On the complexity of the dynamic Steiner problem

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1. Dynamic graphs
2. Steiner problem
3. Possible extensions
4. Focus on a special case: Two terminals, no weight
5. Conclusion

1 Dynamic graphs

2 Steiner problem

3 Possible extensions

4 Focus on a special case: Two terminals, no weight

5 Conclusion

Why dynamic graphs ?

- Time can be an important variable
- Static graphs not sufficient

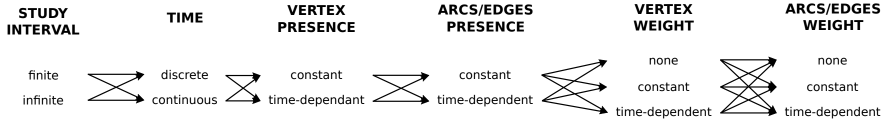
Various application fields

- Transportation networks
  - *Roads temporarily unavailable*
- Communication networks
  - *Sensor networks*
- Social networks
  - *Evolving relationships*

Various terminology

- temporal networks
- dynamic networks
- time varying graphs
- evolving graphs
- temporal graphs
- dynamic graphs
- link streams

Various models



Dynamic graph

- Succession of static graphs: $G = (G_i)_{i \in \mathcal{T}}$, where:
  - $\mathcal{T} = \{1, \ldots, T\}$ is the study interval
  - T is the time horizon
  - $G_i = (V, E_i)$ is a t-graph

Example

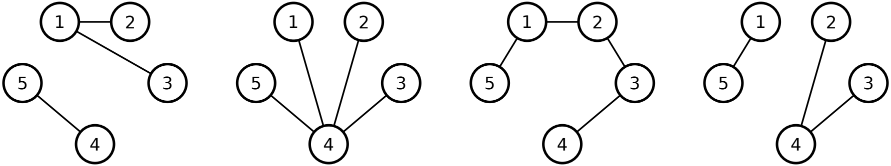


Figure: G_1 Figure: G_2 Figure: G_3 Figure: G_4

1 Dynamic graphs

2 Steiner problem
- Reminder
- What about dynamic graphs?

3 Possible extensions

4 Focus on a special case: Two terminals, no weight

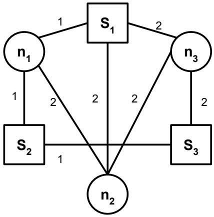
5 Conclusion

Context

- (Static) graph $G = (V, E)$
- Edge weight $w_{(i,j)} \geqslant 0 \ \forall (i, j) \in E$
- Terminal set $S \subset V$

Goal

- Find a tree with minimum weight containing all vertices from S



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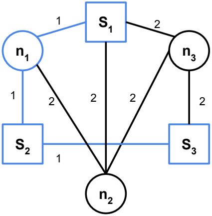

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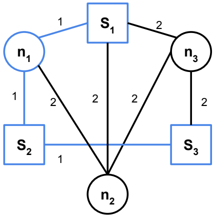

Decision problem

- Is there a subgraph of G containing all vertices from S with total weight lower to K?

Complexity proof

- NP-complete
- Polynomial transformation from *exact cover by 3-sets*

Context

- Dynamic graph: $G = (V, E)$
- Study interval of G: $\mathcal{T} = \{1, \dots, T\}$
- Edges have time-dependent weight: $w_{(i,j),t} \geqslant 0 \ \forall (i,j) \in E, t \in \mathcal{T}$
- Terminal set $S \subset V$

Questions

- What is a Steiner Tree in a dynamic graph ?
  - A "dynamic tree" containing all vertices of S with minimum total weight on $\mathcal{T}$
- How is that tree ?
- Can special cases be identified ?

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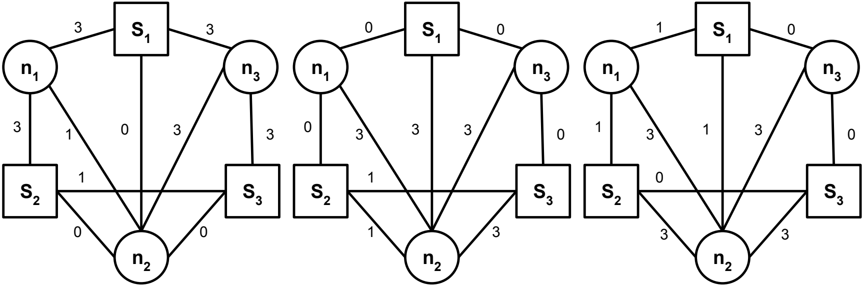
Possibility 1 : Fully Connected Set

