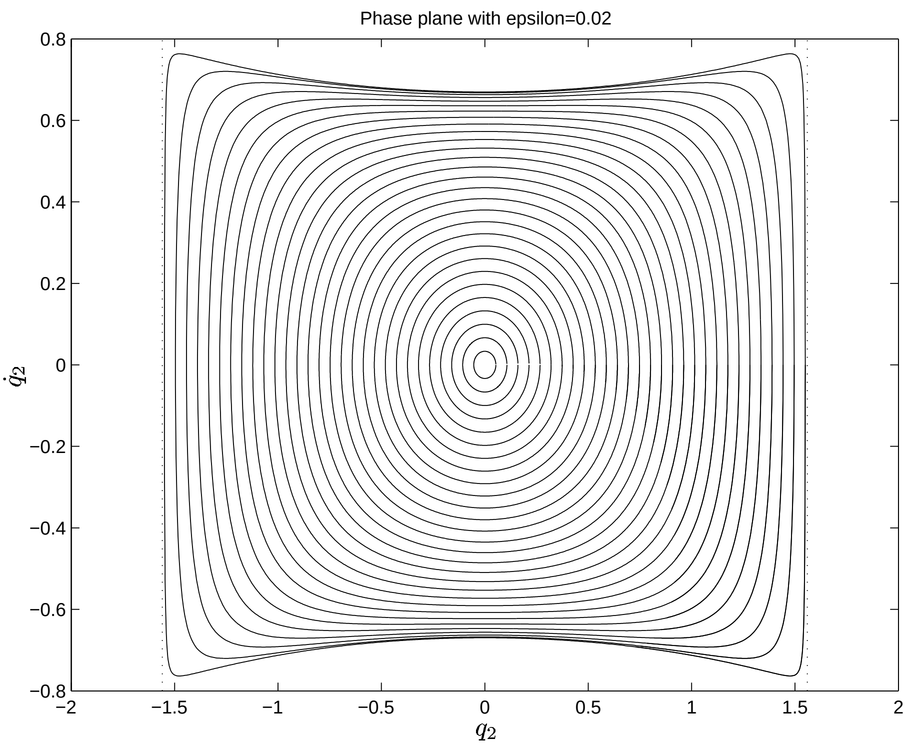

Fig. 7. Phase plane of the zero dynamics (17) for $\epsilon = 0.02$.

Integral function

As (17) fits in the format (8), we can deduce from Lemma 9 the form of the integral function that is kept constant along the solutions of (17):

$$\begin{aligned} V_{(q_2^{*a},q_2^{*b})}(q_2,\dot q_2) &= exp\left\{2\int_{q_2^{*a}}^{q_2}\frac{-L_1\sin(\tau)(1-\epsilon)^2}{(1-\epsilon)L_1\cos(\tau)-\epsilon l_2}d\tau\right\}\dot q_2^2 \\ &+\int_{q_2^{*b}}^{q_2} exp\left\{2\int_{q_2^{*a}}^{s}\frac{-L_1\sin(\tau)(1-\epsilon)^2}{(1-\epsilon)L_1\cos(\tau)-\epsilon l_2}d\tau\right\}\frac{2g\sin(\epsilon s)}{(1-\epsilon)L_1\cos(s)-\epsilon l_2}ds \\ &= \left(\frac{(1-\epsilon)L_1\cos(q_2)-\epsilon l_2}{(1-\epsilon)L_1\cos(q_2^{*a})-\epsilon l_2}\right)^{2(1-\epsilon)}\dot q_2^2 \\ &+\int_{q_2^{*b}}^{q_2}\left(\frac{(1-\epsilon)L_1\cos(s)-\epsilon l_2}{(1-\epsilon)L_1\cos(q_2^{*a})-\epsilon l_2}\right)^{2(1-\epsilon)}\frac{2g\sin(\epsilon s)}{(1-\epsilon)L_1\cos(s)-\epsilon l_2}ds \end{aligned}$$

