Quand Hadwiger rencontre Cayley

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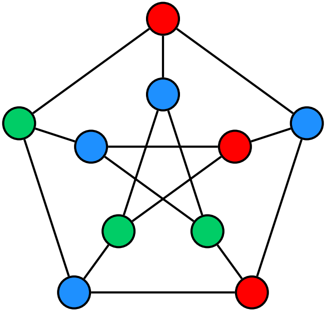
Mineurs et coloration

Coloration de graphes

Colorer les sommets sans donner la même couleur à des sommets adjacents

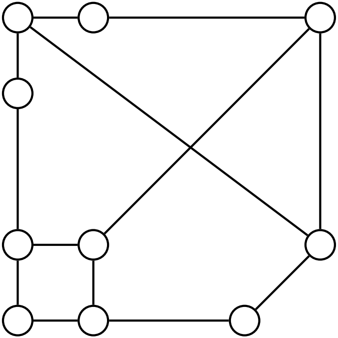


Nombre minimal de couleurs nécessaires $= \chi(G)$.

Mineurs

Opérations :

- Suppression de sommets
- Suppression d'arêtes
- Contraction d'arêtes



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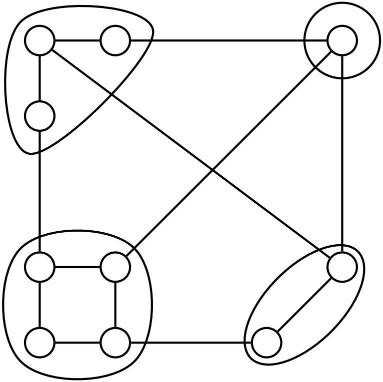

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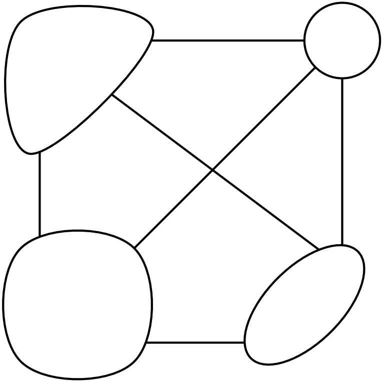

Conjecture d'Hadwiger

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Si $\chi(G) \geqslant k$, alors K_k est un mineur de G.

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Conjecture d'Hadwiger, 1943 – Version 2

Tout graphe sans mineur K_k est $(k-1)$-colorable.

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Wagner, 1936

Hadwiger $k = 5 \Leftrightarrow$ 4CT.

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Kostochka, 1984

G sans mineur $K_k \Rightarrow \chi(G) = O\left(k\sqrt{\log(k)}\right)$.

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Formule de Cayley

Borchardt 1860, Cayley 1889

K_k a k^{k-2} arbres couvrants.

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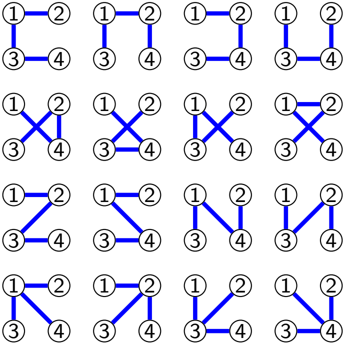

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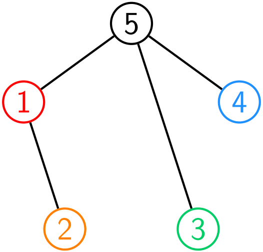

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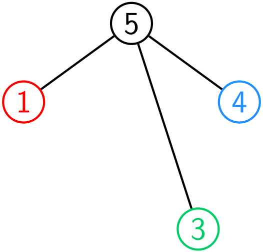

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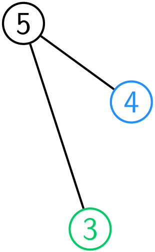

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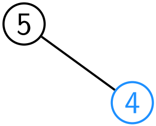

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1555

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1555

Arbre couvrant de $K_n \leftrightarrow$ mot de taille $n-1$ qui finit par n.

Hadwiger + Cayley

Question [Sivaraman, Banff 2020]

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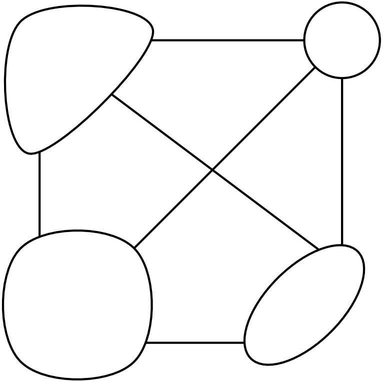

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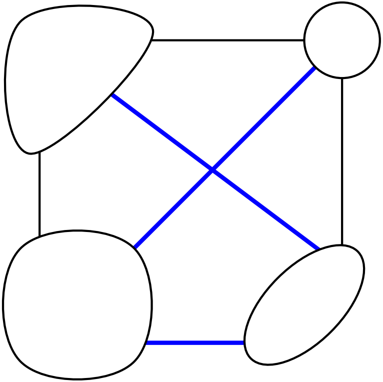

