Linear transformations between dominating sets in the TAR-model

JGA 2020

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November 18, 2020

Reconfiguration of dominating sets in the TAR-model

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Dominating set D of G: $\forall v \in V$, $v \in D$ or $\exists u \in D$ with $uv \in E$

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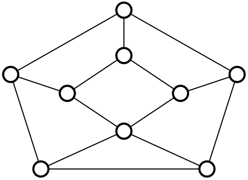

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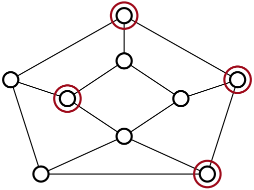

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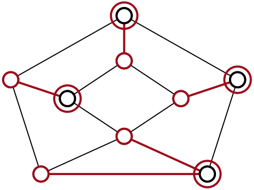

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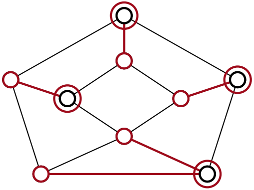

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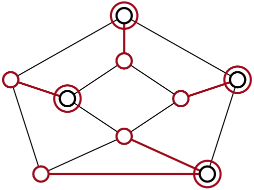

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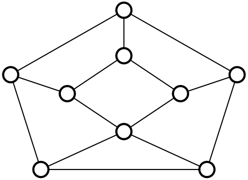

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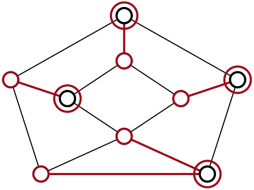

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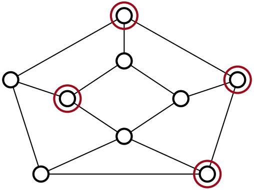

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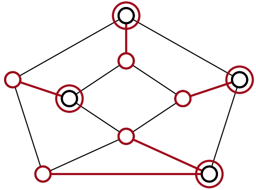

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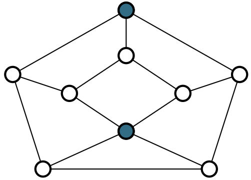

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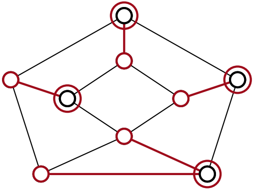

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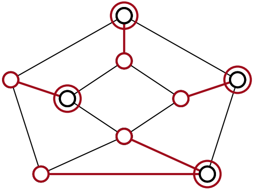

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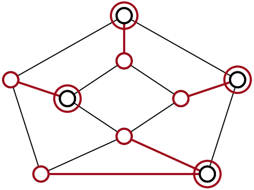

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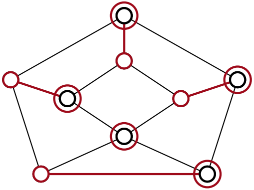

D_2

Reconfiguration of dominating sets in the TAR-model

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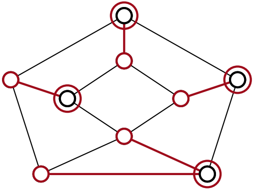

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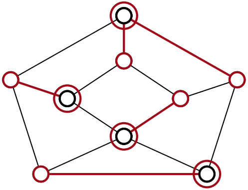

D_3

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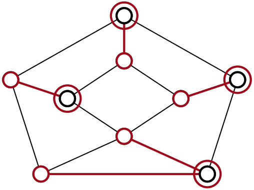

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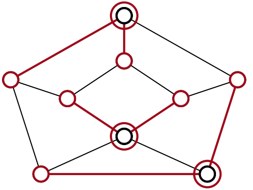

D_4

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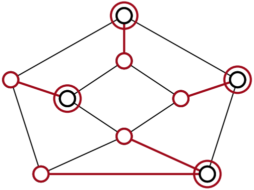

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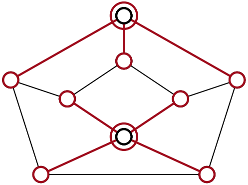

