On Vizing's edge-colouring question

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J. Narboni

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For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

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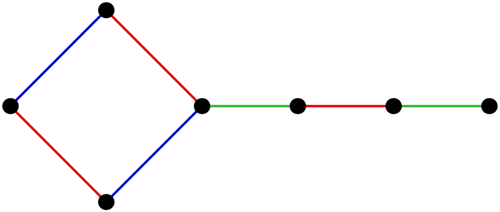

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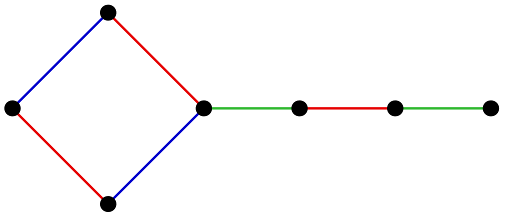

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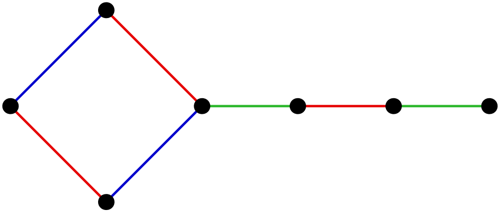

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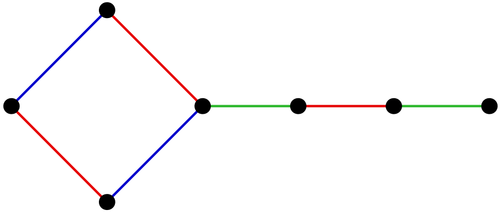

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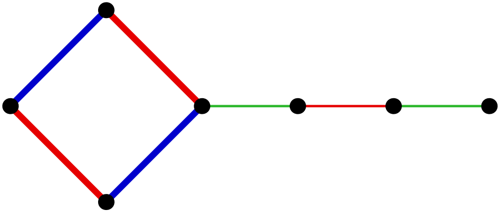

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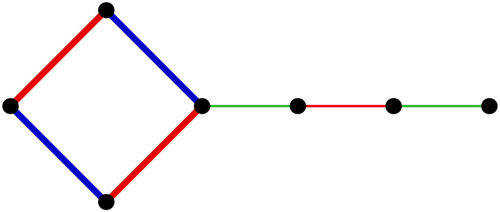

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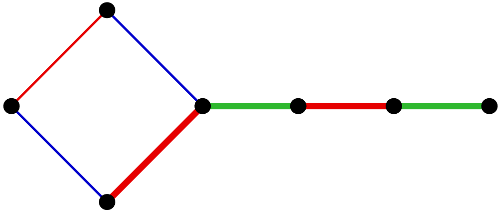

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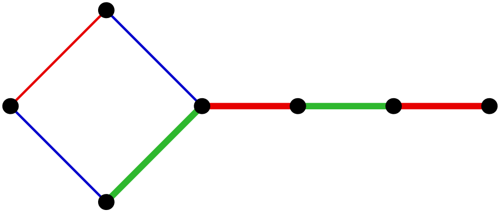

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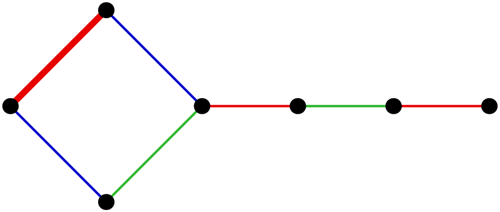

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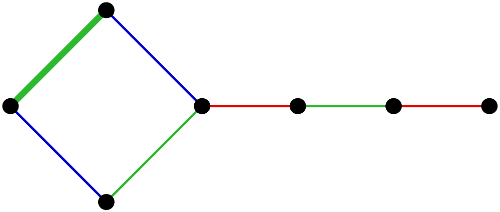

Edge-coloring reconfiguration

Starting from an edge-coloring β, can we reach an other edge-coloring β' using only Kempe-switches?

β is equivalent to β'.

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Theorem (Mc Donald, Mohar, Scheide, 12)

If $\Delta(G)=3$, then all $(\chi'(G)+1)$-colorings are equivalent.

Theorem (Asratian, Casselgren, 16)

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For any triangle-free graph G, any k-coloring of G (with $k > \chi'(G)$) is equivalent to any $\chi'(G)$-colouring of G.

