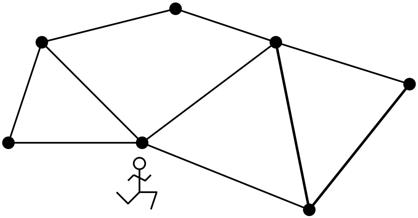
Anonymous Graph Exploration with Binoculars

Jérémie Chalopin Emmanuel Godard Antoine Naudin

LIF, CNRS & Aix-Marseille Université

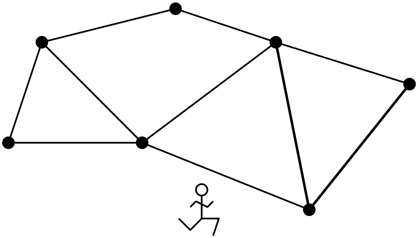
GRASTA-MAC 2015

Graph Exploration



- An agent is moving along the edges of a graph
- Goal : visit all the nodes and stop

Graph Exploration



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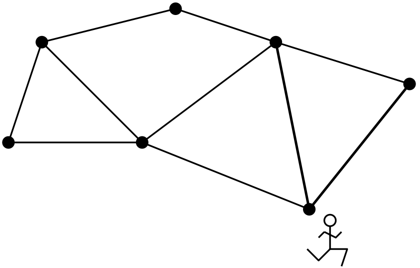

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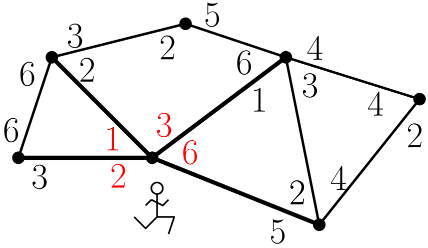
How to navigate in the graph ?



- Anonymous graph
- Port-numbering
- The agent knows its incoming port number
- It has an infinite memory

Exploration without information

Exploration of a graph G

Visit every node of G and stop

Question

What graphs can we explore without information ?

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An algorithm $\mathcal{A}$ is an exploration algorithm

- for every graph G, if $\mathcal{A}$ stops, then the agent has visited all the nodes of G

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Question

What graphs can we explore without information ?

An algorithm $\mathcal{A}$ is an exploration algorithm for a family $\mathcal{F}$

- for every graph G, if $\mathcal{A}$ stops, then the agent has visited all the nodes of G
- for every graph $G \in \mathcal{F}$, $\mathcal{A}$ visits all nodes of G and stops

Known Results [Folklore]

If nodes can be marked :

- every graph is explorable by a DFS in $O(m)$ moves

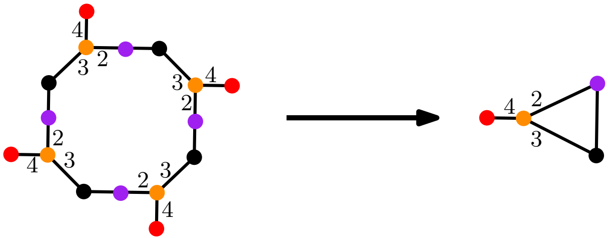
If nodes cannot be marked :

- Trees can be explored by a DFS in $O(n)$ moves
- Non tree graphs : it is impossible to detect when all nodes have been visited

Graph Coverings

Definition

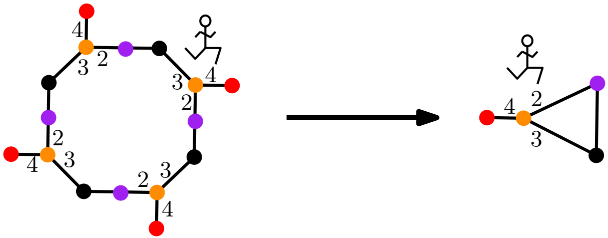
A graph covering is a locally bijective homomorphism $\varphi : G \to H$



Lifting Lemma

Lifting Lemma (from Angluin)

If G is a graph cover of H, then an agent cannot decide if it starts on $v \in V(G)$ or on $\varphi(v) \in V(H)$