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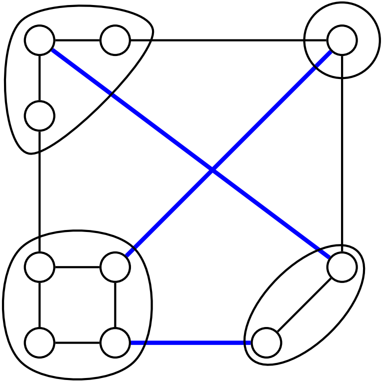

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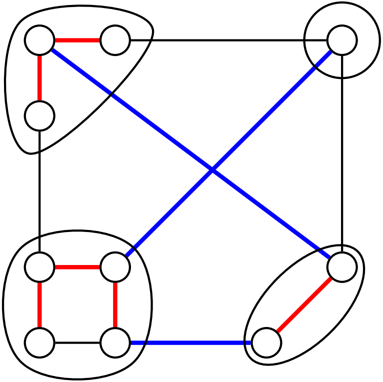
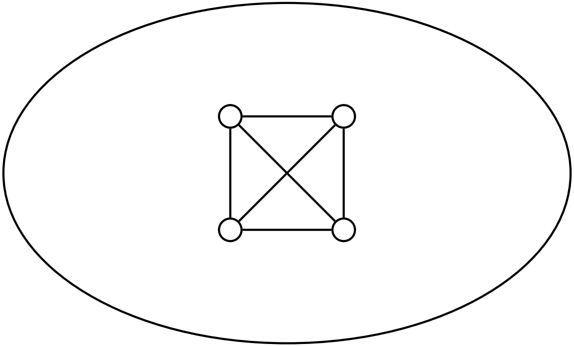


Une preuve

Graphes contenant K_k



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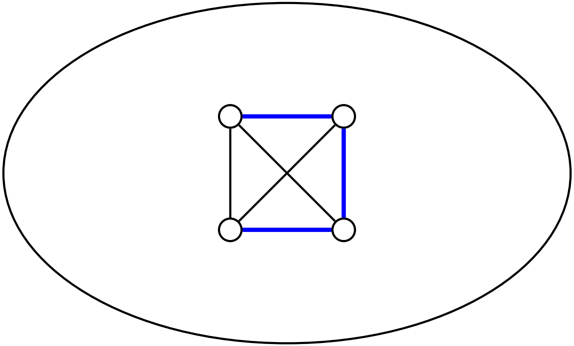

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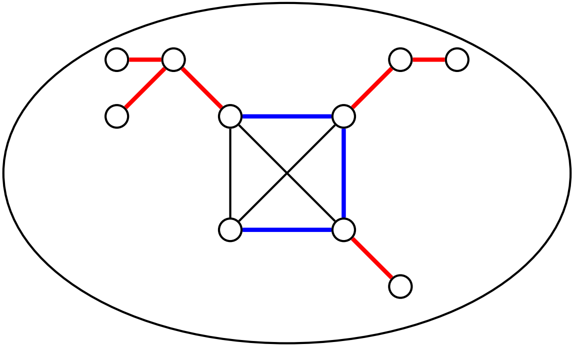

$\#AC = k^{k-2}$ $\#\text{ext} \geqslant 1$

Idée générale

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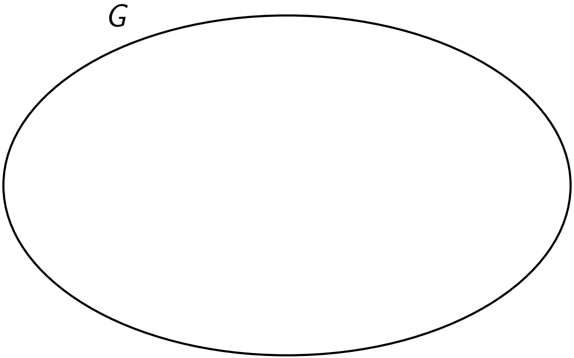

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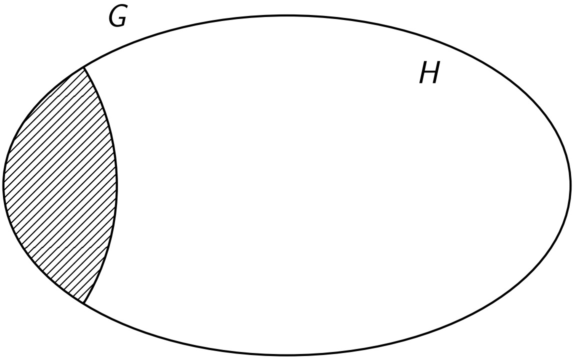

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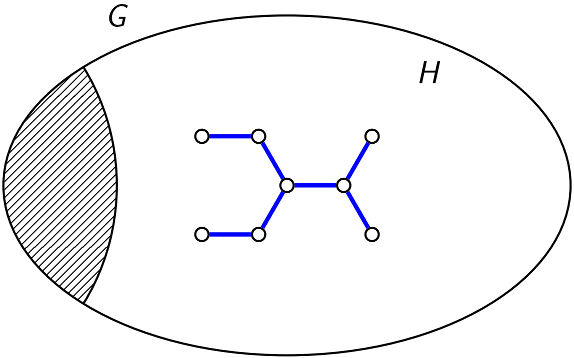

