Extending Brooks' theorem to directed graphs

Joint work with Pierre Aboulker

Guillaume Aubian

TALGO/IRIF

18 november 2020

TALGO

INSTITUT
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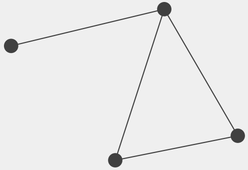
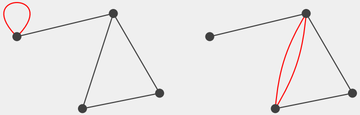
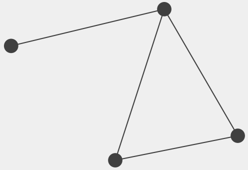
Background and context

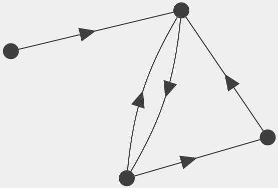


(V, E) with $E \subseteq \{\{u, v\}, u, v \in V\}$



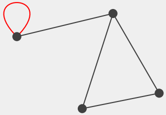
(DI)GRAPHS

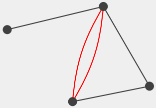


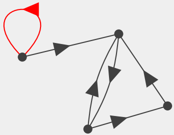


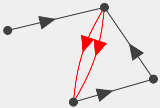
(V, E) with $E \subseteq \{\{u, v\}, u, v \in V\}$

(V, E) with $E \subseteq V \times V$





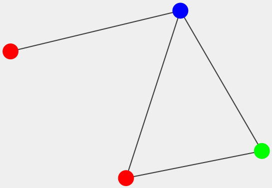




Definition

$G = (V, E)$ k-colorable iff

$V = \bigcup_{i=1}^{k} V_i$ and $G[V_i]$ are edgeless.



$$\chi(G) = \min_{k}\{k | G \ k\text{-colorable}\} = 3$$

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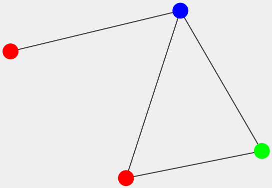

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Definition

$G = (V, E)$ k-colorable iff
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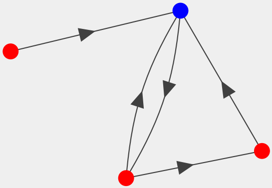
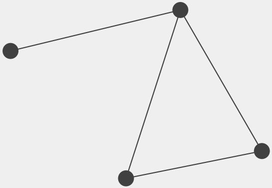
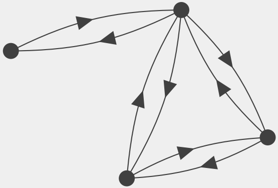


$$\vec{\chi}(G) = \min_k \{k | G\ k\text{-colorable}\} = 2$$

G



$\overleftrightarrow{G}$

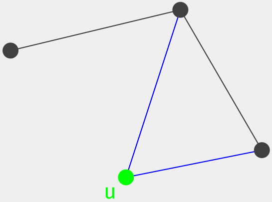


Theorem

$$\chi(G) = \vec{\chi}(\overleftrightarrow{G})$$

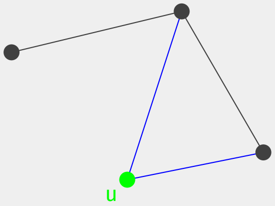
Goal

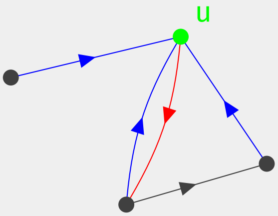
Generalize to directed graphs results that apply to graphs



$$d(u) = \{v | uv \in E\} = 2$$

$$\Delta(G) = max_{v \in V} d(v) = 3$$





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$$\Delta(G) = max_{v \in V} d(v) = 3$$

$$d^{-}(u) = \{v | vu \in E\} = 3$$

$$d^{+}(u) = \{v | uv \in E\} = 1$$

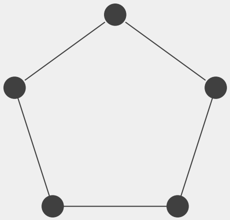
$$\Delta_{MAX}(G) = max_{v \in V} d_{MAX}(v) = 3$$