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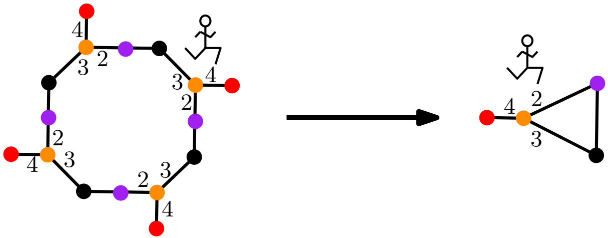
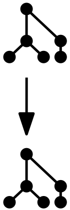


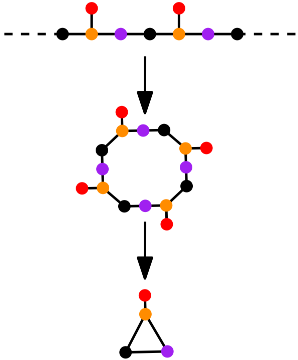
Corollary

If an exploration algorithm $\mathcal{A}$ stops in r steps in H, $r \geq |V(G)|$

Explorable graphs without global information

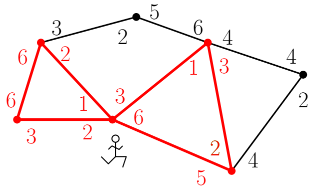
G is explorable
$\iff$ G has a unique graph cover (itself)
$\iff$ G has no infinite graph cover
$\iff$ G is a tree





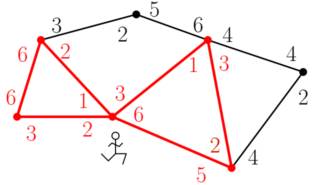
Our model : Mobile Agent with Binoculars

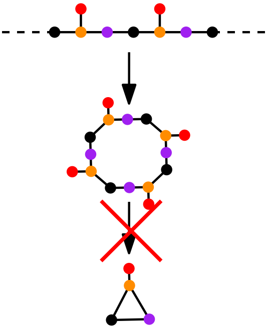
- the agent sees the graph induced by its neighbors



Our model : Mobile Agent with Binoculars

- the agent sees the graph induced by its neighbors
- One can detect triangles
- Graph coverings are no longer the good notion





What can we do with binoculars ?

- Can we explore every graph ?
  - NO
    Cycles of length ≥ 4 cannot be explored

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- Can we characterize explorable graphs ?
  - YES
  - using clique complexes and simplicial coverings
  - a universal exploration algorithm

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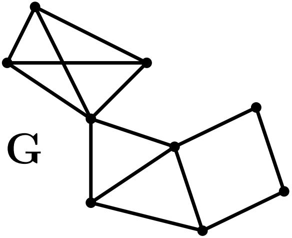
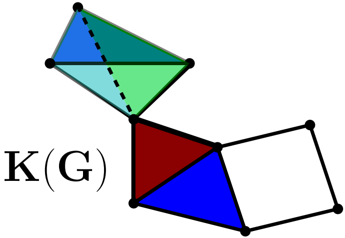
- Can we characterize explorable graphs ?
  - YES
  - using clique complexes and simplicial coverings
  - a universal exploration algorithm
- Can we find an efficient universal algorithm for explorable graphs ?
  - NO
  - the exploration time cannot be bounded by a computable function

Clique complexes

Definition

The clique complex $K(G)$ of G is a simplicial complex s.t. the simplices of $K(G)$ are the cliques of G