D_t

Problematic

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Reconfiguration graph $\mathcal{R}(G)$: the vertices are the dominating sets of G, two dominating sets are adjacent if they differ by an addition/deletion

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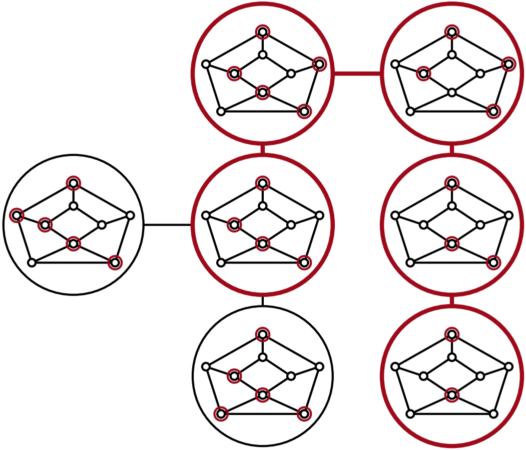

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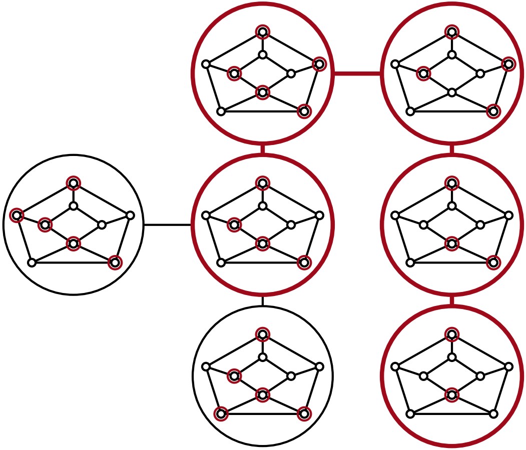

Is $\mathcal{R}(G)$ connected ? What is its diameter ?

Threshold

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If $\mathcal{R}(G)$ contains all the dominating sets, it is connected:

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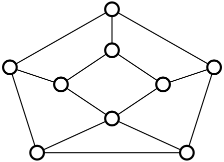

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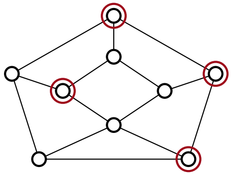

D_s

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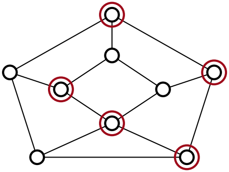

D_2

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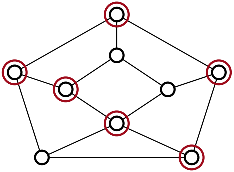
If $\mathcal{R}(G)$ contains all the dominating sets, it is connected:



D_3

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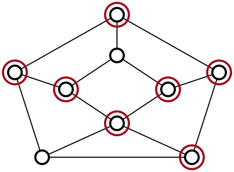

D_4

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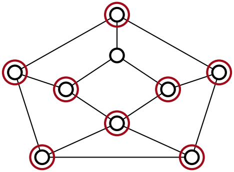
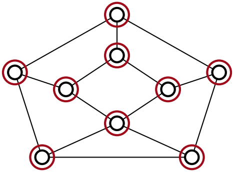


D_5

Threshold

If $\mathcal{R}(G)$ contains all the dominating sets, it is connected:



D_6

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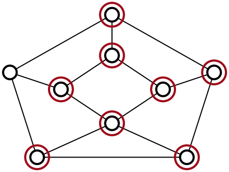
If $\mathcal{R}(G)$ contains all the dominating sets, it is connected:



D_7

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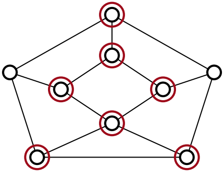

D_8

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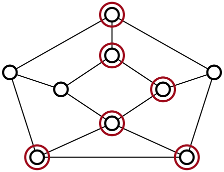