- Find $V' \subset V$ such that $S \subset V'$
- Find spanning tree of V' of minimum weight



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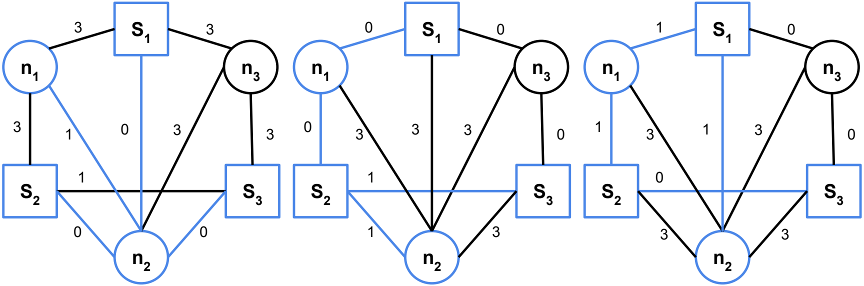

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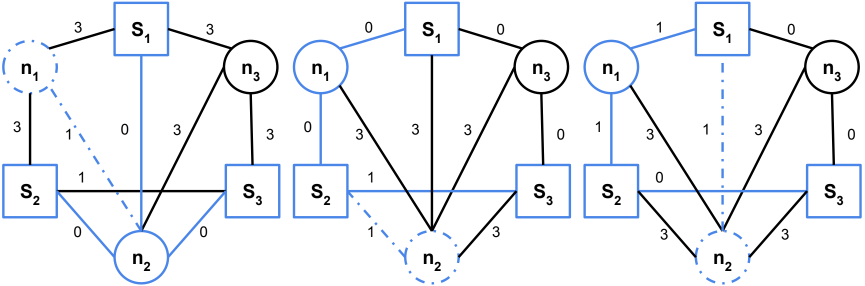

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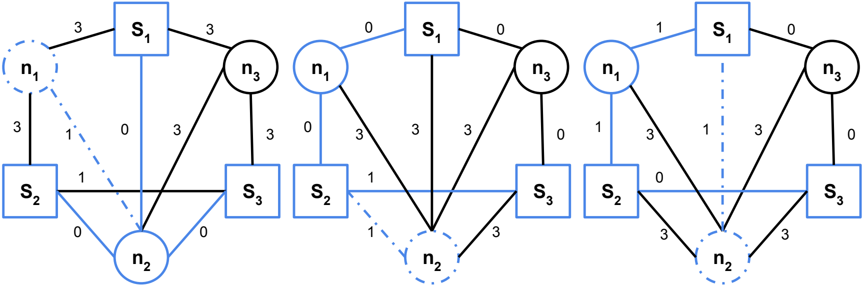
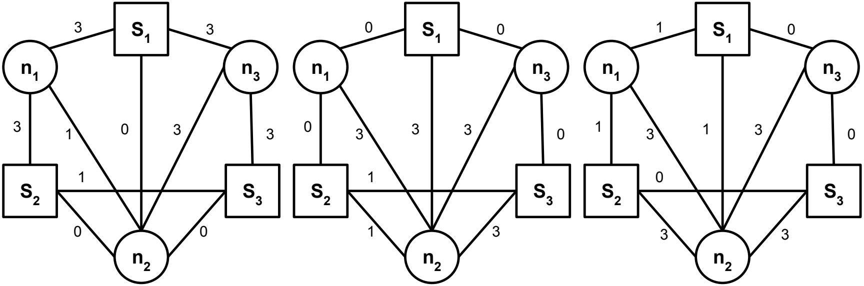


Problem

- As long as the terminals are connected, what is the point of connecting V'?

Possibility 2: Partially connected Set

- Find $V' \subset V$ such that $S \subset V'$
- Minimize the weight of edges connecting S



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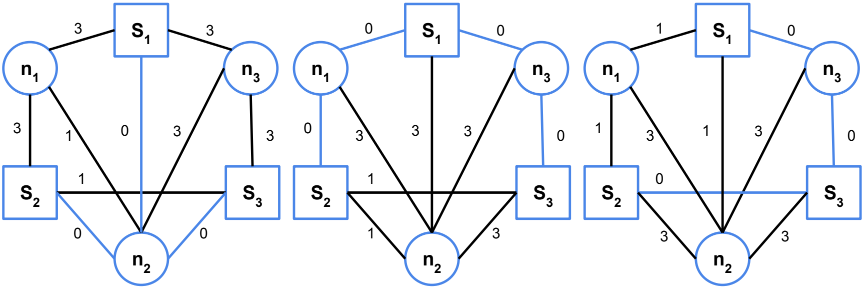