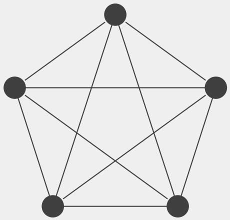
$$\Delta_{MIN}(G) = max_{v \in V} d_{MIN}(v) = 1$$

Theorem

Let G be a connected graph.
$\chi(G) \leq \Delta(G) + 1$ *and equality occurs if and only if G is :*

- *a cycle on an odd number of vertices or*
- *a complete graph on* $\Delta(G) + 1$ *vertices.*





Lovasz' proof

$$\chi(G) \leq \Delta(G) + 1$$

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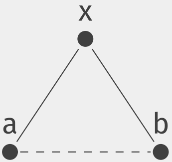
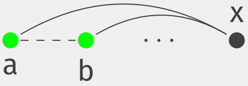
Lovasz' idea :





Theorem

$\vec{\chi}(G) \leq \Delta_{MIN}(G) + 1 \leq \Delta_{MAX}(G) + 1$

Proof : Consider a vertex of minimum indegree/outdegree. Color the rest of the graph, and color it with a color not assigned to any of its in/outneighbours

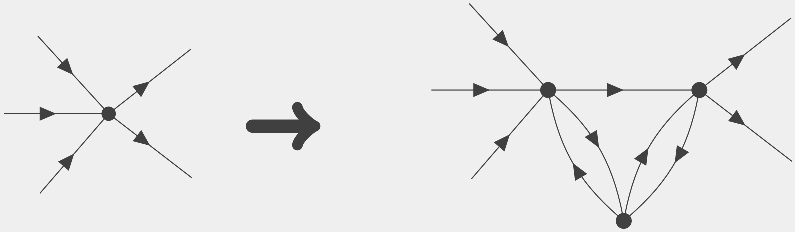
Also : G connected and not regular $\implies \vec{\chi}(G) \leq \Delta_{MAX}(G)$

Brooks' theorem for Δ_{min}

Theorem

Let $k \geq 2$. The problem :
Input: *a digraph G with $\Delta_{MIN}(G) = k$.*
Output: *Does there exist a k-dicoloring of G.*
is NP-complete.

Proof



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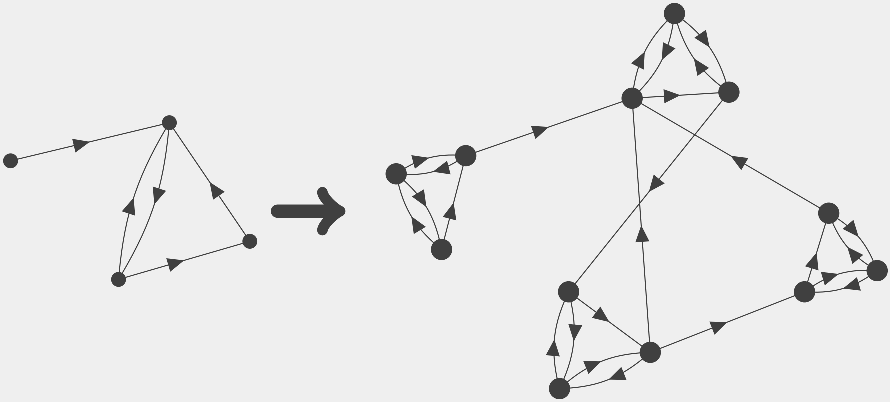