D_9

Threshold

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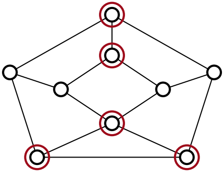
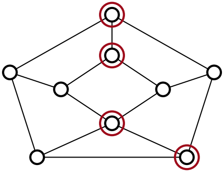


D_{10}

Threshold

If $\mathcal{R}(G)$ contains all the dominating sets, it is connected:



D_{11}

Threshold

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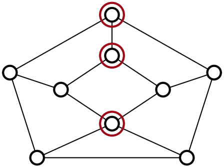

D_{12}

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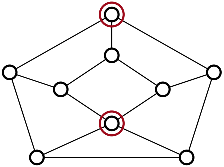

D_t

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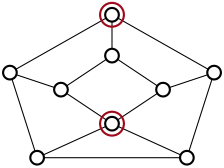

Threshold: maximum size k of the dominating sets in $\mathcal{R}_k(G)$

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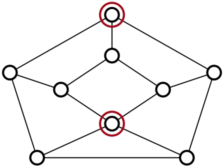

D_t

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Remark: $\mathcal{R}_k(G)$ connected $\not\Rightarrow \mathcal{R}_{k+1}(G)$ connected:

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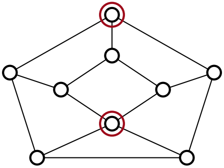

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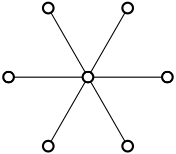

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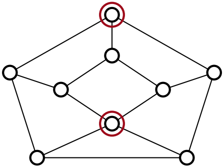

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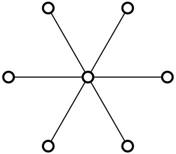

$k = n-2 : \mathcal{R}_k(S_n)$ is connected

Threshold

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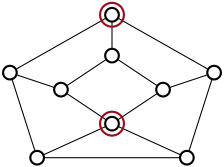

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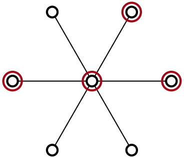

D_s

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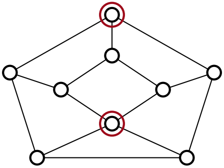

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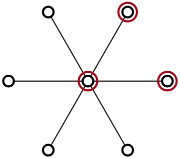

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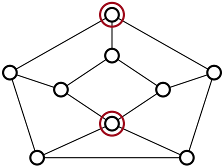

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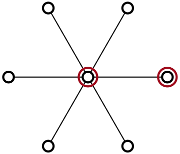

D_3

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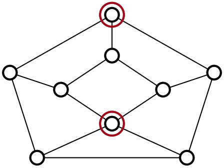

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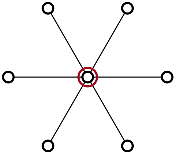

D_4

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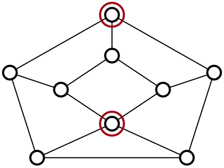

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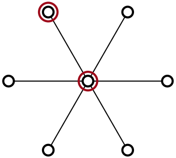

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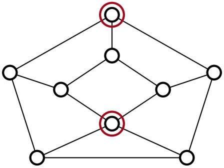

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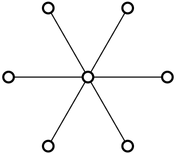

$k = n-2 : \mathcal{R}_k(S_n)$ is connected

$k = n-1 : \mathcal{R}_k(S_n)$ is disconnected

Threshold

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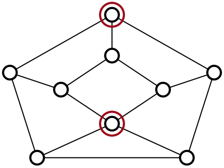

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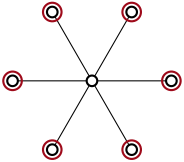
Threshold: maximum size k of the dominating sets in $\mathcal{R}_k(G)$

Remark: $\mathcal{R}_k(G)$ connected $\not\Rightarrow \mathcal{R}_{k+1}(G)$ connected:



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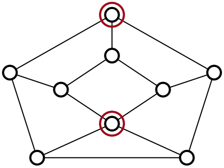

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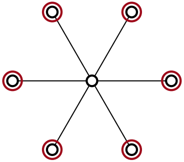

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What is the smallest d_0 s.t. $\mathcal{R}_k(G)$ is connected for any $k \geq d_0$?