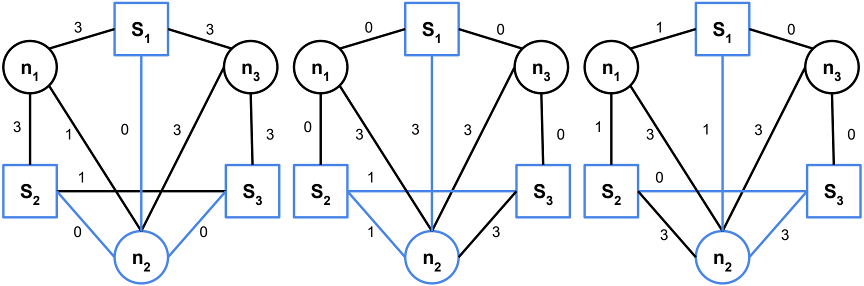
Problem

- Take $V' = V$ and look for Steiner tree at each time step ignoring vertices from $V' \setminus S$

Partially connected Minimum Steiner Set

Find V' with $S \subset V' \subset V$ and $E'_t \subset E_t$ $\forall t \leqslant T$ such that

- All vertices of S in same connected component in $G'_t = (V', E'_t)$
- Cardinality of V' is minimum
- $\sum_e \sum_t w_{e'_t}$ with $e'_t \in E'_t$



1 Dynamic graphs

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3 Possible extensions

4 Focus on a special case: Two terminals, no weight
- Definition
- Extreme examples
- Complexity proof

5 Conclusion

Hypothesis

- No weight on edges
- $|S| = 2$: Connect optimally two vertices
- Minimize number of vertices keeping the terminals connected

Problem

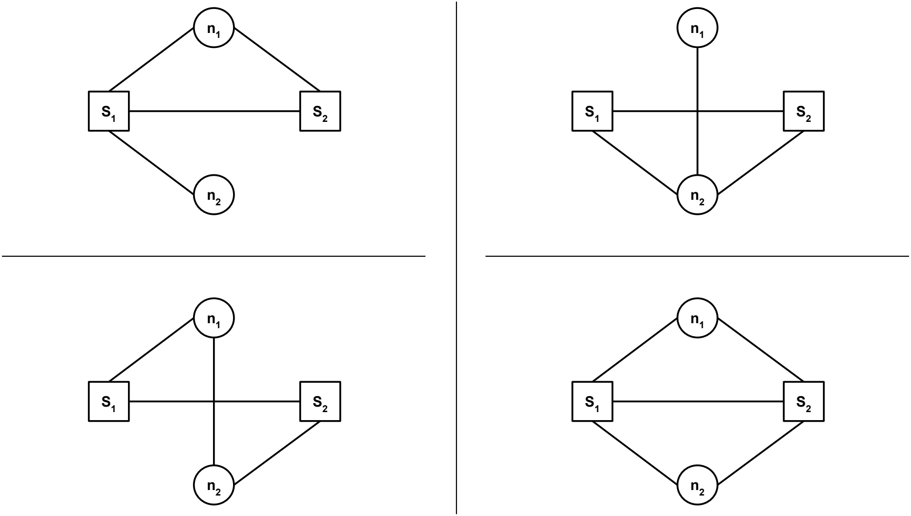
Find V' with $S \subset V' \subset V$ ($|S| = 2$), and $E'_t \subset E_t$ $\forall t \leqslant T$ such that

- All vertices of S in same connected component in $G'_t = (V', E'_t)$
- Cardinality of V' is minimum

Remarks

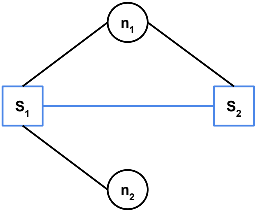
- Polynomial in static graphs
- NP-complete on dynamic graphs