where the parameters q_2^{*a} and q_2^{*b} can be arbitrarily chosen inside the interval defined by (18). The denominators containing q_2^{*a} are scalar factors, so that they can be taken out, and the integral function becomes:

$$\begin{aligned} V_\epsilon(q_2,\dot q_2) &= ((1-\epsilon)L_1\cos(q_2)-\epsilon l_2)^{2(1-\epsilon)}\dot q_2^2 \\ &+2g\int_{q_2^*}^{q_2}((1-\epsilon)L_1\cos(s)-\epsilon l_2)^{1-2\epsilon}\sin(\epsilon s)\,ds \end{aligned} \tag{19}$$

For any cycle, there exists $\bar V_\epsilon$ such that $V_\epsilon = \bar V_\epsilon$ along the cycle. On the other hand, the span of values for $\bar V_\epsilon$ that we can use is limited to those that are such that $V_\epsilon = \bar V_\epsilon$ corresponds to a cycle.

The expression of the integral is computable, though no obvious analytical expression is available. However, it simplifies when $\epsilon = \frac{1}{2}$, so that, in that case,

$$V_{0.5}(q_2,\dot q_2) = \frac{L_1\cos(q_2)-l_2}{2}\dot q_2^2 - 4g\cos(\frac{q_2}{2}) \tag{20}$$

where the q_2^* term has been dropped.

Also, an approximation of (19) can be given when ϵ is small by using

$$\sin(\epsilon q_2)\approx\epsilon\sin(q_2)$$

so that the integral becomes

$$\hat V_\epsilon(q_2,\dot q_2) \;=\; [(1-\epsilon)L_1\cos(q_2)-\epsilon l_2]^{2(1-\epsilon)}(\dot q_2^2 - \frac{\epsilon g}{L_1(1-\epsilon)^2}) \tag{21}$$

In the sequel, we will denote by V_ϵ the exact value of the integral function, and by $\hat V_\epsilon$, its approximation.

Differentiating $\hat V_\epsilon$ along the solutions of (17) results in

$$\dot{\hat V}_\epsilon = 2\,[\epsilon l_2-(1-\epsilon)L_1\cos(q_2)]^{1-2\epsilon}\,g\dot q_2(\sin(\epsilon q_2)-\epsilon\sin(q_2))$$

which confirms that $\hat V_\epsilon$ is constant along the cycles as long as the approximation $\epsilon\sin(q_2)\approx\sin(\epsilon q_2)$ is valid.

On Figure 8, it appears that the approximation (21) is very good when the cycle is small ($\bar{V}_\epsilon \leq -0.05$), which is what we expected: indeed, small cycles imply small amplitudes for the movement of q_2, which is required for the approximation to be valid. It is also to be noted that, when the cycles are large, the approximation gives a good representation of the behavior of the system, though with a slightly larger amplitude.