State of the art

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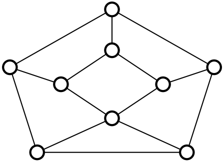

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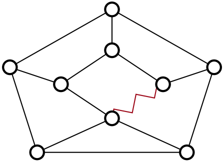

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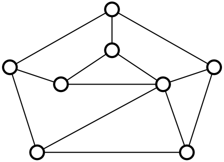

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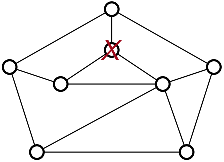

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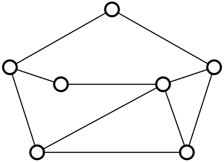

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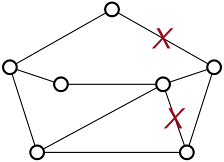

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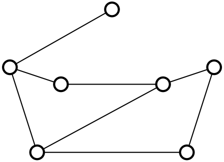

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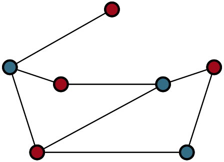

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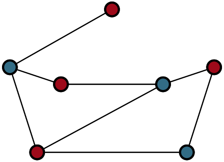

7 vertices, 8 edges $\rightarrow d > \frac{16}{7}$

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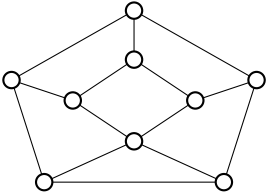

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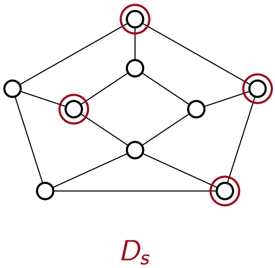

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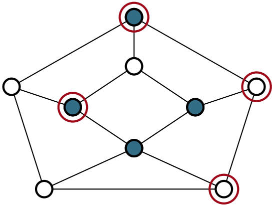

D_t

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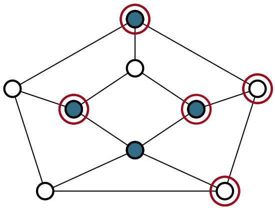

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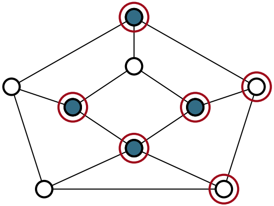

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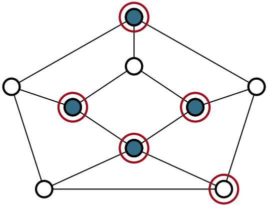

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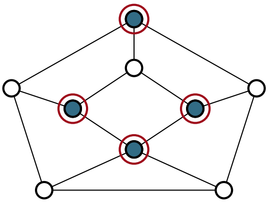

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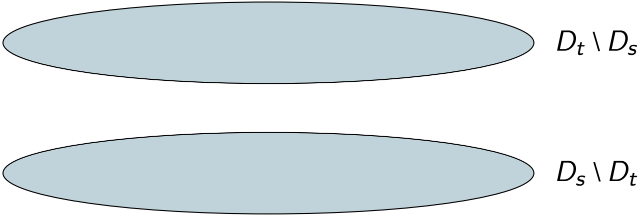

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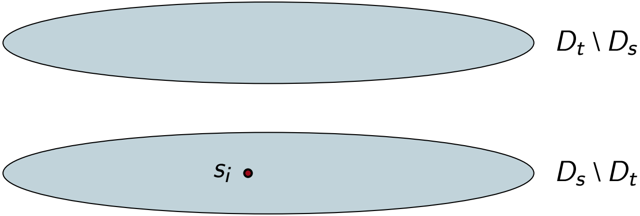