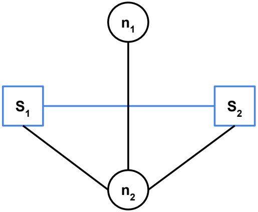
Example 1

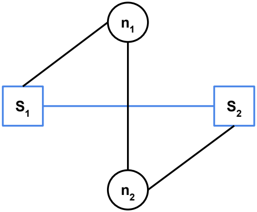


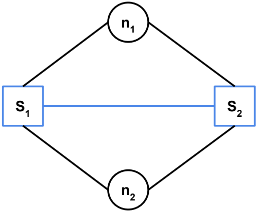
Example 1

No extra vertex is necessary

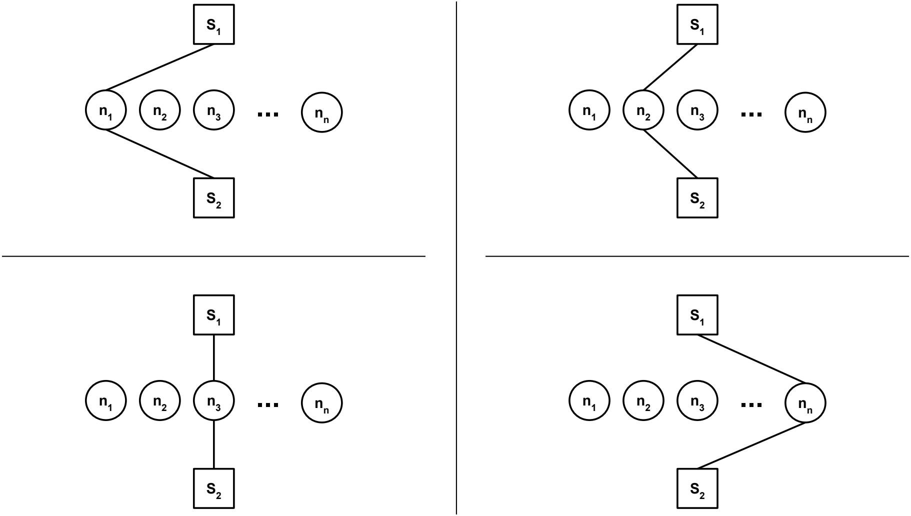






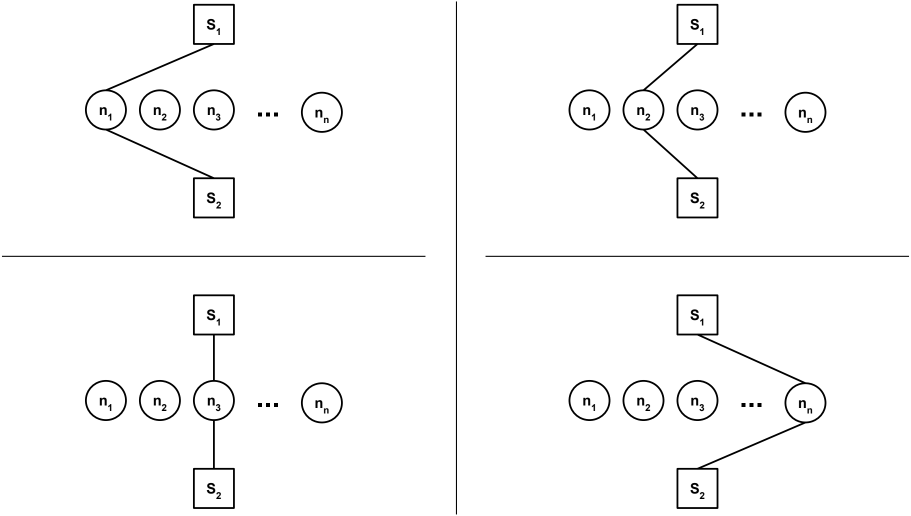


Example 2

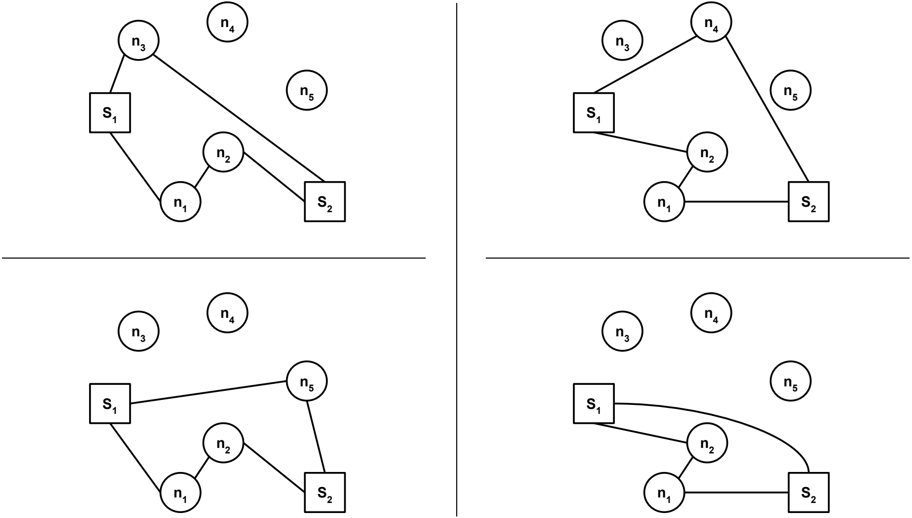


Example 2

All vertices of the graph are necessary



Example 3


n4
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n5
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n2
n1
S2
n4
n3
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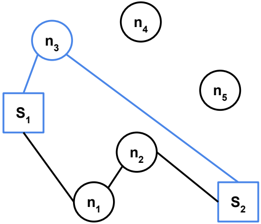
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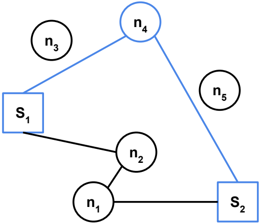
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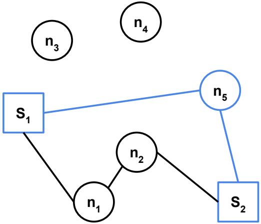
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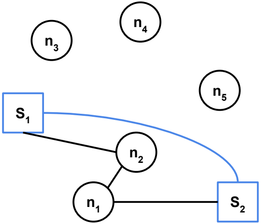
n3
n4
n5
S1
n2
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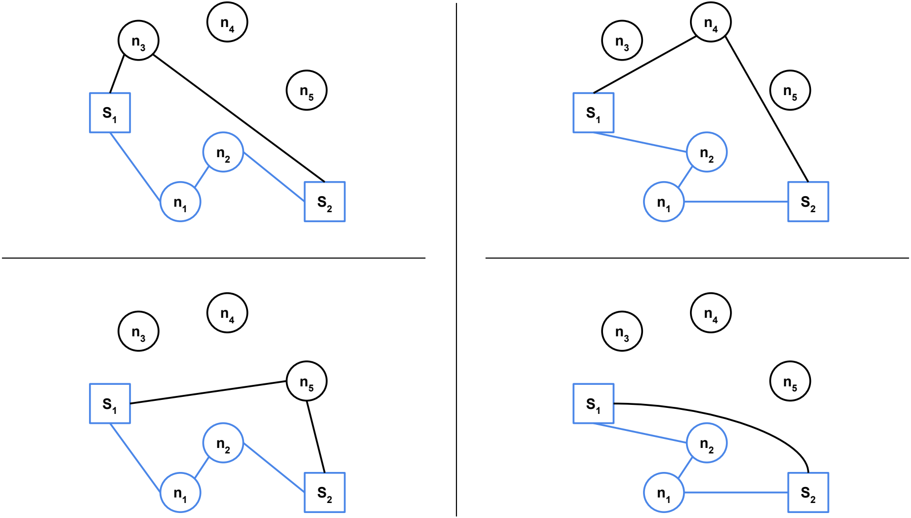
S2

Example 3

The shortest path is not a good idea