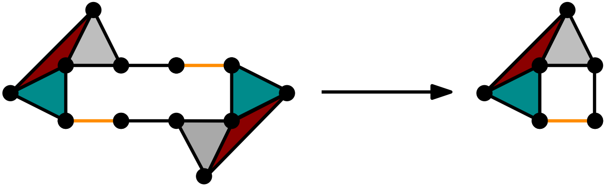




Simplicial coverings

Definition

A simplicial covering is a locally bijective simplicial map $\psi : K \to K'$

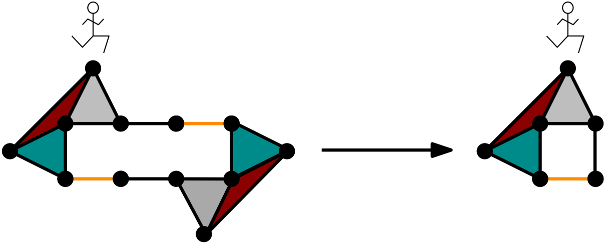


If $K(G)$ is a simplicial cover of $K(H)$, we say that G is a simplicial cover of H

Simplicial Lifting Lemma

Simplicial Lifting Lemma

If $K(G)$ is a simplicial cover of $K(H)$, then an agent cannot decide if it starts on $v \in V(G)$ or on $\varphi(v) \in V(H)$



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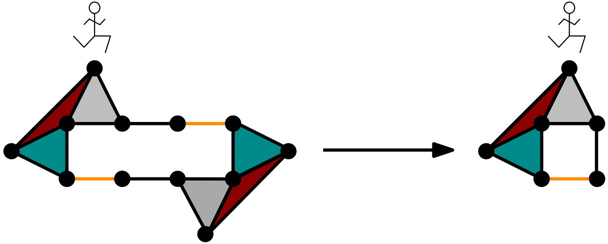
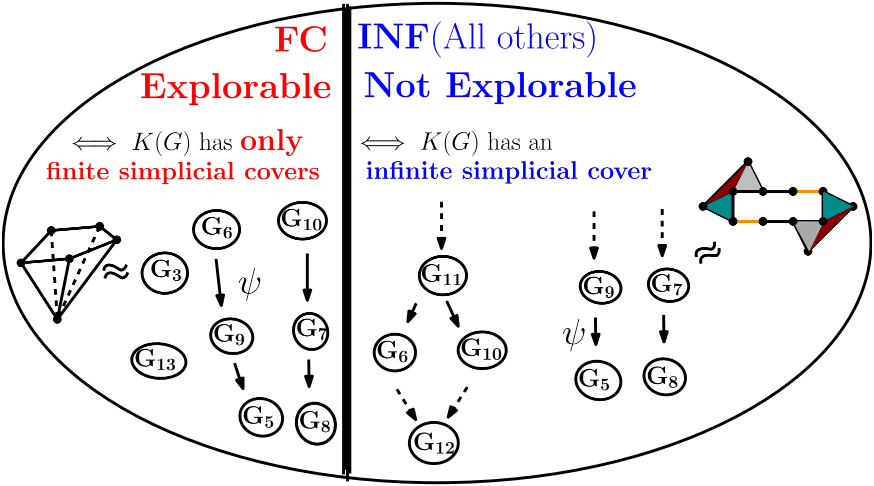


Corollary

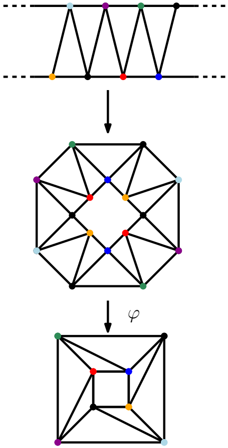
If an exploration algorithm $\mathcal{A}$ stops in r steps in H, $r \geq |V(G)|$

Exploration with binoculars : Characterization



Examples

INF : $K(G)$ has an infinite simplicial cover



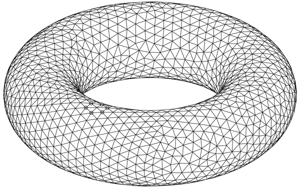
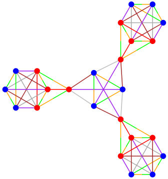
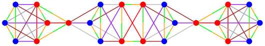


Fig. by Ag2gaeh via Wikimedia Commons

Examples

SC : $K(G)$ has a unique cover (itself)