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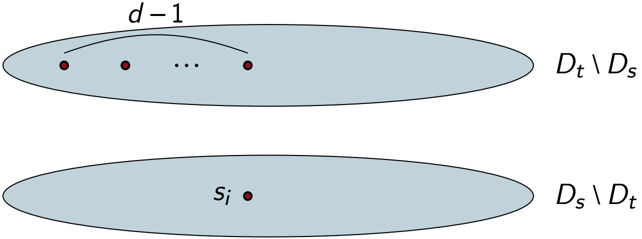

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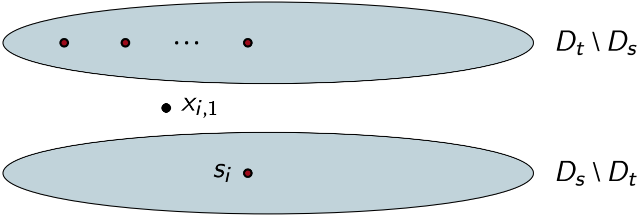

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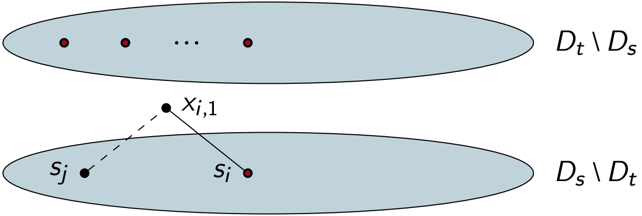

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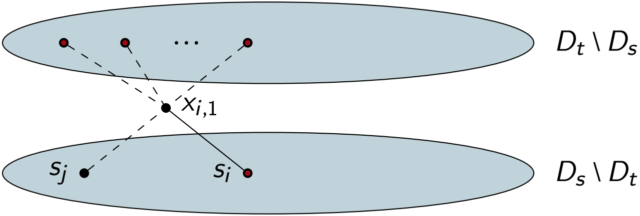

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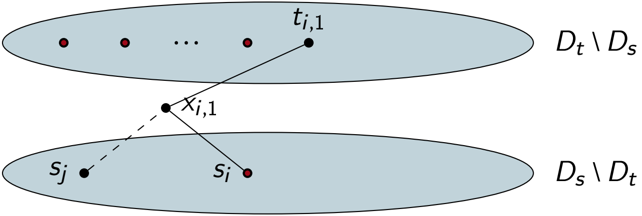

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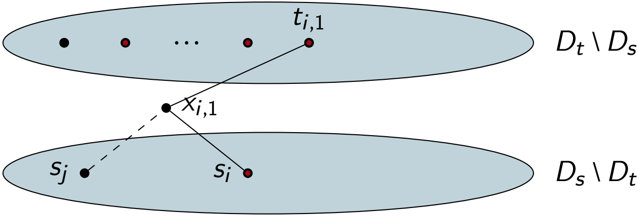

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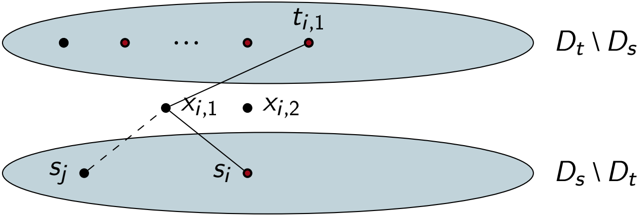

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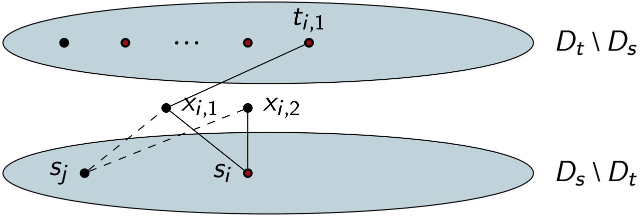