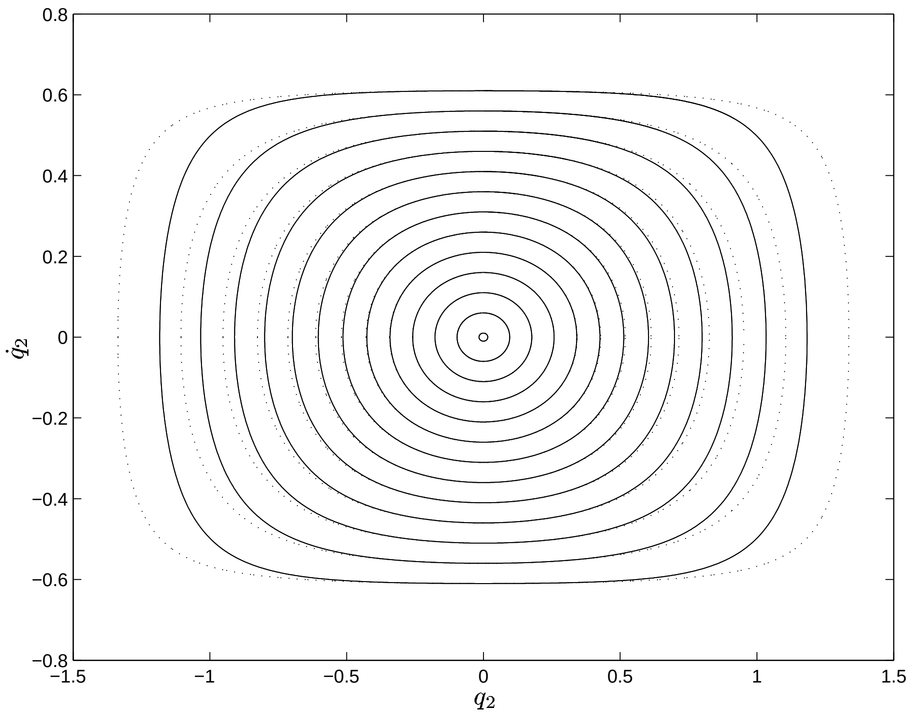

Fig. 8. Phase plane of the zero dynamics for $\epsilon = 0.02$: comparison of the actual cycles (solid lines) and the approximation (21) (dotted line)

In the case of ϵ small, we have $\hat{V}_\epsilon = [(1-\epsilon)L_1 - \epsilon l_2]^{2(1-\epsilon)}\,(-\frac{\epsilon g}{L_1(1-\epsilon)^2})$ at the equilibrium $(q_2, \dot{q}_2) = (0, 0)$ and $\hat{V}_\epsilon = 0$ on the hypothetical cycle that would touch the constraint (18). Therefore, $\hat{V}_\epsilon = \bar{V}_\epsilon$ represents a cycle if and only if

$$-\frac{\epsilon g}{L_1(1-\epsilon)^2}\,[(1-\epsilon)L_1 - \epsilon l_2]^{2(1-\epsilon)} < \bar{V}_\epsilon < 0$$

In the case of $\epsilon = 0.5$, (20) indicates that $V_{0.5} = -4g$ at the equilibrium and $V_{0.5} = -4g\sqrt{\frac{l_2+L_1}{2L_1}}$ when an hypothetical cycle touches the constraint. Therefore, $V_{0.5} = \bar{V}_{0.5}$ represents a cycle if and only if

$$-4g < \bar{V}_{0.5} < -4g\sqrt{\frac{l_2+L_1}{2L_1}}$$

4. Controlled oscillations

In the previous section, we have shown that we could build a linear output such that its regulation would generate neutrally stable oscillations of the pendubot around the upper vertical. This regulation however would result in an oscillating pendubot having unknown amplitude, as the phase plane of the zero-dynamics is illustrated on Figure 7. We now will present a control strategy to generate oscillations having pre-specified amplitude and period.

4.1. *Specifications*

As stated in the previous section, we have the following specifications for the control problem:

(i) The angle q_1 must oscillate between $\pi - \alpha_s$ and $\pi + \alpha_s$ (with $\alpha_s < \frac{\pi}{2}$).
(ii) The period of oscillation must be equal to T_s

We will base this analysis on equation (17). For any ϵ, the angle q_2 can at most oscillate between $-arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right)$ and $+arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right)$, due to the constraint (18). This translates into maximal oscillations of q_1 between $\pi - (1-\epsilon)arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right)$ and $\pi + (1-\epsilon)arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right)$. Specification (i) can then only be achieved if ϵ is such that

$$\alpha_s \leq (1-\epsilon)arccos\left(\frac{\epsilon l_2}{(1-\epsilon)L_1}\right) = \alpha_{max}(\epsilon)$$

It can easily be seen that $\frac{d\alpha_{max}}{d\epsilon} < 0$ for all $\epsilon \in (0,1)$, that $\alpha_{max}(0) = \frac{\pi}{2}$ and that $\alpha_{max}(1) = 0$. Therefore, there exists $\epsilon_{max} > 0$ such that for all $0 \leq \epsilon \leq \epsilon_{max}$, the desired angle of oscillation can be achieved. Therefore, for each $\epsilon < \epsilon_{max}$, there exists $\bar{V}_\epsilon$ such that the oscillation along the level $V_\epsilon(q_2, \dot{q}_2) = \bar{V}_\epsilon$ satisfies specification (i).