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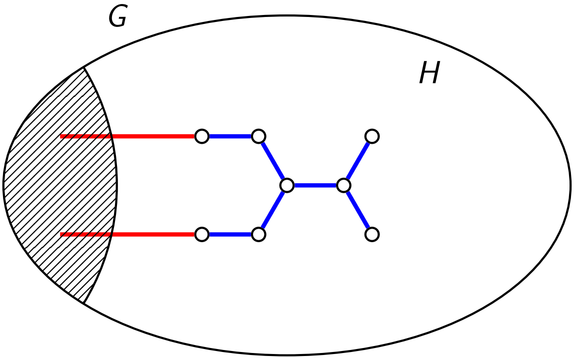

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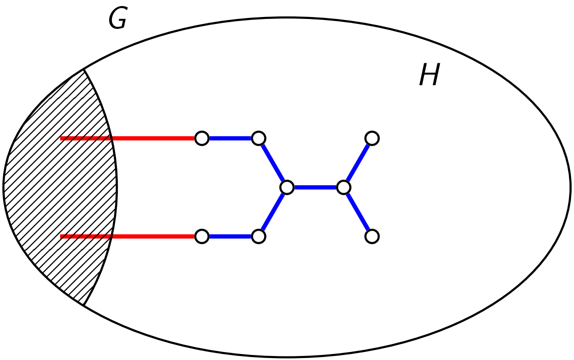
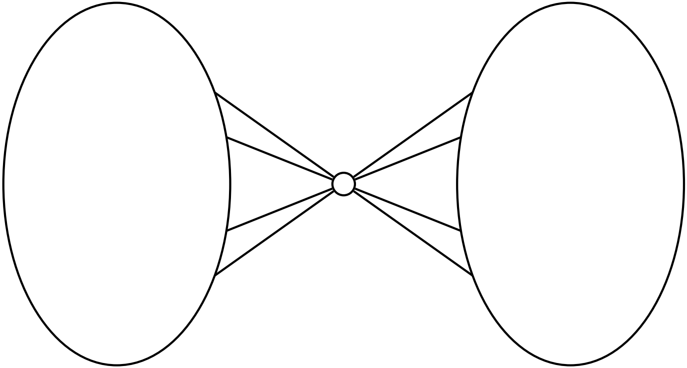


$$\#AC(G) \geqslant \#AC(H) \times \#\text{extensions}$$

Cas non 2-connexe



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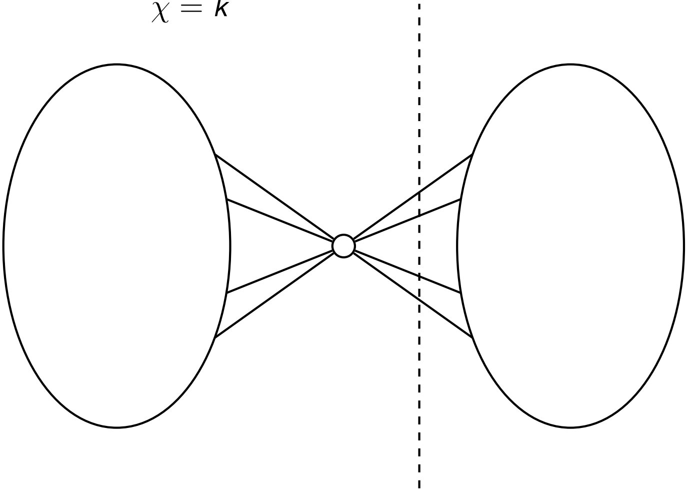

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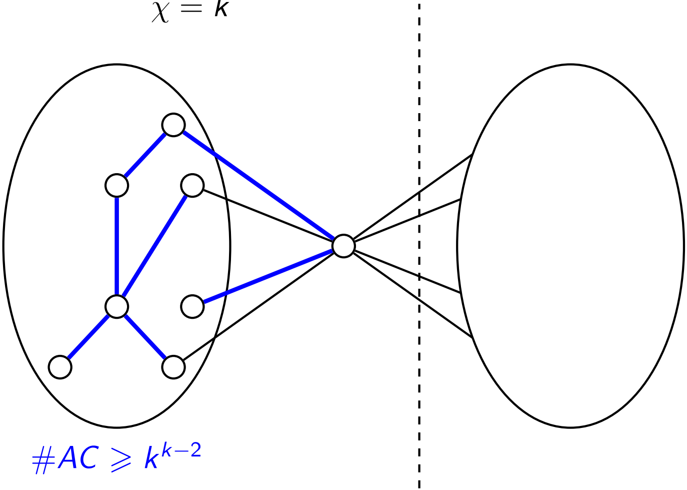

