Connected tree-width of a series-parallel graphs

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**Definition**

A **tree decomposition** $T$ of a graph $G = (V,E)$ is a tree where nodes are subset of $V_G$. Each vertex and each edge appear in $T$ and $\forall x \in V$, the set of nodes containing $x$ has to induce a connected sub-tree of $T$.

**Treewidth**

Let $G$ be a graph. The width of a tree decomposition $T$ of $G$ is $\text{width}(T) = \max\{|X| - 1 | X \in T\}$. The **treewidth** of the graph $G$ is $\text{tw}(G) = \min_T\{\text{width}(T) \text{ with } T \text{ a tree decomposition of } G\}$.

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*Source: Wikipedia*
A tree decomposition $T$ is connected if there exists $r$ such that for every path $p$ from $r$, the subgraph of $G$ induces by the vertices in $p$ is a connected subgraph of $G$. The connected treewidth of a graph $G$ is $\text{ctw}(G) = \min_T \{ \text{width}(T) \mid T \text{ a connected tree decomposition of } G \}$.
Let $G$ be a graph. A *layout* $\sigma$ is a permutation of the vertices of $G$.

![Diagram showing a graph with a layout $\sigma$ and support set $S_\sigma(i)$]

**Support set**

$\forall i \in [1, n]$, we define $S_\sigma(i) = \{x \in V_G | \sigma(x) < \sigma(i) \land \exists\text{ a path } p \text{ from } i \text{ to } x \text{ with internal vertices in } \sigma_{>i}\}$. 
Support set

Let $G$ be a graph. A layout $\sigma$ is a permutation of the vertices of $G$.

$\forall i \in [1, n], \text{ we define } S_\sigma(i) = \{x \in V_G \mid \sigma(x) < \sigma(i) \land \exists \text{ a path } p \text{ from } i \text{ to } x \text{ with internal vertices in } \sigma_{>i}\}$.

Tree vertex separation number

We denote $\text{tvs}(G) = \min_{\sigma} \max_{i \in [1, n]} |S_\sigma(i)|$. 
Connected layout

Let $G$ be a connected graph. A layout $\sigma$ is a connected layout if one of the two following equivalent properties are satisfied:

- $\forall i \in V_G, G[\sigma \leq i]$ induces a connected subgraph of $G$.
- $\forall i \in V_G, \exists j$ such that $\sigma_j$ is a neighbour of $\sigma_i$ with $j < i$.

Connected tree vertex separation number

We denote $\text{ctvs}(G) = \min_\sigma \max_{i \in [1,n]} |S_\sigma(i)|$ with $\sigma$ a connected layout.

By definition, we have $\text{tvs}(G) \leq \text{ctvs}(G)$. 
Connected rooted layout

Rooted layout

Let \((G, R)\) a rooted graph where \(R \subset V_G\). A rooted layout \(\sigma\) on \((G, R)\) is a layout where \(R\) is a prefix of \(\sigma\).

Connected rooted layout

A rooted layout \(\sigma\) is connected if, \(\forall i \in [1, n]\), every connected component of the subgraph \(G[\sigma_{\leq i}]\) contains a root vertex.
Series-parallel graphs

Definition

A graph $G$ is a **series-parallel** graph if it is an edge $\{x, y\}$ or it can be built from two other series-parallel graphs $G_1$ and $G_2$ by the series composition $\otimes$ or by the parallel composition $\oplus$. 

![Diagram of series-parallel graphs](image)
A series-parallel graph $G$ can be represented by a series-parallel tree where each node is $\otimes$ or $\oplus$ composition.

**Theorem**

If $G$ is a bi-connected series-parallel graph, then:

$$\forall (x, y) \in E_G, (G, (x, y)) = (G_1, (x, y)) \oplus (G_2, (x, y)) \text{ with } G_2 = K_2.$$
Results

Theorem

- [Dendris N., Kirousis L., Thilikos D. TCS’97]: \( \text{tw}(G) = \text{tvs}(G) \).
- [Adler I., Paul C., Thilikos D. FST-TCS’19]: \( \text{ctw}(G) = \text{ctvs}(G) \).
- Price of connectivity: \( \forall k \in \mathbb{N}, \text{there exists } G \text{ such that } \text{ctw}(G) - \text{tw}(G) \geq k \).

Our result

[Mescoff G., Paul C., Thilikos D.]: We can compute the connected treewidth of series-parallel graphs in \( O(n^2 \cdot \log n) \) time where \( n \) is the number of vertices of \( G \).
Extended graph

Let $G = (V, E)$ a graph and $F$ a set of edges disjoint from $E_G$. We denote $G^+F$ the extended graph $G$ with fictive edges from $F$. We said that $G$ is the solid graph of $G^+F$.