We now have to choose among those $\epsilon < \epsilon_{max}$ the value that will ensure the satisfaction of specification (ii).

Let us define $T(\epsilon)$ as the period of an oscillation of amplitude $2\alpha_s$ for a given ϵ. This function $T : (0, \epsilon_{max}) \to I\!R^+$ is continuous inside the interval.

We can see that $T(0) = +\infty$; indeed, when $\epsilon = 0$, the oscillation does not really take place: it is replaced by a continuum of equilibria, so that we say that the period of oscillation is infinite.Also, there exists T_{min} such that $T(\epsilon_{max}) = T_{min}$. By continuity of T, for any $T_s > T_{min}$, there exists $\tilde{\epsilon} \in (0, \epsilon_{max})$ such that $T(\tilde{\epsilon}) = T_{min}$. This determines an oscillation that satisfies both (i) and (ii). We will then build a controller that regulates the output $y = q_1 + (1-\tilde{\epsilon})q_2 - \pi$ and ensures convergences of $V_{\tilde{\epsilon}}$ towards $\bar{V}_{\tilde{\epsilon}}$ so that the specifications are satisfied.

4.2. *Control design*

In this section, we build a controller for system (12) that forces the output $y = q_1 + (1-\epsilon)q_2 - \pi$ to 0 and convergence of V_ϵ to $\bar{V}_\epsilon$ (we have dropped the ˜sign from $\tilde{\epsilon}$). We first rewrite system (12) in new coordinates based on the output: we define $y_1 = y = q_1 + (1-\epsilon)q_2 - \pi$ and $y_2 = \dot{y}_1 = \dot{q}_1 + (1-\epsilon)\dot{q}_2$. The last two coordinates, based on the coordinates of the zero dynamics, are q_2 and $\dot{q}_2$. The system then becomes:

$$\begin{cases} \frac{dq_2}{dt} = \dot{q}_2 \\ \frac{d\dot{q}_2}{dt} = \frac{(l_2+L_1\cos(q_2))(F(q,\dot{q})+G(q,\dot{q})\tau)+L_1\sin(q_2)(y_2-(1-\epsilon)\dot{q}_2)^2-g\sin(y_1+\epsilon q_2)}{(1-\epsilon)L_1\cos(q_2)-\epsilon l_2} \\ \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = F(q,\dot{q}) + G(q,\dot{q})\tau \end{cases}$$

where the expressions of F and G directly come from the model (12). It can be seen that $G(q,\dot{q}) \neq 0$ in the region of interest (where (18) is satisfied). We can then linearize the y part of the system by feedback by imposing

$$\tau = \frac{1}{G(q,\dot{q})}(v - F(q,\dot{q}))$$

with v the new control variable and the system becomes

$$\begin{cases} \frac{dq_2}{dt} = \dot{q}_2 \\ \frac{d\dot{q}_2}{dt} = \frac{(l_2+L_1\cos(q_2))v+L_1\sin(q_2)(y_2-(1-\epsilon)\dot{q}_2)^2-g\sin(y_1+\epsilon q_2)}{(1-\epsilon)L_1\cos(q_2)-\epsilon l_2} \\ \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = v \end{cases} \tag{22}$$