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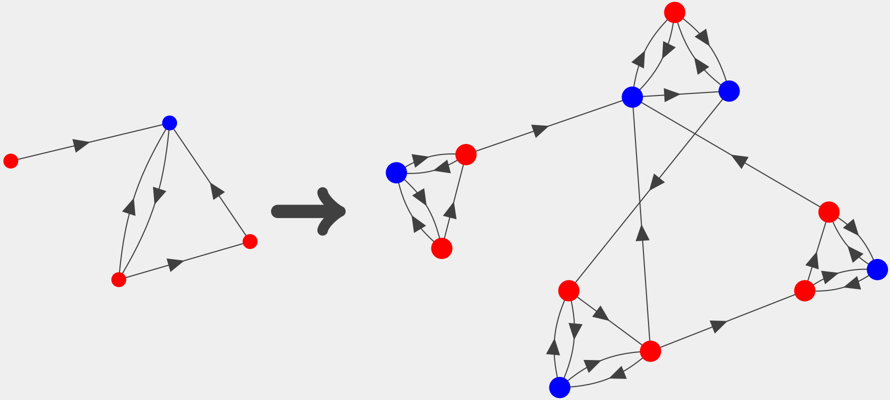
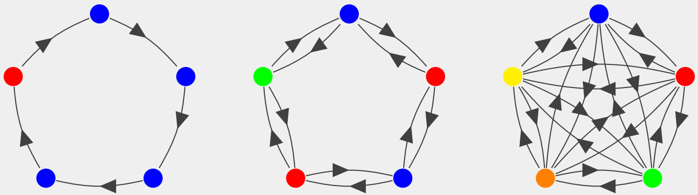


Brooks' theorem for Δ_{MAX}

Theorem (Mohar-Ararat, 2010)

Let G be a connected digraph.
$\vec{\chi}(G) \leq \Delta_{MAX}(G) + 1$ *and equality occurs if and only if G is :*

- *a directed cycle or*
- *a symmetric cycle of odd length or*
- *a complete digraph on $\Delta_{MAX}(G) + 1$ vertices.*



- Adaptation of Lovasz' proof
- Adaptation of Rabern's proof
- Proof using "*k*-trees"
- Proof by smart partition

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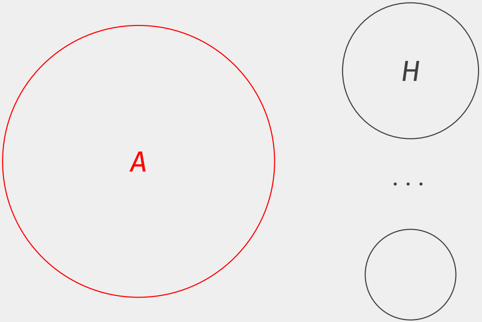
- Adaptation of Rabern’s proof
- Proof using "k-trees"
- Proof by smart partition

Sketch of proof for $\Delta_{MAX}(G) \geq 3$

Let G a minimum counterexample

Lemma

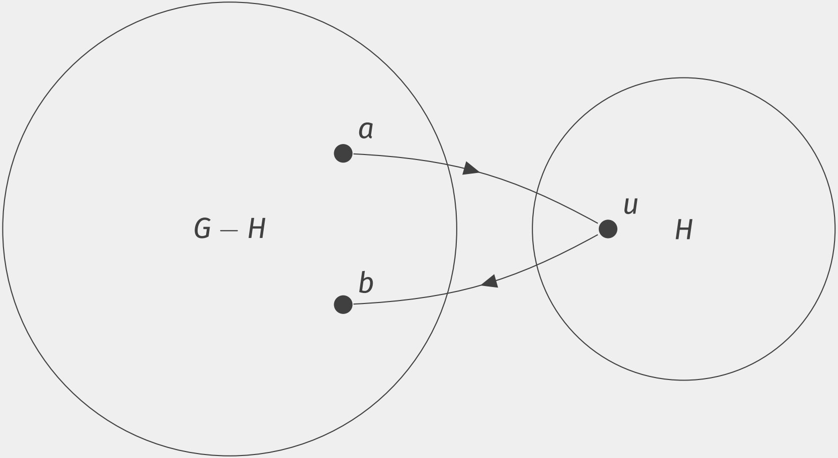
G does not contain $\overleftrightarrow{K}_{\Delta_{MAX}(G)+1}$ less an arc.
G does not contain $\overleftrightarrow{K}_{\Delta_{MAX}(G)+1}$ less a digon.