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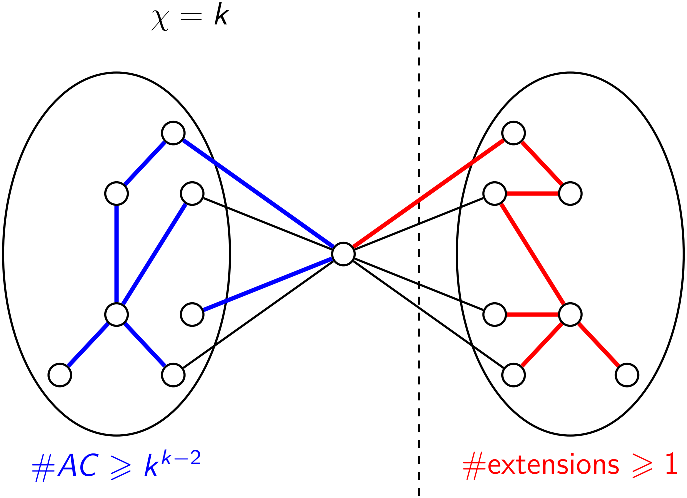

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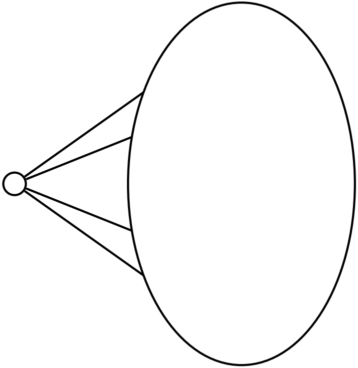

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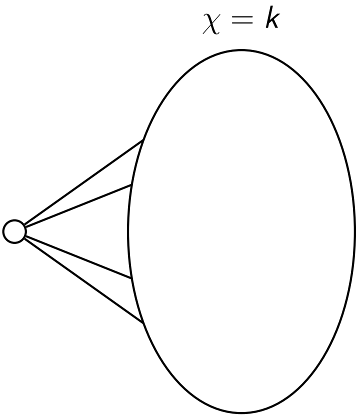

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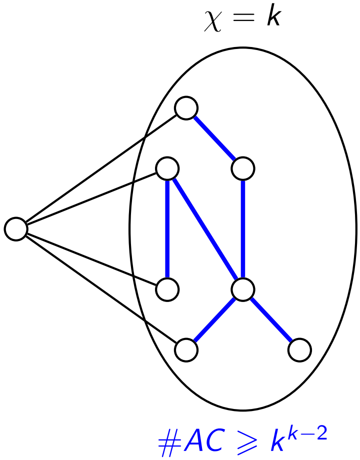

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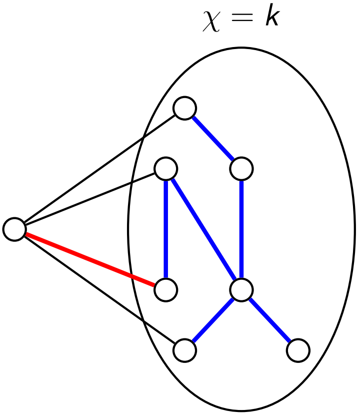

$\#\mathrm{ext} \geqslant 1$ $\#AC \geqslant k^{k-2}$

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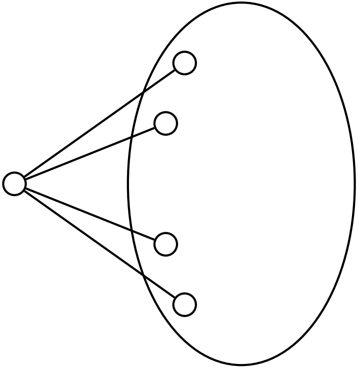

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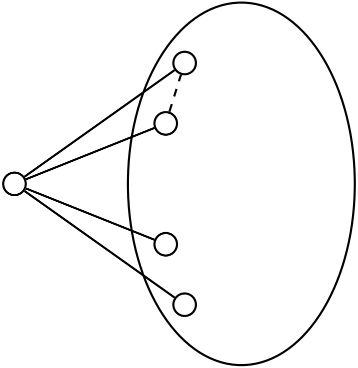

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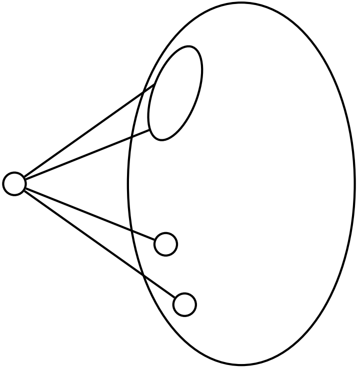

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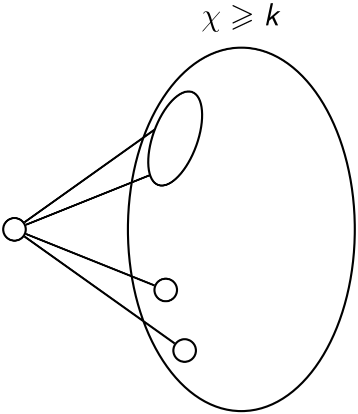

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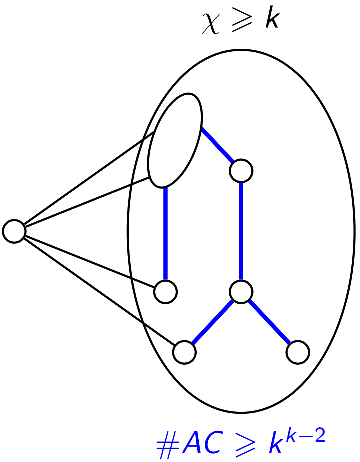