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Corollary

If G is triangle-free, then all $(\chi'(G) + 1)$-edge-colourings are equivalent.

Three key ingredients

- Reduce to the case of χ'- regular graphs.
- Induction on χ': successively remove color classes.
- Vizing's fans.

Simplify our lives

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  - Apply the induction on $G \setminus M$.

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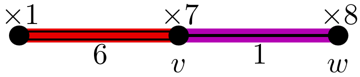
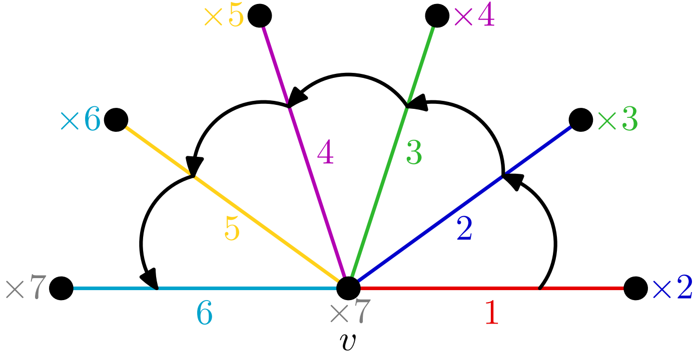


Fan-Like tool

For every vertex v, we consider the directed graph D_v:

- Each vertex represents an edge vv_i incident with v.
- There is an arc between vv_i and vv_j if $m(v_i) = \beta(vv_j)$
- So every vertex has outdegree 0 or 1.
- A fan starting at vv_i is the maximum subgraph reachable from vv_i.



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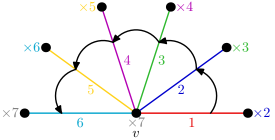

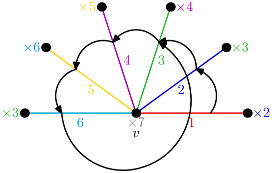

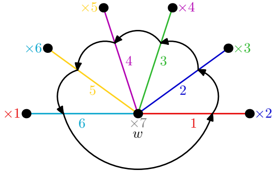

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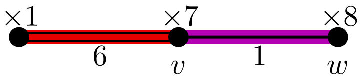

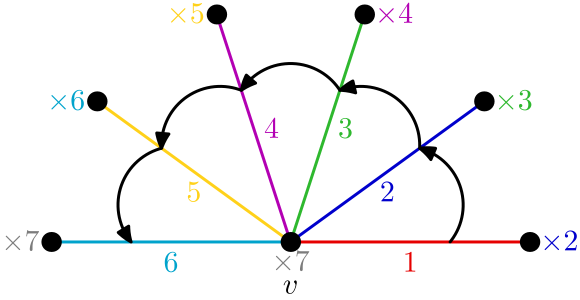

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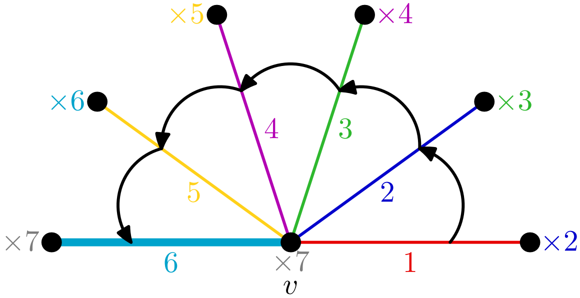

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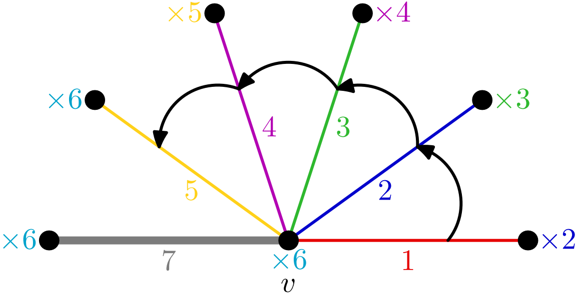