Note that $\frac{d\dot{q}_2}{dt}$ equation can be rewritten as

$$\begin{aligned} \frac{d\dot{q}_2}{dt} = & \frac{L_1\sin(q_2)(1-\epsilon)^2\dot{q}_2^2-g\sin(\epsilon q_2)}{(1-\epsilon)L_1\cos(q_2)-\epsilon l_2} \\ & + \frac{(l_2+L_1\cos(q_2))v+L_1\sin(q_2)(y_2^2-2(1-\epsilon)\dot{q}_2 y_2)-g[\sin(y_1+\epsilon q_2)-\sin(\epsilon q_2)]}{(1-\epsilon)L_1\cos(q_2)-\epsilon l_2} \end{aligned}$$

where the first term contains the zero-dynamics, and that the evolution of V_ϵ is as follows:

$$\frac{dV_\epsilon}{dt} = 2\dot{q}_2((1-\epsilon)L_1\cos(q_2)-\epsilon l_2)^{1-2\epsilon}(L_1\sin(q_2)(y_2^2-2(1-\epsilon)\dot{q}_2y_2)-g[\sin(y_1+\epsilon q_2)-\sin(\epsilon q_2)]) \\ +2\dot{q}_2((1-\epsilon)L_1\cos(q_2)-\epsilon l_2)^{1-2\epsilon}(l_2+L_1\cos(q_2))v$$

and it is easily seen that, when $y_1 = y_2 = 0$, the control law

$$v = -k_V\dot{q}_2(V_\epsilon - \bar{V}_\epsilon)$$

with $k_V > 0$ steers $(q_2, \dot{q}_2)$ to the cycle corresponding to $V_\epsilon = \bar{V}_\epsilon$ (except if $(q_2(0), \dot{q}_2(0)) = (0,0)$).

On the other hand, the control law

$$v = -k_1y_1 - k_2y_2$$

with $k_1, k_2 > 0$ steers (y_1, y_2) to $(0,0)$ and results in neutral stability of the oscillations in the zero-dynamics.

In order to achieve both at the same time, we could apply the method that was presented in Shiriaev and Canudas de Wit[16], which requires the analysis of the local controllability of an auxiliary time-varying system around the target orbit, but we prefer to simply sum both control laws and analyze the resulting behavior:

$$v = -k_V\dot{q}_2(V_\epsilon - \bar{V}_\epsilon) - k_1y_1 - k_2y_2 \tag{23}$$

We would like to check if the cycle corresponding to $V_\epsilon = \bar{V}_\epsilon$ and $y_1 = y_2 = 0$ is exponentially stable in (22). Exponential stability of a limit-cycle can be verified[8] by first applying a (diffeomorphic) change of coordinates

$$(q_2, \dot{q}_2, y_1, y_2) \longrightarrow (\theta, \rho)$$

where ρ represents coordinates that are transverse to the considered limit cycle and equal to zero on the limit cycle, and θ represents the evolution of the solution along the limit cycle so that the system is rewritten in the form

$$\begin{cases} \dot{\theta} = 1 + f_1(\theta, \rho) \\ \dot{\rho} = A(\theta)\rho + f_2(\theta, \rho) \end{cases}$$

with $f_1(\theta, 0) = 0$, $f_2(\theta, 0) = 0$ and $\frac{\partial f_2(\theta,0)}{\partial \rho} = 0$. The limit cycle is then an exponentially stable orbit if and only if the transverse linearization

$$\frac{d\rho}{d\theta} = A(\theta)\rho \tag{24}$$

is asymptotically stable. Ideally, we should build a Lyapunov function $\rho^T P(\theta)\rho$ with P, positive definite, satisfying

$$\frac{dP}{d\theta} = -A(\theta)^T P - PA(\theta) - Q(\theta), \quad \text{with } Q(\theta) > 0$$

to show this stability, but we are not able to; however, we can at least show that the matrix A is Hurwitz along the cycle, which is a good indication for stability (note, however, that this is not sufficient for asymptotic stability of the non-autonomous system (24), as shown in Khalil[11]).