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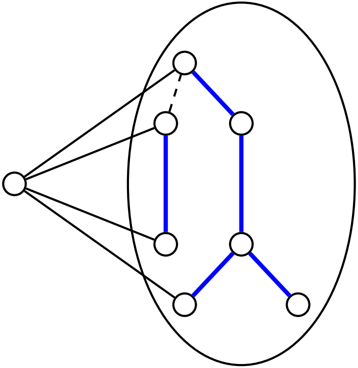

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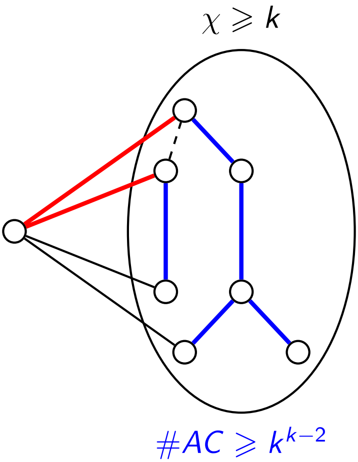

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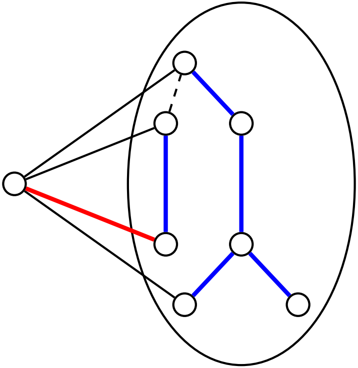

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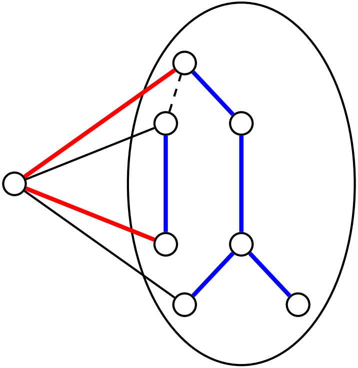

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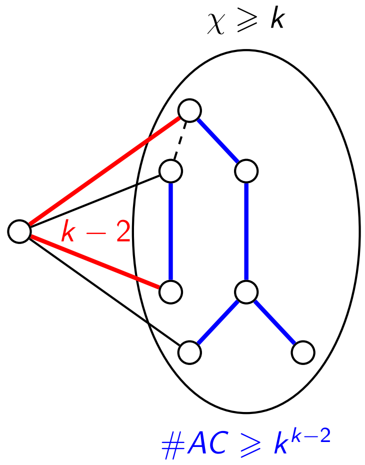

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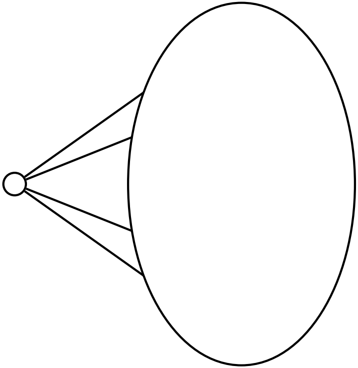

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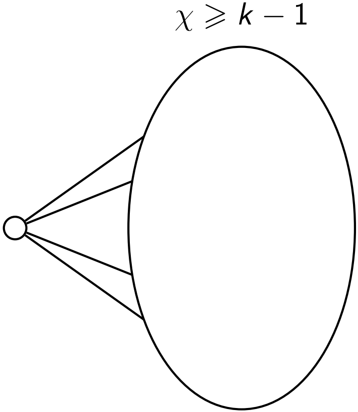

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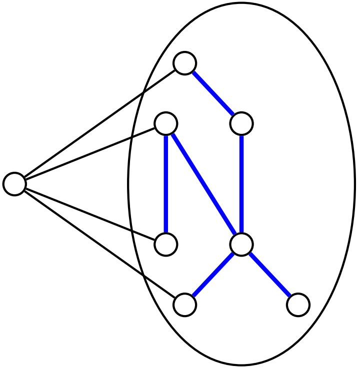

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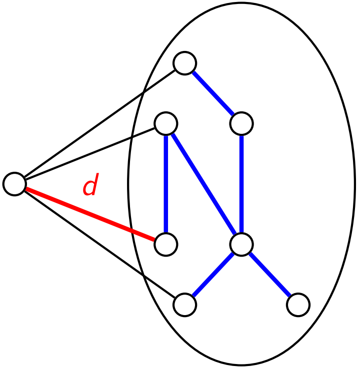

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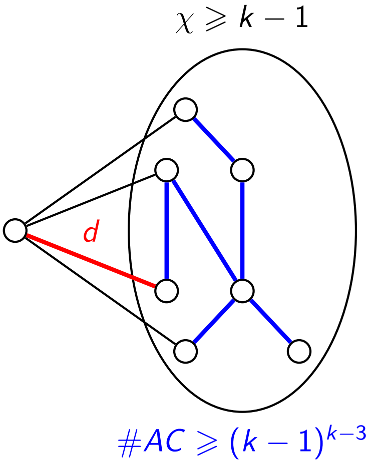