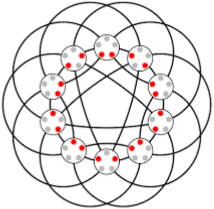
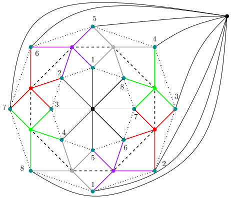
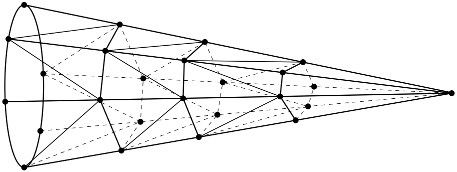


Fig. by Tilman Piesk via Wikimedia Commons

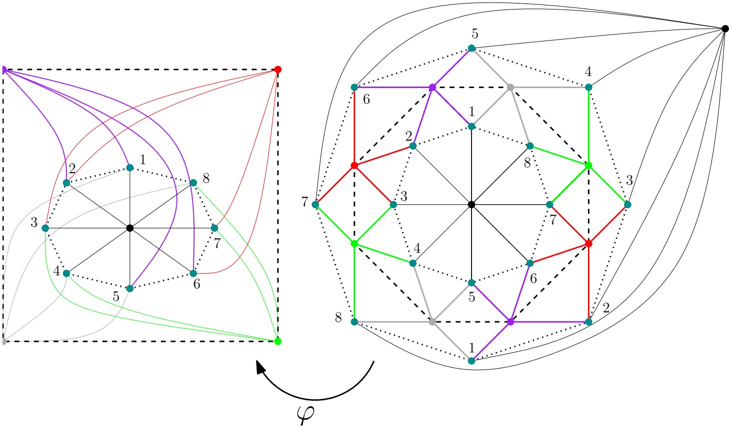




Examples

FC : $K(G)$ has only finite covers

- in this case, $K(G)$ has a finite number of covers
- SC $\subsetneq$ FC



How to distinguish the two classes ?

- INF : $\{G \mid K(G)$ has an infinite simplicial cover$\}$
- FC : $\{G \mid K(G)$ has only finite simplicial covers$\}$
- SC : $\{G \mid K(G)$ has a unique finite simplicial cover$\}$

Proposition (from Topology)

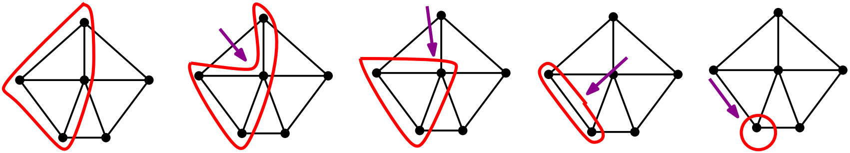
G is in FC $\iff$ G has a finite simplicial cover in SC

Theorem (from Topology)

G is in SC $\iff$ $K(G)$ is simply connected

Contractible Cycles

- a cycle is contractible if it is related with the empty cycle (a vertex) by a sequence s of elementary deformations :
  - Pushing across a triangle
  - Pushing across an isolated vertex



- c is k-contractible if $|s| \leq k$

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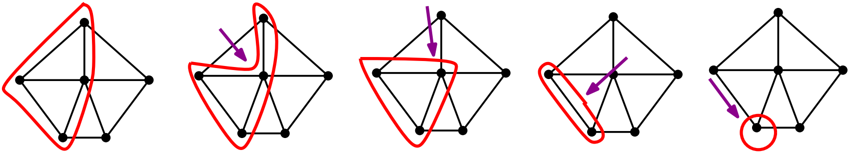