NP-completeness proof

- Polynomial transformation from the Vertex Cover Problem

Reminder: Vertex Cover

- Graph $G = (V, E)$
- Vertex Cover Set $V_c \subset V$ such that $\forall (u, v) \in E$, $u \in V_c$ or $v \in V_c$

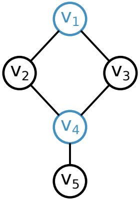
For a given integer $k \geq 0$, is there a set V_c of size k ?

Transformation

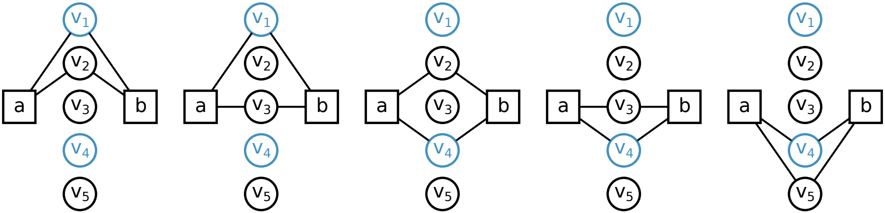
- $\forall u \in V$, there is a vertex u in the dynamic graph G^{DYN}
- G^{DYN} has two extra vertices a and b
- $\forall e = (u, v) \in E$, there is a time step i_e in G^{DYN} and $G^{DYN}_{i_e}$ has 4 edges; (a, u), (u, b), (a, v), (v, b)

Example of transformation

Vertex Cover instance:



Corresponding instance on dynamic graph:



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Summary

- Extension of Steiner Problem to dynamic graphs
- Special case proven to be NP-complete

Future work

- Exact algorithms efficient in specific cases
- Approximation algorithms

Thank you for your attention

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