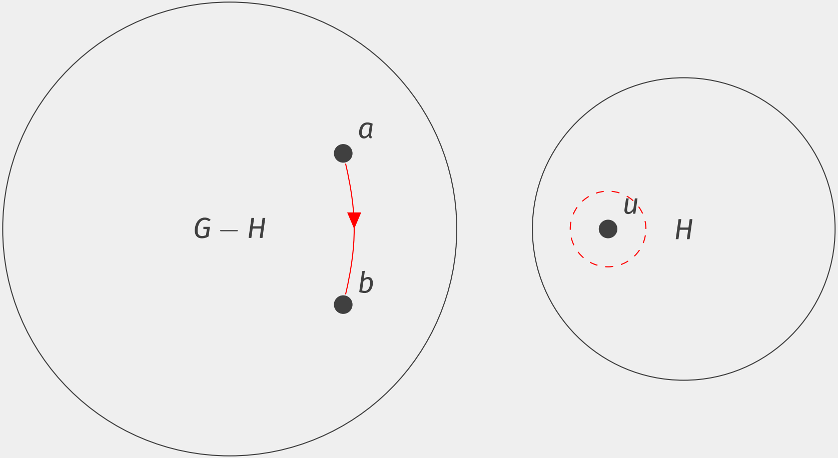
A is a maximal DAG

H is an exception for $\Delta_{MAX}(G) - 1$

A VERTEX OF H HAS DISTINCT NEIGHBOURS OUTSIDE H



A vertex of H has distinct neighbours outside H



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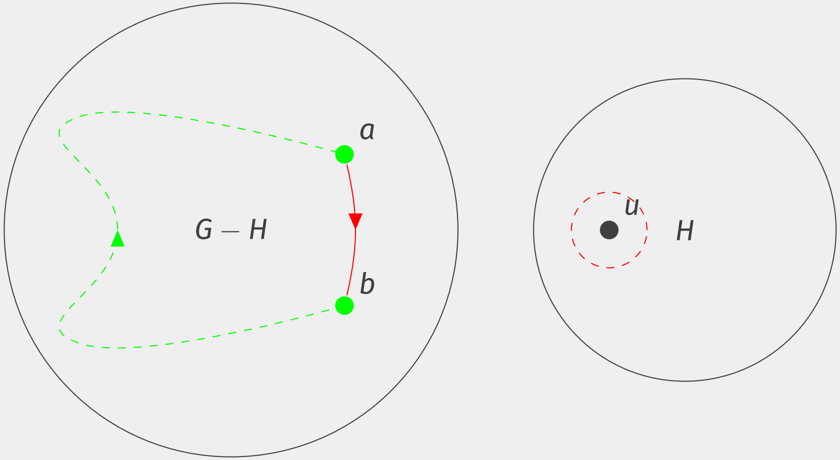