Connexity of $G^+F$

Fictive edges do not increase connexity of $G^+F$. Connected components of $G^+F$ are exactly the connected components of the solid graph $G$. 
Extended Path

An extended path is a path of $G^+F$ containing fictive edges.

**Extended cost**

- $\forall i \in V_G$ we define $S^+_{\sigma}(i) = \{x \in V_G | \sigma(x) < \sigma(i) \land \exists \text{ a extended path } p \text{ from } i \text{ to } x \text{ with internal vertices in } \sigma_{>i}\}.$
- $\text{ectvs}(G) = \min_{\sigma} \max_{i \in [1,n]} |S^+_{\sigma}(i)|$ with $\sigma$ a connected layout.
Lemma

Let \((G^+∅, R)\) be a rooted extended graph such that \(R = \{x, y\}\) and \(G = G_1 \oplus G_2\) with \(G_1 = (G_1, (x, y))\) and \(G_2 = (G_2, (x, y))\). Then,

\[
\text{ectvs}(G^+∅, R) = \max\{\text{ectvs}(G_1^+∅, R), \text{ectvs}(G_2^+∅, R)\}.
\]
Let \((G^+\emptyset, R)\) be a rooted extended graph such that \(R = \{x, y\}\) and \(G = G_1 \oplus G_2\) with \(G_1 = (G_1, (x, y))\) and \(G_2 = (G_2, (x, y))\). Then,

\[
\text{ectvs}(G^+\emptyset, R) = \max\{\text{ectvs}(G_1^+\emptyset, R), \text{ectvs}(G_2^+\emptyset, R)\}.
\]
Lemma

$$\text{ectvs} (G^+\emptyset, R) = \min \left\{ \max \left\{ \text{ectvs}(\tilde{G}_1^+\{zy\}, R), \text{ectvs}(G_2^+\emptyset, R_2) \right\}, \max \left\{ \text{ectvs}(\tilde{G}_2^+\{zx\}, R), \text{ectvs}(G_1^+\emptyset, R_1) \right\} \right\}. $$

Figure: Decomposition of an extended graph resulting from a series composition.
Series composition without fictive edges

Lemma

\[ \text{ectvs}(G^+ \emptyset, R) = \min \left\{ \begin{array}{l} \max \left\{ \text{ectvs}(\tilde{G}_1^+ \{zy\}, R), \text{ectvs}(G_2^+ \emptyset, R_2) \right\} \\ \max \left\{ \text{ectvs}(\tilde{G}_2^+ \{zx\}, R), \text{ectvs}(G_1^+ \emptyset, R_1) \right\} \end{array} \right\}. \]

Figure: Rearranging a layout \( \sigma^* \) of \( G = G_1 \otimes G_2 \) of minimum cost into
\[ \sigma = \langle x, y \rangle \odot \sigma^*[V_1 \setminus \{x\}] \odot \sigma^*[V_2 \setminus \{y, z\}] \].
Parallel and series composition with extended graph
Composition with the fictive edge $(x,y)$

\begin{align*}
G_1 & \rightarrow G_2 \\
\quad & \rightarrow \\
\quad & +
\end{align*}
Let $G$ a biconnected graph.

- $\text{ctw}(G) = \min_{(x,y) \in G} (G^+ \emptyset, \{x, y\}) \leftarrow \mathcal{O}(n)$.
- We have at most $2n$ steps in our algorithm $\leftarrow \mathcal{O}(n)$.
- For each step, we compute at most $\alpha n$ results with some constant $\alpha \leftarrow \mathcal{O}(n)$.

Which gives a total time complexity in $\mathcal{O}(n^3)$. With a better complexity analysis, we can show that the real time complexity is $\mathcal{O}(n^2 \cdot \log n)$. 
**Generalization**

**Treewidth at most 2**

A graph $G$ has treewidth at most 2 iff its biconnected components are series-parallel graphs. So, $G$ contains a cut vertex or $G$ is a biconnected series-parallel graph.

$G_1 \xrightarrow{X} G_2 + G_3$

**Complexity**

Since the complexity is $\mathcal{O}(n^2 \cdot \log n)$ for every biconnected component and since we try for every starting vertex, the total time complexity is $\mathcal{O}(n^3 \cdot \log n)$. 
Conclusion

We see in this presentation how compute the connected treewidth for graph with treewidth at most 2. The complexity of the general problem is still open:

- **Conjecture 1:** Connected treewidth can be computed by an $\mathcal{O}(n^{f(tw(G))})$-time algorithm.
- **Conjecture 2:** Connected treewidth bounded by $k$ can be decided by an $\mathcal{O}(n \cdot f(k, tw(G)))$-time algorithm.
- **Conjecture 3:** Connected treewidth bounded by $k$ can be decided by an $\mathcal{O}(n^{f(k)})$-time algorithm.