In our case, the ρ coordinate is already available. It suffices to take $\rho = (V_\epsilon - \bar{V}_\epsilon, y_1, y_2)$. We will not explicitly build the θ coordinates, because it will not change anything for us if we show that $A(\theta)$ or $A(q_2, \dot{q}_2)$ is Hurwitz. Both matrices are identical. Let us now write the ρ dynamics, with $h(q_2, \dot{q}_2) = 2((1-\epsilon)L_1\cos(q_2) - \epsilon l_2)^{1-2\epsilon}$ (> 0 in the region of interest):

$$\begin{cases} \frac{dW}{dt} = 2\dot{q}_2 h(q_2,\dot{q}_2)\left(L_1\sin(q_2)(y_2^2 - 2(1-\epsilon)\dot{q}_2 y_2) - g\left[\sin(y_1+\epsilon q_2) - \sin(\epsilon q_2)\right]\right) \\ \qquad +2\dot{q}_2 h(q_2,\dot{q}_2)(l_2 + L_1\cos(q_2))(-k_V\dot{q}_2 W - k_1 y_1 - k_2 y_2) \\ \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = -k_V\dot{q}_2 W - k_1 y_1 - k_2 y_2 \end{cases}$$

with $W = V_\epsilon - \bar{V}_\epsilon$. We can linearize the ρ dynamics around $(W, y_1, y_2) = (0, 0, 0)$ and obtain

$$\begin{pmatrix} \dot{W} \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -\dot{q}_2^2 a_{11}(q_2,\dot{q}_2) & -\dot{q}_2 a_{12}(q_2,\dot{q}_2) & -\dot{q}_2 a_{13}(q_2,\dot{q}_2) \\ 0 & 0 & 1 \\ -k_V\dot{q}_2 & -k_1 & -k_2 \end{pmatrix} \begin{pmatrix} W \\ y_1 \\ y_2 \end{pmatrix}$$

where

$$\begin{aligned} a_{11}(q_2,\dot{q}_2) &= k_V(l_2 + L_1\cos(q_2))h(q_2,\dot{q}_2) \\ a_{12}(q_2,\dot{q}_2) &= (g\cos(\epsilon q_2) + k_1(l_2 + L_1\cos(q_2)))h(q_2,\dot{q}_2) \\ a_{13}(q_2,\dot{q}_2) &= (2(1-\epsilon)L_1\sin(q_2)\dot{q}_2 + k_2(l_2 + L_1\cos(q_2)))h(q_2,\dot{q}_2) \end{aligned}$$

and the $\dot{}$ derivative now represents the derivative with respect to the time evolution θ along the target cycle, with q_2 and $\dot{q}_2$ being the value of those states along the target limit-cycle at time θ.

The characteristics polynomial of this linear system, for fixed $(q_2, \dot{q}_2)$ on the target orbit is

$$\begin{aligned} p_{CA}(s) &= s^3 + (k_2 + \dot{q}_2^2 a_{11})s^2 + (k_1 + k_2\dot{q}_2^2 a_{11} - k_V\dot{q}_2^2 a_{13})s + k_1\dot{q}_2^2 a_{11} - k_V\dot{q}_2^2 a_{12} \\ &= s^3 + (k_2 + \dot{q}_2^2 k_V(l_2 + L_1\cos(q_2))h)s^2 \\ &\quad +(k_1 - 2(1-\epsilon)L_1 k_V\dot{q}_2^3\sin(q_2)h)s - k_V\dot{q}_2^2 g\cos(\epsilon q_2)h \end{aligned}$$

The first conclusion that can be drawn is that, to the contrary of what we expected, k_V needs to be negative for the last term to be positive when