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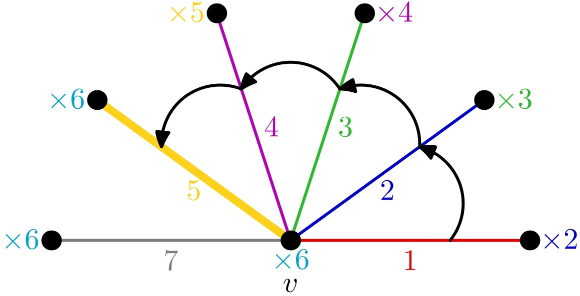

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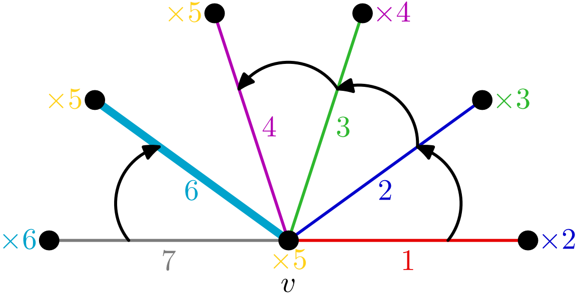

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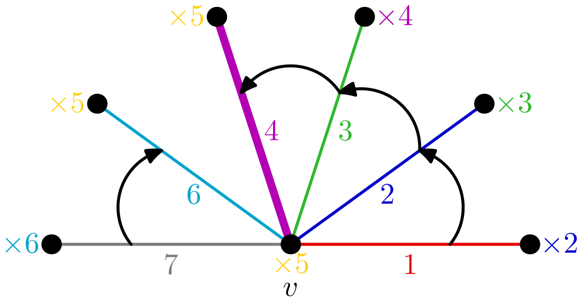

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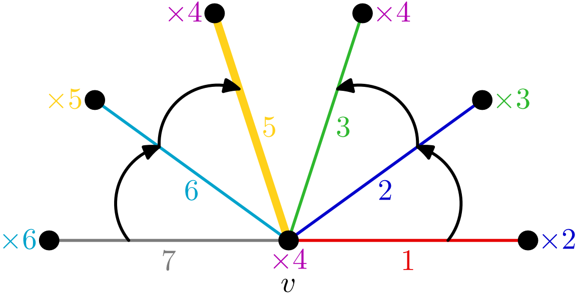

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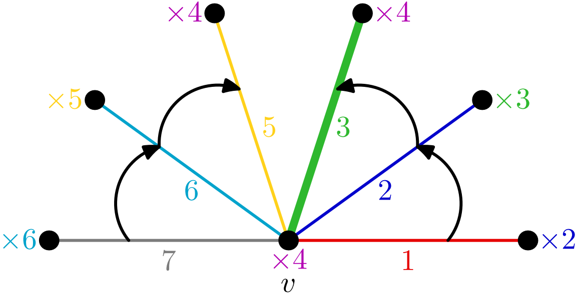

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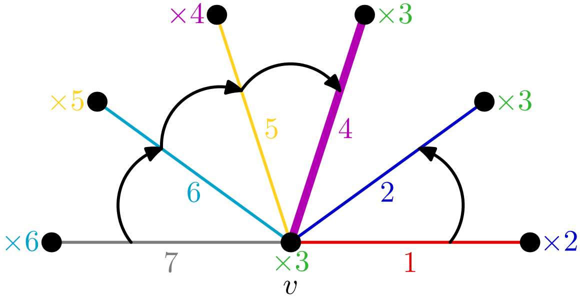

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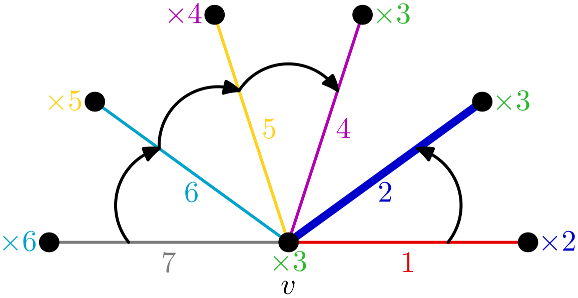

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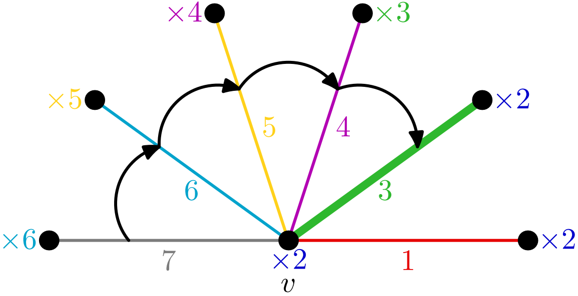

