A Generalized Model for Games of Cops and Robbers with Randomness

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Introduction

Cops and Robbers games have been known as models for **fully discretized perfect information pursuit games**.
They assume

- two players oppose each other,
- discrete time,
- discrete space,
- players know everything that has been played.

Introduction

These games come in many instances

- the cop and robber game of Nowakowski, Winkler and Quilliot (**solved in 1983**),
- the $k-$cops and robber game (**solved around 2012**),
- the helicopter cops and robber (**solved in 1993**),
- the tandem-cops and robber (**solved in 2005**),
- etc.

The one model to rule them all

Instead of studying each game seperately, one could want to regroup them under a single model. In fact, we note that

- each game is played on a fixed discrete structure,
- each turn is described with a precise configuration,
- each turn necessitates a defined way to modify the configuration,
- at the end of each turn we can verify if the robber is free.

Thus,

We should be able to describe all cops and robbers game with a single model!

The one model to rule them all

Bonato and MacGillivray's generalization

Bonato and MacGillivray described a generalization of cops and robbers games in a 2014. However it *doesn't* include random elements such as **random walks, capture probabilities, etc.**

The one model to rule them all

Indeed, we want to incorporate the **cop and drunk robber game** into our model.

Cop and drunk robber game

- Same rules as the classic game,
- the robber walks according to a transition matrix M.

It was solved recently with the following recurrence relation

$$w_0(r,c) := 1 \iff r = c; \text{ otherwise it is } 0;$$

$$w_n(r,c) = \begin{cases} 1, & \text{if } c \in N[r]; \\ \max\limits_{c' \in N[c]} \sum\limits_{r' \in N[r]} M(r,r') w_{n-1}(r',c'), & \text{if } c \notin N[r]. \end{cases}$$

The one model to rule them all

We wish to integrate stochasticity (**random robbers, random captures, random events, etc.**).
We thus talk abstractly of

- sets of game states;
- sets of actions;
- transition probabilities.

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A generalized Cops and Robbers game

We say $\mathcal{G} := (S, I, F, A, T)$ is a Cops and Robbers game if

- $\mathcal{G}$ is played with two-players (cops and robbers), with perfect information and turn-based;

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A generalized Cops and Robbers game

We say $\mathcal{G} := (S, I, F, A, T)$ is a Cops and Robbers game if

- $S = S_c \times S_r \times S_o$ is the set of **possible game states**;

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A generalized Cops and Robbers game

We say $\mathcal{G} := (S, I, F, A, T)$ is a Cops and Robbers game if

- $I \subset S$ is the set of initial states;

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A generalized Cops and Robbers game

We say $\mathcal{G} := (S, I, F, A, T)$ is a Cops and Robbers game if

- $F \subset S$ is the set of **final game states**;

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A generalized Cops and Robbers game

We say $\mathcal{G} := (S, I, F, A, T)$ is a Cops and Robbers game if

- $A_c(s), A_r(s')$ are the sets of **playable actions** from states s, s';

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A generalized Cops and Robbers game

We say $\mathcal{G} := (S, I, F, A, T)$ is a Cops and Robbers game if

- $T_c : S \times A_c \times S \to [0, 1]$ and $T_r : S \times A_r \times S \to [0, 1]$ are **transition probabilities** seen as $\mathbb{P}[s' \mid s, a]$.

Let's recap

- $\mathcal{G} = (S, I, F, A, T)$ is an abstract game, said to be of **cops and robbers** if it is two-player, perfect information and turn-based.
- $\mathcal{G}$ is described by a sequence of states $s \in S$. Two sets I, F are included in S.
  - $\Rightarrow$ The game ends whenever the robbers cannot exit F!
- Each player must choose an action $a_c \in A_c(s), a_r \in A_r(s')$ defined from states s, s'.
- Whenever a player $x \in \{r, c\}$ chooses an action a from state s, the resulting state s' is chosen randomly according to $T_x(s, a, s')$.