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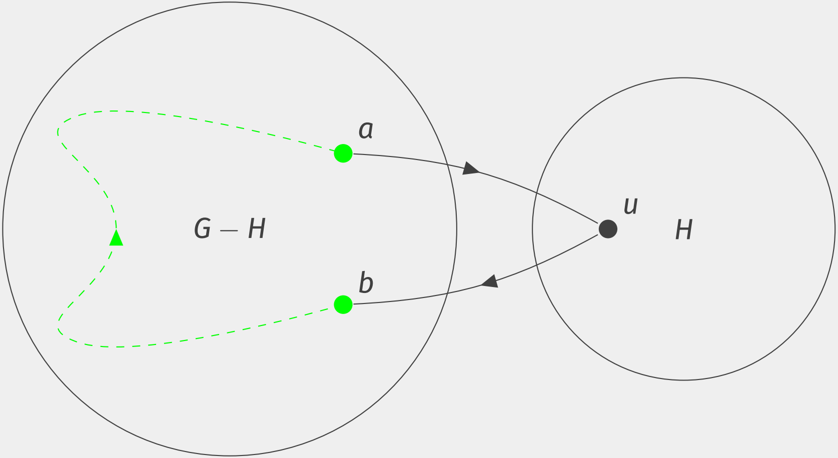
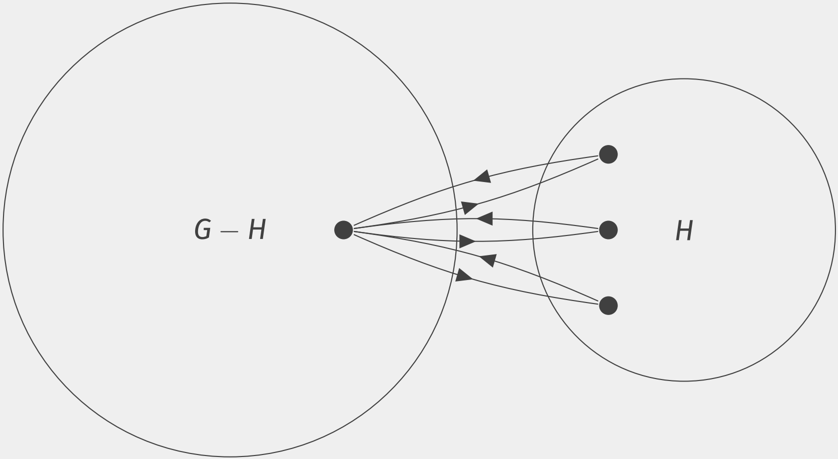
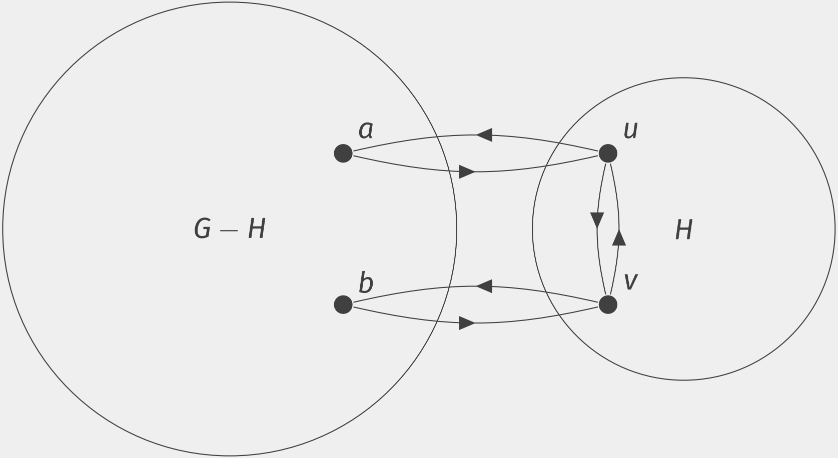


This situation cannot happen



Thus, this situation necessarily arises



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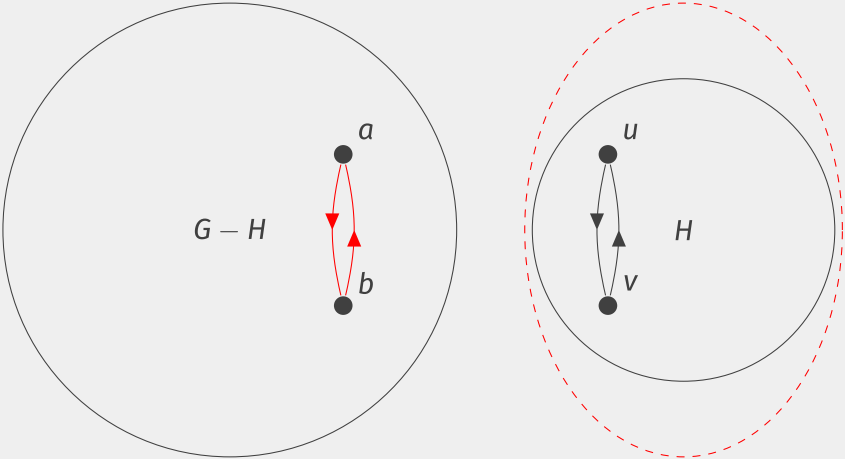