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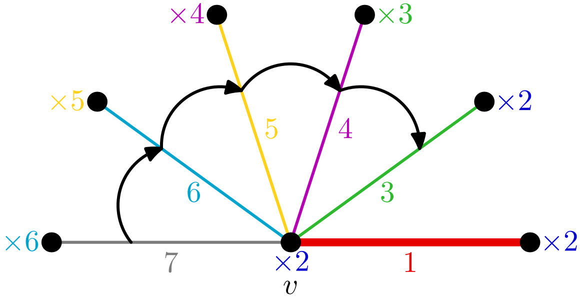

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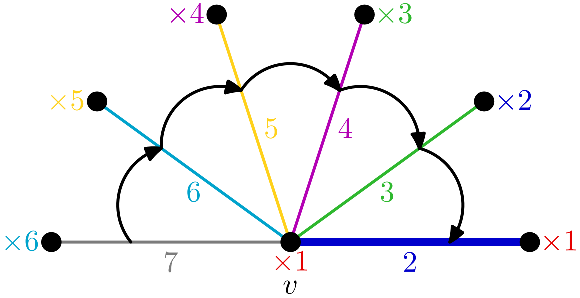

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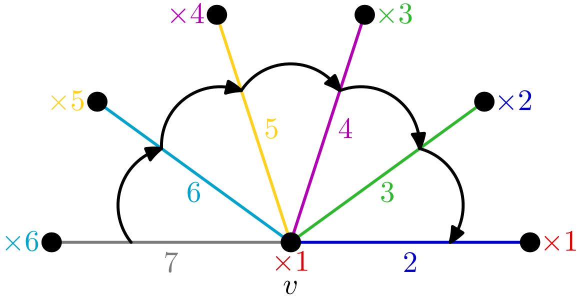

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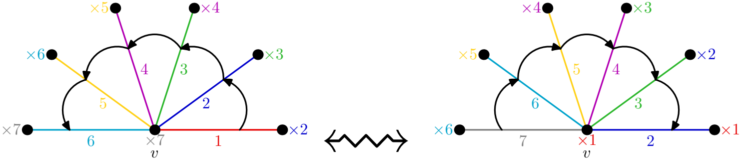

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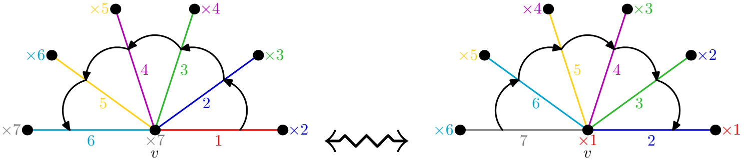
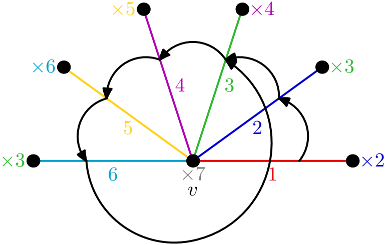
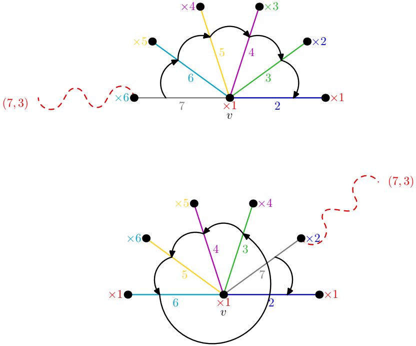