$\dot q_2 \neq 0$ (if $\dot q_2 = 0$, the polynomial has one root in $s = 0$). The Routh criterion also indicates that we need to have

$$k_2 > \dot q_2^2 |k_V|(l_2 + L_1 \cos(q_2))h$$

for all $(q_2, \dot q_2)$ on the target orbit. This can be achieved because, for a given $k_V < 0$ the right-hand side is bounded (it then suffices to take k_2 large enough). The Routh criterion also imposes

$$(k_2+\dot q_2^2 k_V(l_2+L_1\cos(q_2))h)(k_1-2(1-\epsilon)L_1 k_V \dot q_2^3 \sin(q_2)h) > |k_V|\dot q_2^2 g \cos(\epsilon q_2)h$$

for all $(q_2, \dot q_2)$ on the target orbit. This can be achieved by taking k_1 large enough.

To summarize this analysis, the parameters of the control law have to be picked as follows: k_V needs to be taken negative while k_1 and k_2 need to be taken large enough. The efficiency of the control law is confirmed in the following simulations.

4.3. *Simulations*

We will now present simulations of the control law for the pendubot with the parameters that were given in Canudas de Wit *et al.*[4], that is

$$L_1 = 0.52\,m \;\; l_1 = 0.30\,m \;\; l_2 = 0.29\,m \;\; m_1 = 6\,kg \;\; m_2 = 4\,kg \;\; g = 9.81\,\frac{m}{s^2}$$

We first apply the procedure of Section 4.1 in order to find a cycle that has an angular amplitude of $2\alpha_s = \frac{\pi}{3}$ and a period of 10 seconds. This yields that we should take $\epsilon = 0.0195$ and $\bar V_\epsilon = -0.074$ (we obtain that value by computing the approximate value $\hat V_\epsilon$ of V_ϵ along the target cycle). The pendubot then has the desired behavior. If the pendubot starts at rest in a position close to the vertical $(q_2(0), \dot q_2(0), y_1(0), y_2(0)) = (0.01, 0, 0, 0)$, the controlled system (with $k_V = -100$, $k_1 = 10$, $k_2 = 20$ and $\hat V_\epsilon$ instead of V_ϵ) behaves as shown on Figures 9 and 10. The time-evolutions of the states, of V_ϵ and the torque τ are shown on Figure 9 while the projection of the solution on the $(q_2, \dot q_2)$ phase plane is shown on Figure 10. Note that the time evolution of $\hat V_\epsilon$ is not monotonous and does not exactly settle at a constant value because we display the approximate value of the integral function $(\hat V_\epsilon)$ and not its exact value (V_ϵ) and we use this approximate value in the control law. See also that the torque oscillates with time and that (y_1, y_2) stays close to $(0, 0)$ during the whole time-span (of 60 seconds). Figure 10 shows that the solution converges quite directly towards the desired orbit.