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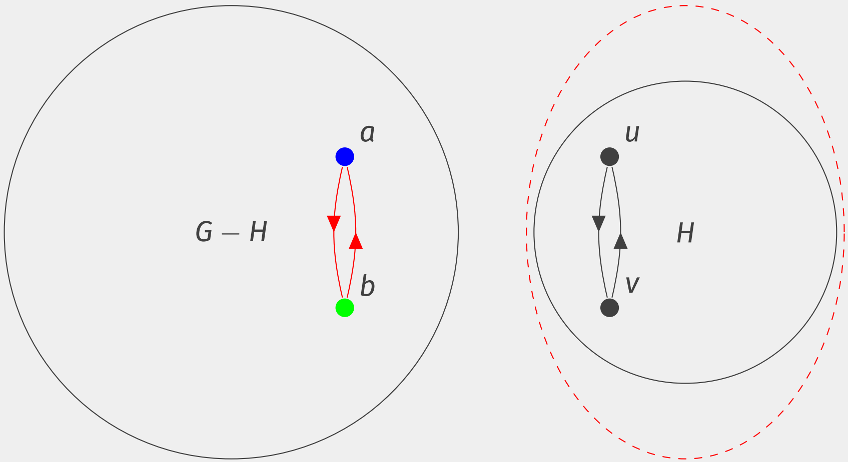

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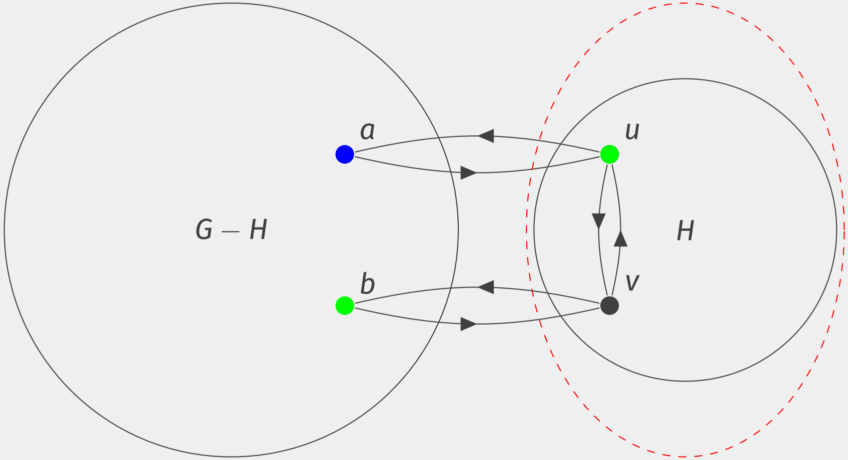
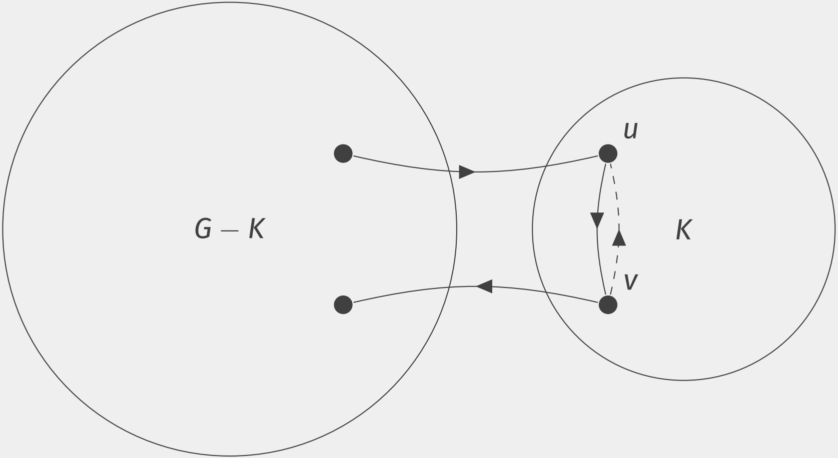