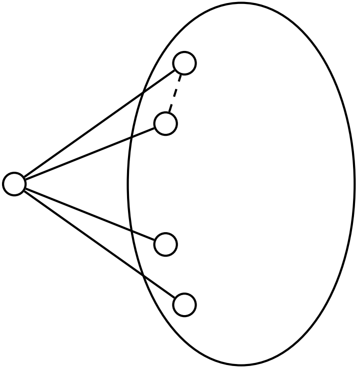

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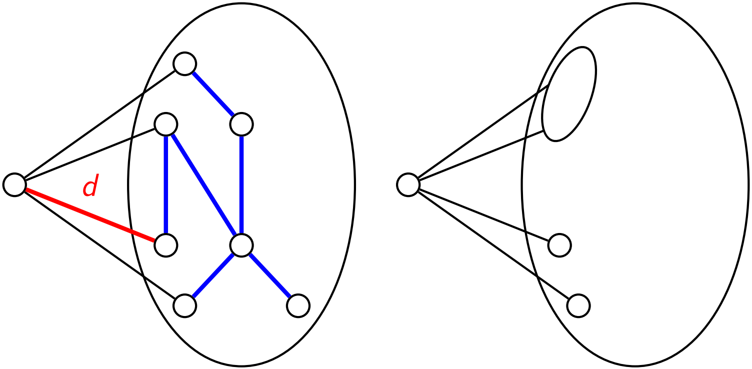

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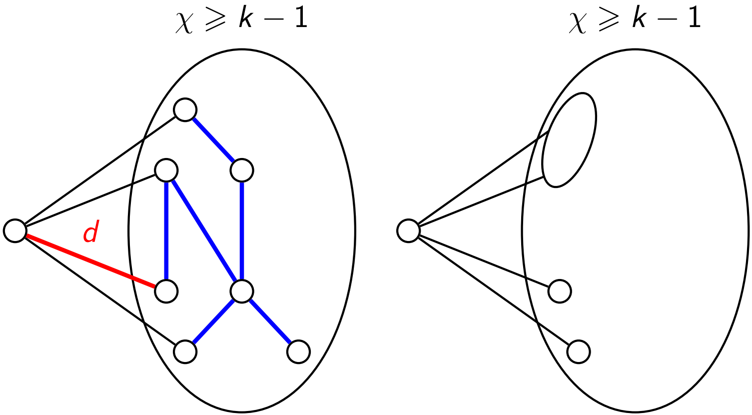

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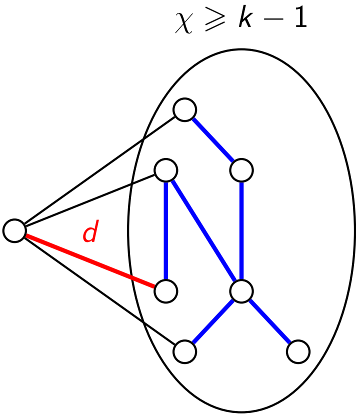

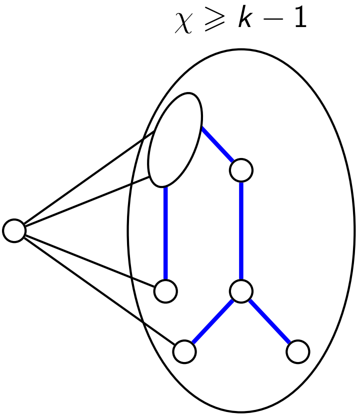

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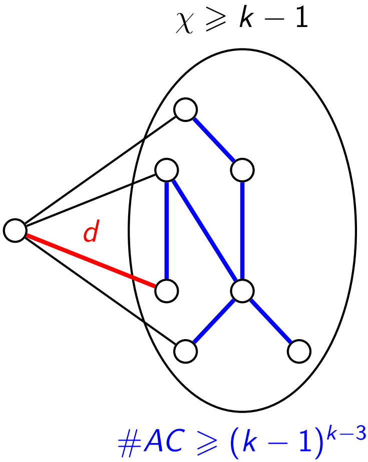

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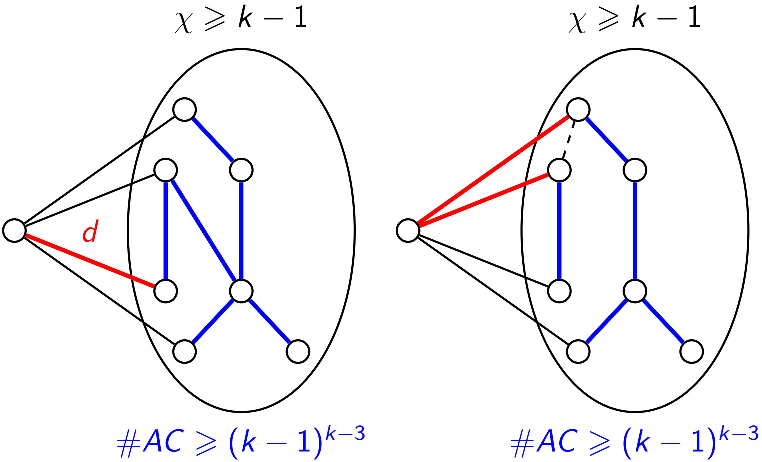