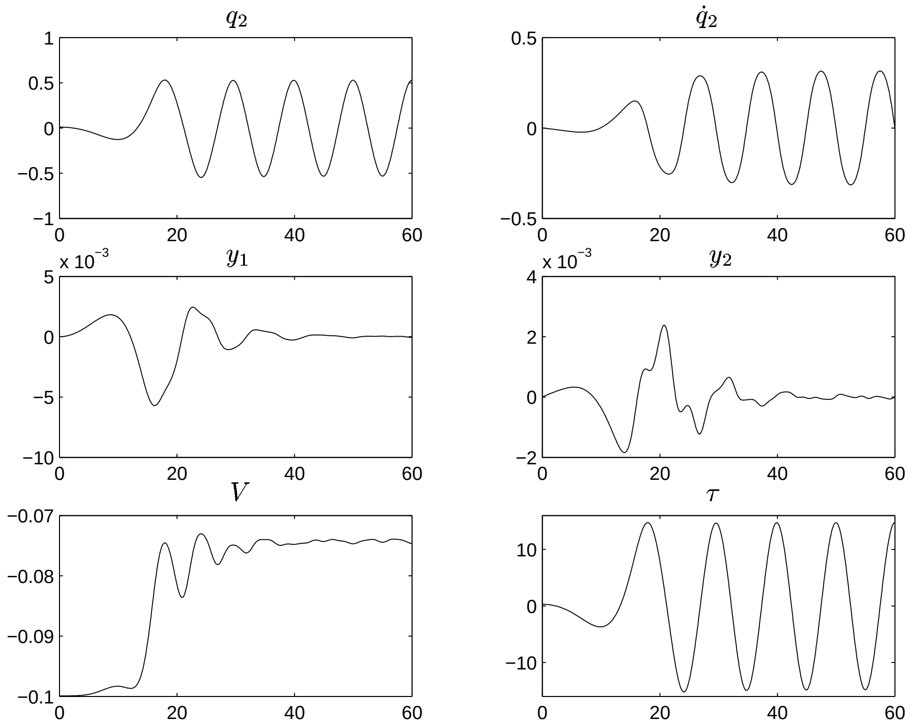

Fig. 9. Time evolution of the states, the integral function V, and the torque along the solution of the controlled pendubot with $(q_2(0), \dot{q}_2(0), y_1(0), y_2(0)) = (0.01, 0, 0, 0)$

Conclusion

In this chapter, we have presented two results about the stability of periodic orbits in cascade systems when there is a periodic orbit in the zero-dynamics. After giving some results about periodic orbits in two-dimensional systems, we have shown how oscillations can be generated in the pendubot through the design and regulation of an output. A detailed analysis shows that a prespecified behavior can be achieved by tuning the available parameters (ϵ and $\bar{V}_\epsilon$).

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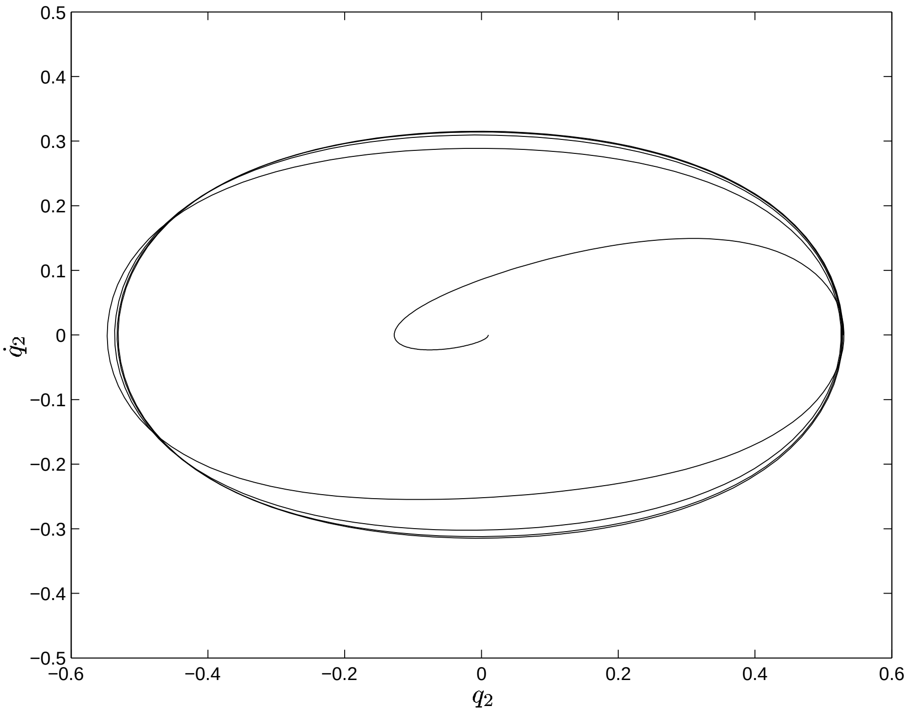

Fig. 10. Projection of the solution on the $(q_2, \dot{q}_2)$ phase-plane with $(q_2(0), \dot{q}_2(0), y_1(0), y_2(0)) = (0.01, 0, 0, 0)$

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