No induced $\overleftrightarrow{K}_{\Delta_{max}(G)+1}$ less an arc



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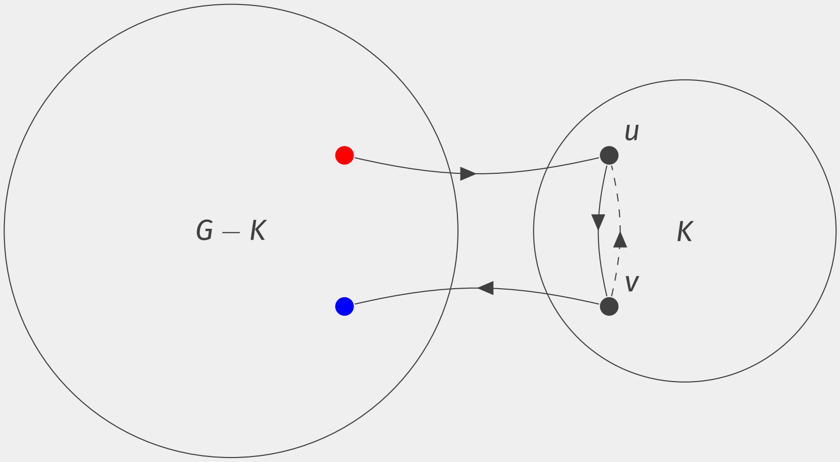

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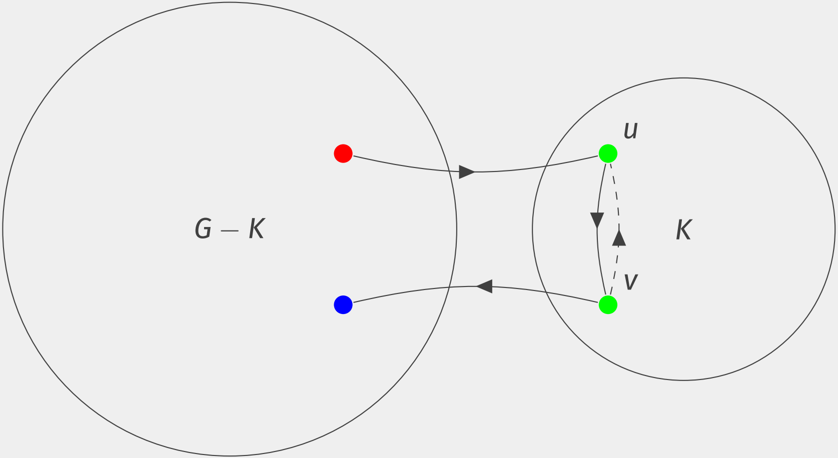
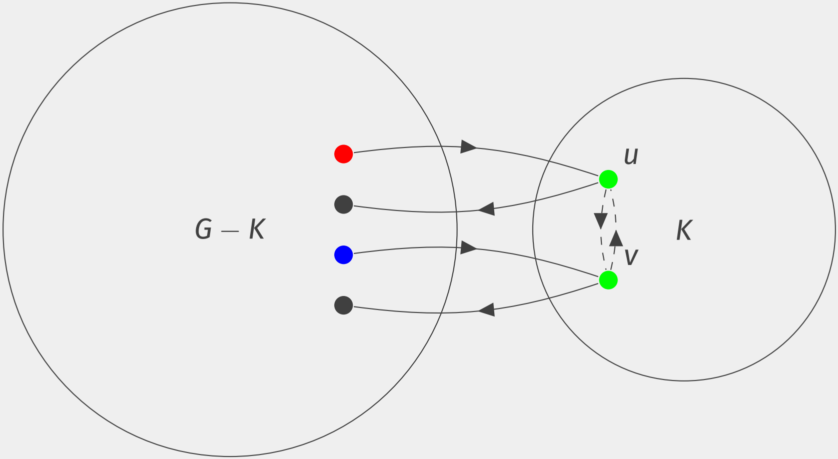