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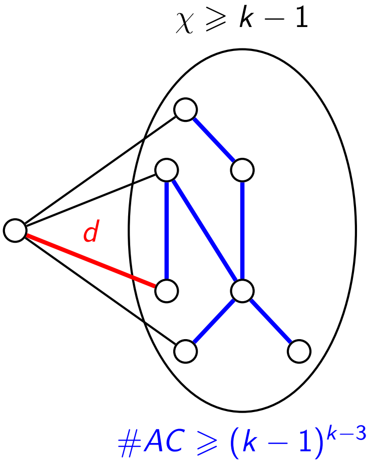

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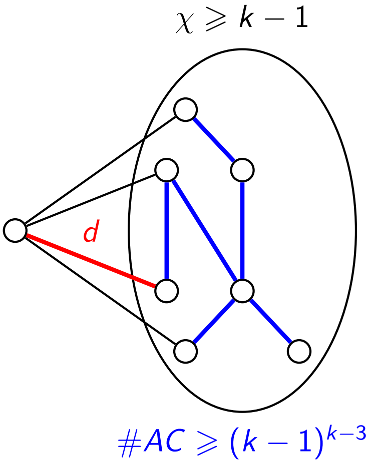

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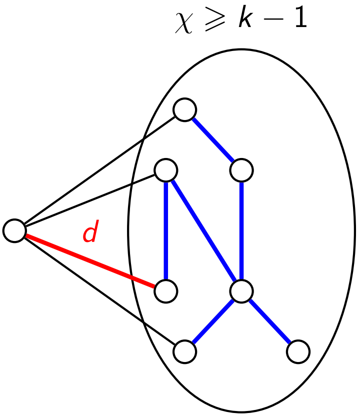
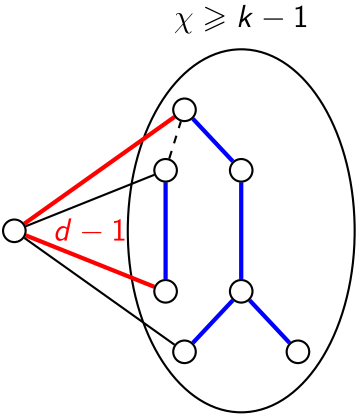




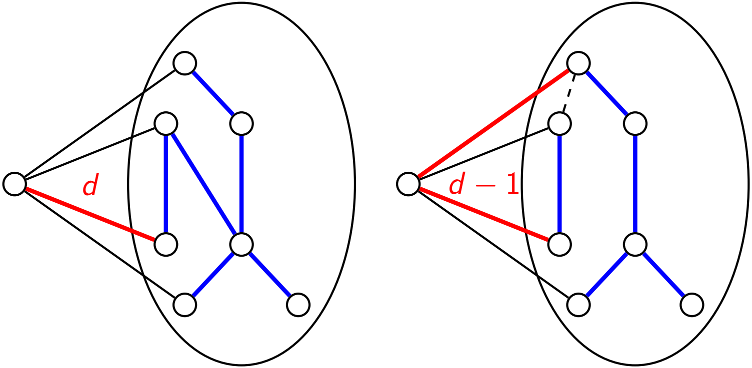
$\#AC \geqslant (k-1)^{k-3}$ $\quad$ $\#AC \geqslant (k-1)^{k-3}$

$$d(k-1)^{k-3} + (d-1)(k-1)^{k-3}$$

Degré minimum $d \geqslant k$

Supposons que G contient .

$\chi \geqslant k-1$ $\chi \geqslant k-1$



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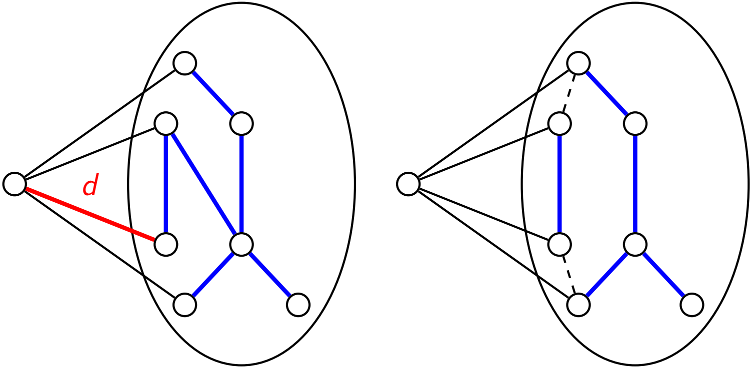