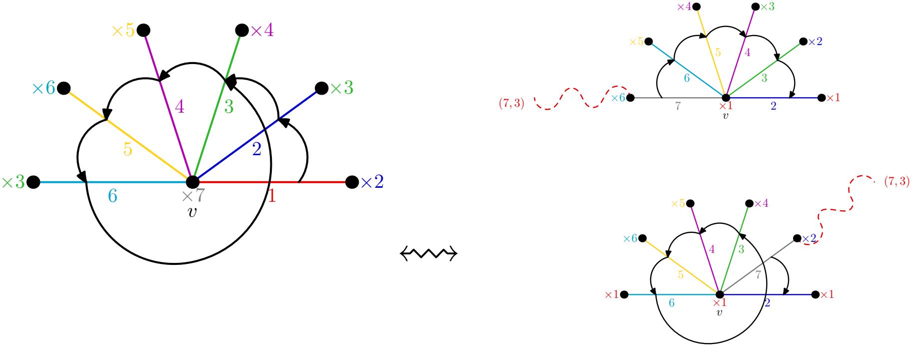
Paths are invertible

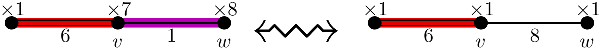
Comets are "almost" invertible



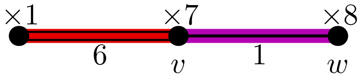


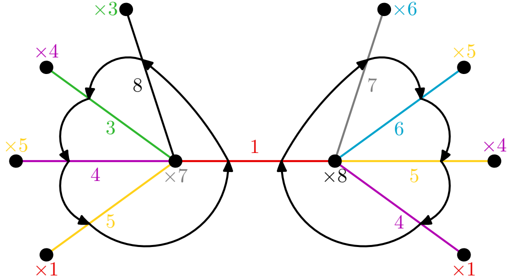
Comets are "almost" invertible



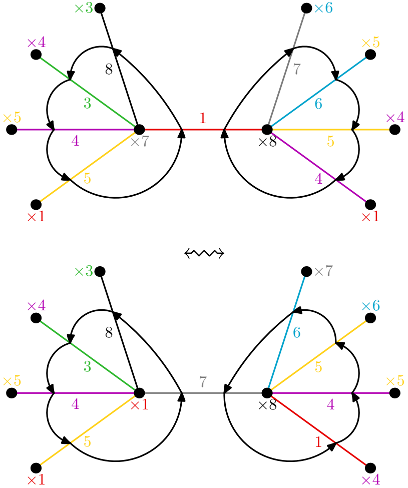


So double cycle everywhere





Double cycle are also invertible



We can always reduce the number of bad or ugly edges.

Further: triangles are bad

Theorem

For any triangle-free graph G, any k-coloring of G (with $k > \chi'(G)$) is equivalent to any $\chi'(G)$-colouring of G.

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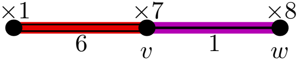
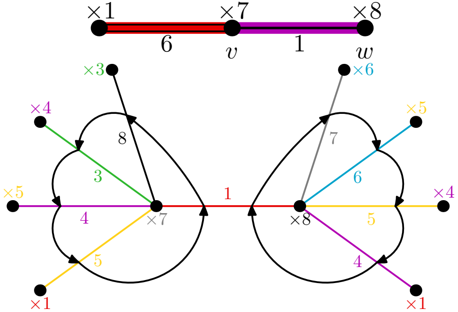


Further: triangles are bad

Theorem

For any *triangle-free* *graph G, any k-coloring of G (with $k > \chi'(G)$) is equivalent to any $\chi'(G)$-colouring of G.*



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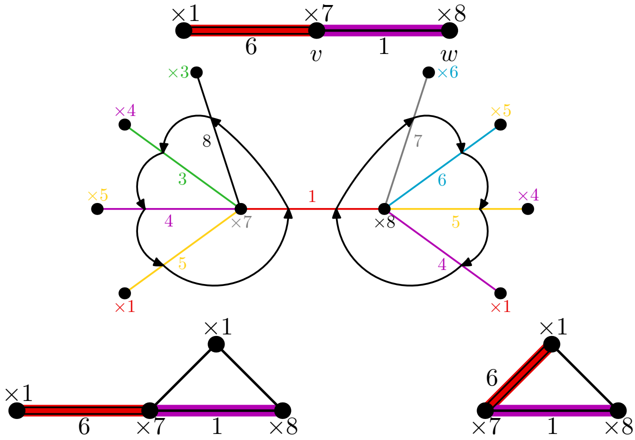

Open questions

- If G is diamond-free, all $(\chi'(G)+1)$ are equivalent?
- If G is $K_{1,1,(\Delta-1)}$-free ?
- Are all $(\chi'+1)$-coloring equivalent?
- Generalizing to multigraphs?

Merci!