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Simple Connectivity

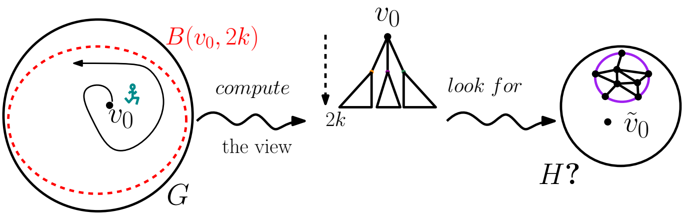
$K(G)$ is simply connected

$\iff$ all cycles of G are contractible

$\iff$ $K(G)$ has a unique simplicial cover

Our Exploration Algorithm

Explore $B(v_0, 2k)$ by computing the view $\mathcal{T}_G(v_0, 2k)$



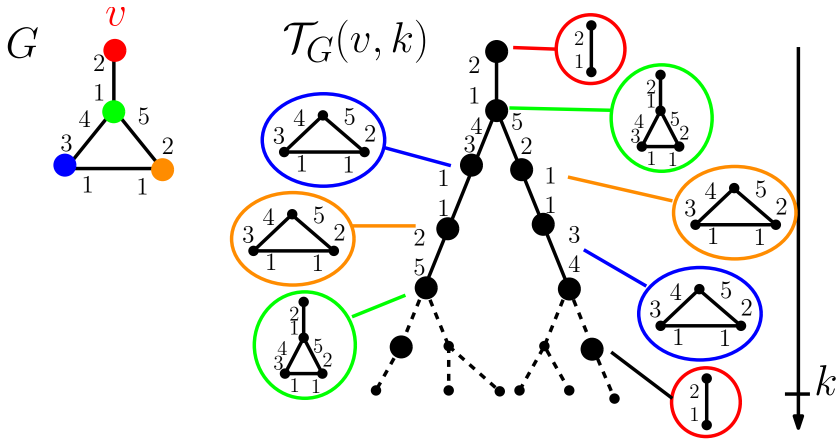
Look for a graph H such that

- $|V(H)| \leq k$
- $\exists \tilde{v}_0 \in V(H)$ s.t. $\mathcal{T}_H(\tilde{v}_0, 2k) \simeq \mathcal{T}_G(v_0, 2k)$
- simple cycles of H are $k-$contractible

If there is no such H, increment k and repeat the procedure

Views

We compute the view $\mathcal{T}_G(v)$ of v in G where every node u is labeled with $B_G(u, 1)$



Correctness of the algorithm

When the algorithm stops

- $B_G(v_0, 2k)$ explored
- $\exists H$ is found s.t.
  - $|V(H)| < k$
  - $\exists \tilde{v}_0 \in V(H)$ s.t. $\mathcal{T}_H(\tilde{v}_0, 2k) \simeq \mathcal{T}_G(v_0, 2k)$
  - simple cycles of H are $k-$contractible

Lemma

$K(H)$ is a simplicial cover of $K(G)$

Correctness

- coverings are surjective : $|V(G)| \leq |V(H)| < k$
- all nodes of G have been visited
- we have an Exploration Algorithm

Termination

For $G \in FC$, consider a simplicial cover $\widehat{G}$ of G such that $K(\widehat{G})$ is simply connected

- $\widehat{G}$ is finite
- there exists $s(\widehat{G})$ s.t. all cycles of $\widehat{G}$ are $s(\widehat{G})$-contractible
- if $k \geq |V(\widehat{G})|$ and $k \geq s(\widehat{G})$, then $\widehat{G}$ satisfies the halting conditions

Theorem

Our algorithm is an Exploration Algorithm for FC

Lower bound for the exploration with binoculars

Our algorithm seems to be terribly inefficient but . . .

Theorem

For any Exploration Algorithm $\mathcal{A}$ for SC, for any computable function $f : \mathbb{N} \to \mathbb{N}$, there exists $G \in$ SC such that $\mathcal{A}$ performs strictly more than $f(|V(G)|)$ moves on G

By a reduction from the following problem that is undecidable [Haken, 1973]

- INPUT : A finite simplicial complex K
- QUESTION : Is K simply connected ?

Conclusion

Summary

- Binoculars are a natural and interesting enhancement
- A large class of explorable graphs
  - Triangulations of the sphere , Chordal graphs, Planar triangulation of the projective plane, ...
- An amazing but unavoidable complexity

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Perspectives

- What happens if we enlarge the vision of the agent ?
  - we believe the results would be qualitatively the same
- Find large subclasses that can be explored more efficiently (with a linear or polynomial number of moves)