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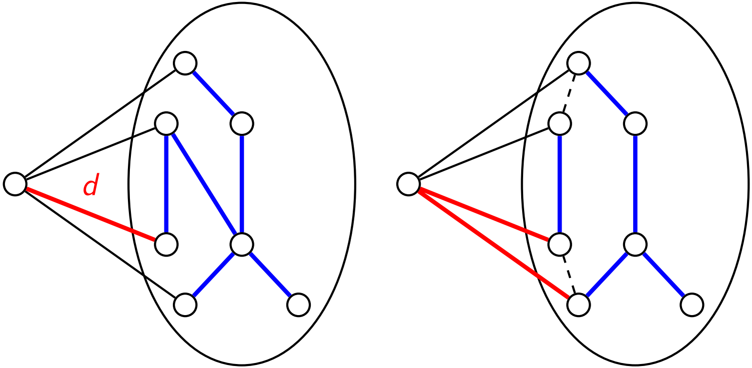
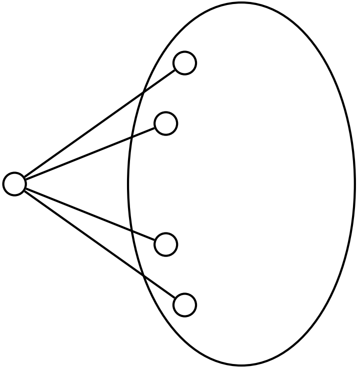


$\#AC \geqslant (k-1)^{k-3}$ $\#AC \geqslant (k-1)^{k-3}$

$$d(k-1)^{k-3} + (d-1)(k-1)^{k-3} + (d-3)(k-1)^{k-3} \geqslant k^{k-2}$$

Sans

Then $d = k$ and there is



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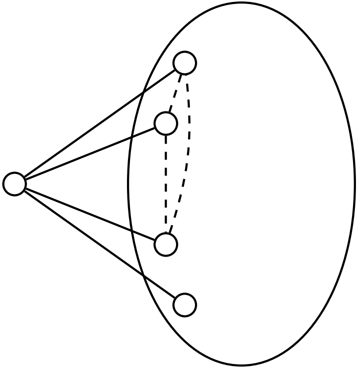
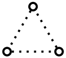
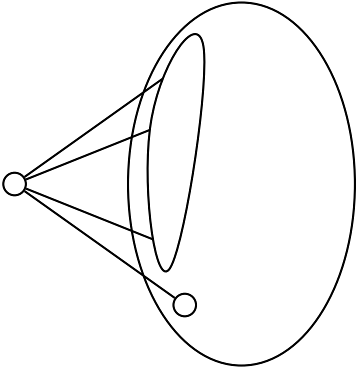


Sans 

Then $d = k$ and there is 



Sans

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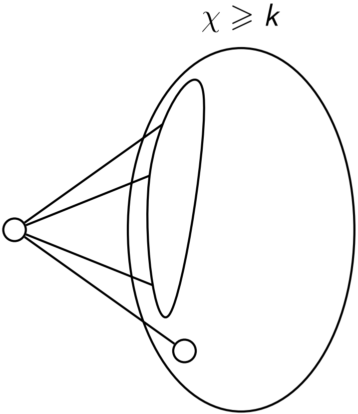
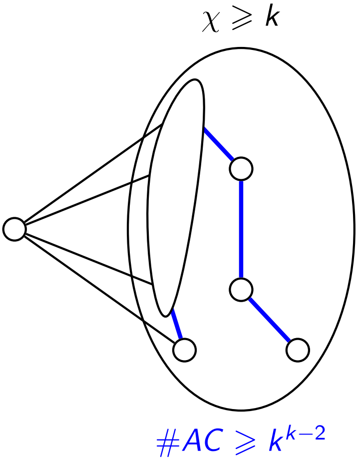


Sans

Then $d = k$ and there is



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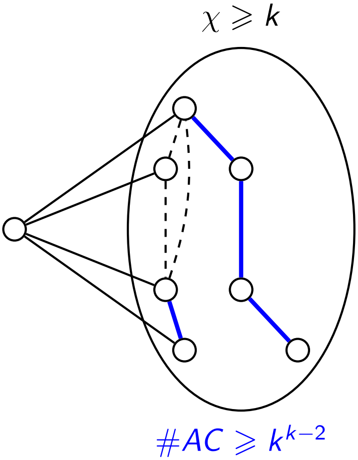
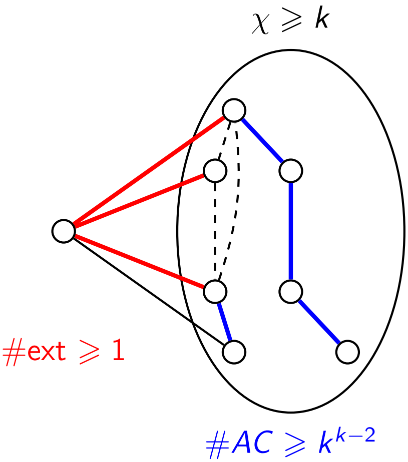


Sans

Then $d = k$ and there is



Conclusion

Cooper, Kabela, Král', P., 2020

Tout graphe k-chromatique a au moins k^{k-2} arbres couvrants.

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- Atteint seulement pour K_k + arbres pendants.

Conclusion

Cooper, Kabela, Král', P., 2020

Tout graphe k-chromatique a au moins k^{k-2} arbres couvrants.

- Atteint seulement pour K_k + arbres pendants.
- Si $\#AC > k^{k-2}$, alors $\#AC \gg k^{k-2}$.

Merci pour votre attention.