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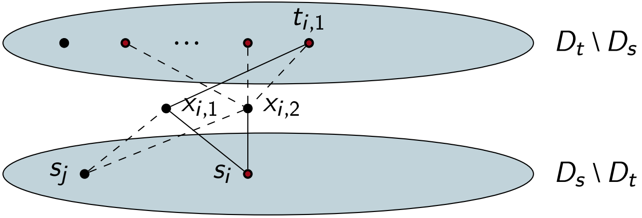

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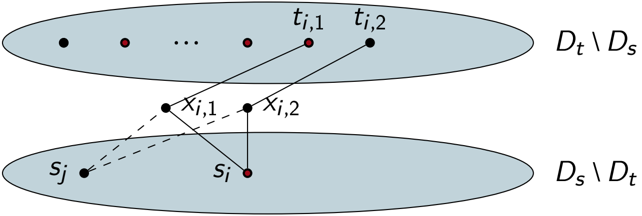

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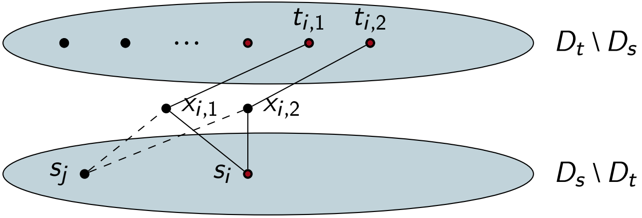

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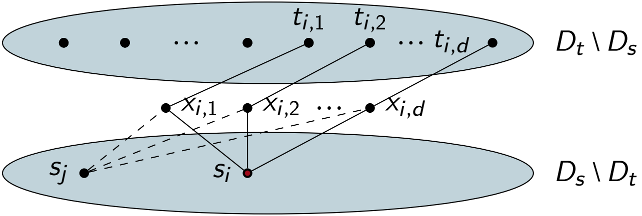

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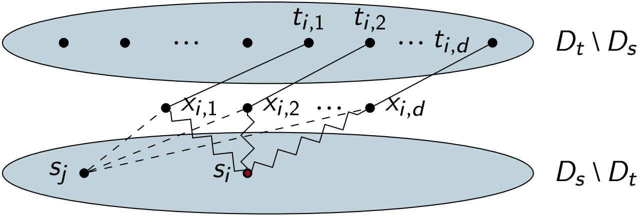

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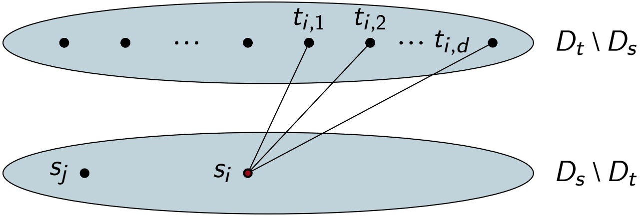

K_ℓ-minor free graphs and planar graphs

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Theorem: For G K_ℓ-minor free, there exists C s.t. if $k \geq \Gamma(G) + C\ell\sqrt{\log \ell}$ then $\mathcal{R}_k(G)$ is connected and has linear diameter

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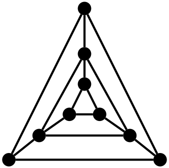

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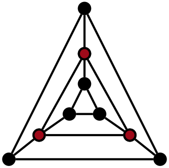

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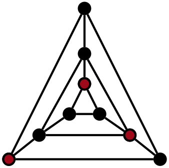

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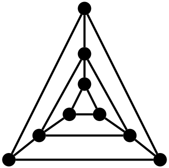

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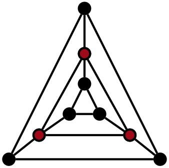

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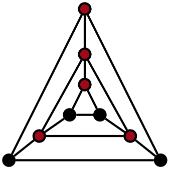

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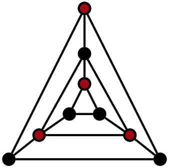

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