Solving abstract games

The capture time

At each turn i is defined a random variable $X_i \in \{0, 1\}$ such that $X_i = 1$ if and only if at **the end of** turn i the current state is **final**. The robbers **capture time** is defined by

$$T_{\omega_c,\omega_r} := \begin{cases} \min_n \left(X_n = 1 \mid \omega_c, \omega_r\right), & \text{if } n \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

ω_c, ω_r are the cops' and the robbers' **strategies**.

The capture probability

The capture time

$$T_{\omega_c,\omega_r} := \begin{cases} \min_n \left(X_n = 1 \mid \omega_c, \omega_r\right), & \text{if } n \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Assume players play **optimally** and define the **probability of capture in n turns** as

$$p_n^* := \max_{\omega_c} \min_{\omega_r} \mathbb{P}\left[T_{\omega_c,\omega_r} \leq n\right].$$

The w_n recursion

In addition to our definition of $\mathcal{G}$, our main contribution is the definition of the following function $w_n : S \to [0, 1]$.

The w_n recursion

$$w_0(s) := 1 \iff s \in F.$$

$$w_n(s) = \begin{cases} 1, & \text{if } s \in F, \\ \max\limits_{a_c \in A_c(s)} \sum\limits_{s' \in S} T_c(s, a_c, s') & \\ \min\limits_{a_r \in A_r(s')} \sum\limits_{s'' \in S} T_r(s', a_r, s'') w_{n-1}(s''), & \text{otherwise.} \end{cases}$$

$w_n(s)$ gives the **probability** a state s leads to a **final** state in n turns or less!

The w_n recursion

The capture probability

The probability the cops capture the robbers in at most n turns is

$$p_n^* := \max_{\omega_c} \min_{\omega_r} \mathbb{P}\left[T_{\omega_c,\omega_r} \leq n\right].$$

A copwin theorem

The recursion w_n is *adequatly* defined and gives the probability the robbers get captured in n turns or less.

$$\max_{s_c \in S_c} \min_{s_r \in S_r} w_n(s_c, s_r) = p_n^*$$

Complexity results

The w_n function is computed using a dynamic programming approach.

Complexity

Say at most N turns are allowed in $\mathcal{G}$. Then computing w_N takes $O\left(|S|^3|A_c||A_r|\right)$ time complexity and $O\left(N|S|\right)$ space complexity.

- N can be upper-bounded and the space complexity is linear in this upper-bound and $|S|$.
- When A_c and A_r are of size **polynomial in S**, then computing w_N takes polynomial time.

Let $G = (V, E)$ be a finite, undirected, reflexive, connected graph and let

$$\begin{array}{ll} S = V^2 & F = \{(c,c) \in S\} \\ A_c(c,r) = N[c] & A_r(c,r) = N[r]. \end{array}$$

Now assume the robber moves uniformly at random and let

$$T_c((c,r), c', (c',r)) = 1 \iff c' \in N[c];$$

$$T_r((c,r), r', (c,u)) = \begin{cases} \mathrm{pod}(r), & \text{if } c = r = u; \\ \frac{(1-\mathrm{pod}(r))}{\deg(r)}, & \text{if } c = r, u \in N(r); \\ \frac{1}{\deg(r)+1}, & \text{if } c \neq r, u \in N[r]; \\ 0, & \text{otherwise.} \end{cases}$$

pod is the **probability of detection on vertex r**.

Some examples

Cop and Drunk Robber with detection probability

$$w_0(c, r) := 1 \iff c = r;$$

$$w_n(c, r) = \begin{cases} 1, \text{ if } c = r, \text{ otherwise} \\ \max_{c' \in N[c]} \min_{r' \in N[r]} \sum_{u \in N[r]} T_r((c', r), r', (c', u)) w_{n-1}(c', u). \end{cases}$$

Conclusion

In conclusion, we

- defined a general model of cops and robbers game with random elements,
- conceived of a recursion formula that solves this game,
- evaluated the complexity of this formula,
- showed how to model known games in our framework.