No induced $\overleftrightarrow{K}_{\Delta_{max}(G)+1}$ less a digon



Could we get a similar result when $\vec{\chi}(G) = \Delta_{MAX}(G)$ **?**

Could we get a similar result when $\vec{\chi}(G) = \Delta_{MAX}(G)$?
Somehow, yes.

Theorem

If $\vec{\chi}(G) \geq \Delta_{MAX}(G) \geq 9$, then $\omega(G) \geq \lceil \frac{\Delta_{MAX}(G)+1}{2} \rceil$.

Partition

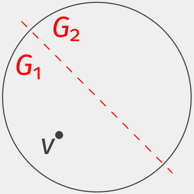
Definition

A (r_1, r_2)-partition of a digraph G is a partition (V_1, V_2) of V which minimizes $r_1|E(G[V_2]| + r_2|E(G[V_1])|$.

Theorem

If $r_1 + r_2 \geq 2\Delta_{MAX}(G) - 1$, then for $i \in \{1, 2\}$:

$$\forall v \in V_i, d^-_{G[V_i]}(v) + d^+_{G[V_i]}(v) \leq r_i$$



$$d^-_{G_1}(v) + d^+_{G_1}(v) > r_i$$

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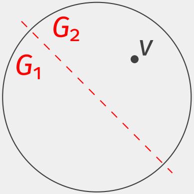

$r_1(r_2 - 1) - r_2(r_1 + 1)$

Theorem

If $\vec{\chi}(G) \geq \Delta_{MAX}(G) \geq 9$, then $\omega(G) \geq \lceil \frac{\Delta_{MAX}(G)+1}{2} \rceil$.

Let $r_1 = \lceil \frac{\Delta_{MAX}(G)-1}{2} \rceil$ and $r_2 = \lfloor \frac{\Delta_{MAX}(G)-1}{2} \rfloor$

Proof : Borodin–Kostochka on digraphs

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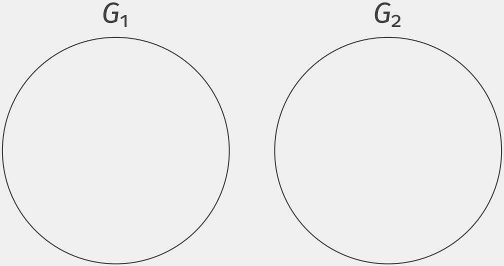

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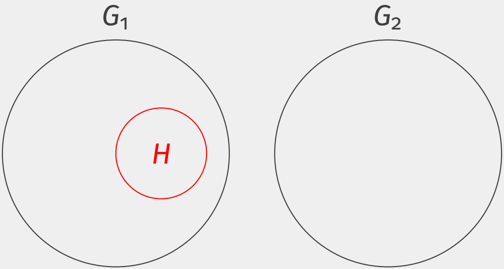

- Borodin–Kostochka : $\vec{\chi}(G) \geq \Delta_{MAX}(G) \geq 9 \implies \vec{\chi}(G) = \omega(G)$
- Reed : $\vec{\chi}(G) \leq \lceil \frac{\omega + \Delta_{MAX}(G) + 1}{2} \rceil$
- Using other invariants instead of $\Delta_{MIN}/\Delta_{MAX}$

Thanks for your attention!